Hom-Lie 2-superalgebras

Chunyue WANG$^1$, Qingcheng ZHANG$^{2,*}$, Jizhu NAN$^3$

$^1$School of Applied Sciences, Jilin Engineering Normal University, Changchun, P.R. China
$^2$School of Mathematics and Statistics, Northeast Normal University, Changchun, P.R. China
$^3$School of Mathematical Sciences, Dalian University of Technology, Dalian, P.R. China

Received: 29.03.2015  •  Accepted/Published Online: 16.06.2015  •  Final Version: 01.01.2016

Abstract: Hom-Lie 2-superalgebras can be considered as the categorification of Hom-Lie superalgebras. We give the definition of Hom-Lie 2-superalgebras and study their superderivations. We obtain the representation, deformation, and abelian extensions related to the 2-cocycle and Hom-Nijenhuis operators. Moreover, we also construct a skeletal (strict) Hom-Lie 2-superalgebra from a Hom-associative Rota-Baxter superalgebra.

Key words: Hom-Lie 2-superalgebras, superderivations, representations, deformations, abelian extensions, Hom-associative Rota-Baxter superalgebras

1. Introduction

Higher categorical structures play an important role in both string theory [2] and physics [9,15]. Some higher categorical structures are obtained by categorifying existing mathematical concepts. One of the simplest higher structures is a categorical vector space, that is, a 2-vector space. A categorical Lie algebra introduced by Baez and Crans [3], which is called a Lie 2-algebra, is a 2-vector space equipped with a skew-symmetric bilinear functor, whose Jacobi identity is replaced by the Jacobiator satisfying some coherence laws of its own. Baez and Crans [3] showed that the category of Lie 2-algebras is equivalent to the category of 2-term $L_1$-algebras, so a Lie 2-algebra is often defined by a 2-term $L_1$-algebra. Recently, Lie 2-algebra theories have been widely developed [4,5,10,12,14,16–19]. In particular, Lie 2-superalgebras were studied in [7,25].

Hom-Lie algebras were initially introduced by Hartwig et al. [6] to study the deformations of the Witt and the Virasoro algebras. A Hom-algebra is also connected with deformed vector fields, so many results about Hom-algebra structures have been investigated [1,8,13,20,22–24]. The categorification of Hom-Lie algebras, which is called a Hom-Lie 2-algebra, was given in [21].

In this paper, we generalize Hom-Lie 2-algebras to Hom-Lie 2-superalgebras, which are regarded as the deformation and categorification of Lie superalgebras. It was proved that the category of Hom-Lie 2-algebras and the category of 2-term $HL_1$-algebras are equivalent in [21]. An analogous result is obtained in the case of Hom-Lie 2-superalgebras, so we define Hom-Lie 2-superalgebras by 2-term Hom-$L_1$-algebras. Motivated by deformations of Lie 2-algebras [11], we give notions of representations and 2-cocycles of Hom-Lie 2-superalgebras, and we prove that a 1-parameter infinitesimal deformation is related to a 2-cocycle with coefficients in adjoint representations. Furthermore, we study Hom-Nijenhuis operators and abelian extensions...
connected with representations and 2-cocycles. In particular, we show that the superderivation of idempotent Hom-Lie 2-superalgebras under a commutator is a strict Lie 2-superalgebra.

The paper is organized as follows. In Section 2, we give notions of Hom-Lie 2 superalgebras and their homomorphisms. In Section 3, we give the definition of superderivations of Hom-Lie 2-superalgebras, and we prove that the superderivation of degree 0 of idempotent Hom-Lie 2-superalgebras is a Lie superalgebra. In Section 4, we show the relation between 1-parameter infinitesimal deformations and 2-cocycles of Hom-Lie 2-superalgebras. In Section 5, the Hom-Nijenhuis operators of Hom-Lie 2-superalgebras are studied. In Section 6, we show that there exists a representation and a 2-cocycle associated to any abelian extensions. Finally, we construct a skeletal (strict) Hom-Lie 2-superalgebra from a Hom-associative Rota-Baxter superalgebra.

The parity of the homogeneous element \( x \) in superalgebras (super vector spaces) is denoted by \( |x| \). The set of all homogeneous elements of Hom-Lie 2-superalgebras \( \mathcal{M} \) is denoted by \( \text{hg}(\mathcal{M}) \).

2. Preliminaries

In this section, we first give the notion of Hom-Lie 2-superalgebras, and then we study some properties of the homomorphism of Hom-Lie 2-superalgebras.

**Definition 2.1** A Hom-Lie 2-superalgebra consists of the following data:

- two super vector spaces \( M_0 \) and \( M_1 \) together with an even linear map \( d: M_1 \to M_0 \),
- an even bilinear map \( [\cdot, \cdot] : M_i \times M_j \to M_{i+j} \) \( (0 \leq i + j \leq 1) \),
- two even linear maps \( \tau_0 : M_0 \to M_0 \) and \( \tau_1 : M_1 \to M_1 \) satisfying \( \tau_0 \circ d = d \circ \tau_1 \),
- an even skew-symmetric trilinear map \( l_3 : M_0 \times M_0 \times M_0 \to M_1 \) satisfying \( l_3 \circ \tau_0 = \tau_1 \circ l_3 \), such that for any \( x, y, z, t \in \text{hg}(M_0) \), \( a, b \in \text{hg}(M_1) \), the following equalities are satisfied:

  1. \( [x, y] = -(-1)^{|x||y|} [y, x] \),
  2. \( [x, a] = -(-1)^{|x||a|} [a, x] \),
  3. \( [a, b] = 0 \),
  4. \( d([x, a]) = [x, da] \),
  5. \( [da, b] = [a, db] \),
  6. \( \tau_0([x, y]) = [\tau_0(x), \tau_0(y)] \),
  7. \( \tau_1([x, a]) = [\tau_0(x), \tau_1(a)] \),
  8. \( dl_3(x, y, z) = [\tau_0(x), [y, z]] + (-1)^{|x||y|+|z|}[\tau_0(y), [z, x]] + (-1)^{|x||y|+|z|}[\tau_0(z), [x, y]] \),
  9. \( l_3(x, y, da) = [\tau_0(x), [y, a]] + (-1)^{|x||y|+|a|}[\tau_0(y), [a, x]] + (-1)^{|x||y|+|a|}[\tau_1(a), [x, y]] \),
  10. \( l_3([t, x], t_0(y), t_0(z)) + (-1)^{|x||y|+|z|}l_3([t, z], t_0(x), t_0(y)) + (-1)^{|x||y|+|z|}l_3([t, y], t_0(z), t_0(y)) \) + \( (-1)^{|x||y|+|z|}l_3([t, y], t_0(z), t_0(x)) + (-1)^{|x||y|+|z|}l_3([t, x], t_0(y), t_0(z)) \) + \( (-1)^{|x||y|+|z|}l_3([t, x], t_0(y), t_0(z)) + (-1)^{|x||y|+|z|}l_3([t, y], t_0(z), t_0(x)) \),

where \( |x|, |y|, |z|, |t_0(x)|, |t_0(y)|, |t_0(z)| \) are the parities of \( x, y, z, t_0(x), t_0(y), t_0(z) \), respectively.

A Hom-Lie 2-superalgebra is denoted by \( (\mathcal{M} : M_1 \xrightarrow{d} M_0, [\cdot, \cdot], l_3, \tau_0, \tau_1) \), simply denoted by \( \mathcal{M} \).

**Example 2.2** Let \( (M, [\cdot, \cdot], B) \) be a multiplicative quadratic Hom-Lie superalgebra. It gives a Hom-Lie
2-superalgebra on the super vector space $M \oplus \mathbb{R}$, denoted by $(M \oplus \mathbb{R} : \mathbb{R} \xrightarrow{d=0} M, [-,], l_3, \beta, I_{\mathbb{R}})$, where $M$ is of degree 0, $\mathbb{R}$ is of degree $-1$, an even linear map $d$ is defined by $0 = d : \mathbb{R} \to M$, an even bilinear map $[-,] : (M \oplus \mathbb{R}) \times (M \oplus \mathbb{R}) \to M \oplus \mathbb{R}$ is defined by $[x + a, y + b] = [x, y]_M$, and an even trilinear map $l_3 : M \times M \times M \to \mathbb{R}$ is defined by $l_3(x, y, z) = B([x, y]_M, z)$.

**Definition 2.3** Let $(\mathbb{M} : M_1 \xrightarrow{d_1} M_0, [,], l_3, \tau_0, \tau_1)$ and $(\mathbb{M}' : M'_1 \xrightarrow{d'_1} M'_0, [,], l'_3, \tau'_0, \tau'_1)$ be two Hom-Lie 2-superalgebras. A Hom-Lie 2-superalgebra homomorphism $g : \mathbb{M} \to \mathbb{M}'$ consists of

- an even linear map $g_0 : M_0 \to M'_0$ satisfying $g_0 \circ \tau_0 = \tau'_0 \circ g_0$,
- an even linear map $g_1 : M_1 \to M'_1$ satisfying $g_1 \circ \tau_1 = \tau'_1 \circ g_1$,
- an even skew supersymmetry bilinear map $g_2 : M_0 \times M_0 \to M'_1$ satisfying $g_2(\tau_0(x), \tau_0(y)) = \tau'_1(g_2(x, y))$

such that the following equalities hold for any $x, y, z \in hq(M_0), a \in hq(M_1)$:

1. $g_0 \circ d = d' \circ g_1$,
2. $g_0([x, y]_M) = [g_0(x), g_0(y)]_{M'} = d'(g_2(x, y))$,
3. $g_1([x, a]_M) = [g_0(x), g_1(a)]_{M'} = g_2(x, da)$,
4. $g_2([x, y]_M, \tau_0(z)) + (-1)^{|x||y|+|z|}g_2([y, z]_M, \tau_0(x)) + (-1)^{|x||y|+|z|}g_2([z, x]_M, \tau_0(y))$
   
   + $g_1(l_3(x, y, z)) - [g_0(\tau_0(x)), g_2(\tau_0(y), z)]_{M'} - (-1)^{|x||y|+|z|}g_0(\tau_0(y)), g_2(z, x)]_{M'}$
   
   $= (-1)^{|x||y|+|z|}[g_0(\tau_0(z)), g_2(x, y)]_{M'} + l'_3(g_0(x), g_0(y), g_0(z))$.

The homomorphism of Hom-Lie 2-superalgebras is denoted by $g = (g_0, g_1, g_2)$.

The homomorphism $g$ is called strict if $g_2 = 0$. The identity homomorphism $I_{\mathbb{M}} : \mathbb{M} \to \mathbb{M}$ is defined by $I_0 : M_0 \to M_0, I_1 : M_1 \to M_1$, and $I_2 = 0$, denoted by $I_{\mathbb{M}} = (I_0, I_1, 0)$.

Let $g : \mathbb{M} \to \mathbb{M}'$ and $g' : \mathbb{M}' \to \mathbb{M}''$ be two homomorphisms of Hom-Lie 2-superalgebras. Their composition $g'g = ((g'g)_0, (g'g)_1, (g'g)_2) : \mathbb{M} \to \mathbb{M}''$ is defined by $(g'g)_0 = g_0 \circ g_0 : M_0 \to M''_0, (g'g)_1 = g'_1 \circ g_1 : M_1 \to M''_1$, and $(g'g)_2 = g_2 \circ (g_0 \times g_0) + g'_1 \circ g_2 : M_0 \times M_0 \to M''_1$. It is clear that $g'g = ((g'g)_0, (g'g)_1, (g'g)_2)$ is a homomorphism of Hom-Lie 2-superalgebras.

**Definition 2.4** A homomorphism of Hom-Lie 2-superalgebras $g : \mathbb{M} \to \mathbb{M}'$ is called an isomorphism if there exists a homomorphism of Hom-Lie 2-superalgebras $h : \mathbb{M}' \to \mathbb{M}$ such that $hg : \mathbb{M} \to \mathbb{M}$ and $gh : \mathbb{M}' \to \mathbb{M}'$ are both identity homomorphisms.

**Proposition 2.5** Let $(\mathbb{M} : M_1 \xrightarrow{d_1} M_0, [,], l_3, \tau_0, \tau_1)$ and $(\mathbb{M}' : M'_1 \xrightarrow{d'_1} M'_0, [,], l'_3, \tau'_0, \tau'_1)$ be two Hom-Lie 2-superalgebras. Let $g = (g_0, g_1, g_2) : \mathbb{M} \to \mathbb{M}'$ be a homomorphism of Hom-Lie 2-superalgebras. If $g_0, g_1$ are invertible, then there exists a map $g^{-1} = (g_0^{-1}, g_1^{-1}, -g_1^{-1}g_2g_0^{-1} \times g_0^{-1})$ such that $g$ is an isomorphism of Hom-Lie 2-superalgebras.

**Proof** For any $x', y', z' \in hq(M_0)$, we have

$$
\begin{align*}
[g_0^{-1}(\tau'_0(x')), & -g_1^{-1}(g_2g_0^{-1}(y'), g_0^{-1}(z'))]_{M'} + (-1)^{|x'||y'|+|z'|}g_0^{-1}(\tau'_0(y')), -g_1^{-1}(g_2g_0^{-1}(z'), g_0^{-1}(x'))]_{M'} \\
+ (-1)^{|x'||y'|+|z'|}g_0^{-1}(\tau'_0(z')) & , -g_1^{-1}(g_2g_0^{-1}(x'), g_0^{-1}(y'))]_{M'} + l_3(g_0^{-1}(x'), g_0^{-1}(y'), g_0^{-1}(z')) \\
= (-1)^{|x'||y'|+|z'|}g_1^{-1}g_2(g_0^{-1}[y, z']_{M'}, \tau_0^{-1}(x')) - (-1)^{|x'||y'|+|z'|}g_1^{-1}g_2(g_0^{-1}[z', x']_{M'}, \tau_0^{-1}(y'))) 
\end{align*}
$$
\[-g_1^{-1}g_2(g_0^{-1}[x', y']_{M'}, \tau_0'(g_0^{-1}(z'))) + g_1^{-1}l'_3(x', y', z').\]

**Proposition 2.6** Let \((\mathcal{M} : M_1 \xrightarrow{\cdot} M_0, \cdot, [\cdot, \cdot]_{M'}, l_3, \tau_0, \tau_1)\) be a Hom-Lie 2-superalgebra. For a graded super vector space \(\mathcal{M}' = M'_0 \oplus M'_1\) with two invertible even linear maps \(g_0 : M'_0 \to M_0,\ g_1 : M'_1 \to M_1\), and an even skew supersymmetry bilinear map \(g_2 : M'_0 \times M'_0 \to M_1\), define

1. \(d' \triangleq g_0^{-1} \circ d \circ g_1\),
2. \([x, y]_{M'} \triangleq g_0^{-1}([g_0(x), g_0(y)]_M + d(g_2(x, y)))\),
3. \([x, a]_{M'} \triangleq g_1^{-1}([g_0(x), g_1(a)]_M + g_2(x, d'a))\),
4. \([a, b]_{M'} \triangleq 0\),
5. \(\tau'_0 \triangleq g_0^{-1} \circ \tau_0 \circ g_0 : M'_0 \to M_0,\ \tau'_1 \triangleq g_1^{-1} \circ \tau_1 \circ g_1 : M'_1 \to M_1\) satisfying

\[g_2(\tau'_0(x), \tau'_0(y)) = \tau_1(g_2(x, y)),\]

\(l'_3(x, y, z) \triangleq g_1^{-1}([g_0(\tau'_0(x)), g_2(y, z)]_M - g_2([x, y]_{M'}, \tau'_0(z)) - (-1)^{|x||y|+|z|}g_2([y, z]_{M'}, \tau'_0(x))
- (-1)^{|z||x|+|y|}g_2([x, y]_{M'}, \tau'_0(z)) + l_3(g_0(x), g_0(y), g_0(z))
- (-1)^{|y||x|+|z|}[g_0(\tau'_0(y)), g_2(z, x)]_M + (-1)^{|z||x|+|y|}[g_0(\tau'_0(z)), g_2(x, y)]_M\).

Then \((\mathcal{M}' : M'_1 \xrightarrow{d'} M_0, [\cdot, \cdot]_{M'}, l'_3, \tau'_0, \tau'_1)\) is a Hom-Lie 2-superalgebra. Furthermore, \(g = (g_0, g_1, g_2) : \mathcal{M}' \to \mathcal{M}\) is an isomorphism of Hom-Lie 2-superalgebras.

**Proof** For any \(x, y, z, t \in hg(M_0)\), since

\[l_3([g_0(t), g_0(x)]_M, \tau_0(g_0(y)), \tau_0(g_0(z))) + (-1)^{|x||y|+|z|}l_3([g_0(t), g_0(z)]_M, \tau_0(g_0(x)), \tau_0(g_0(y)))
+ (-1)^{|x||y|+|z|}l_3([g_0(y), g_0(x)]_M, \tau_0(g_0(t)), \tau_0(g_0(z)))
+ (-1)^{|x||y|+|z|}l_3([g_0(y), g_0(z)]_M, \tau_0(g_0(t)), \tau_0(g_0(x)))
+ (-1)^{|x||y|+|z|}l_3([g_0(x), g_0(y)]_M, \tau_0^2(g_0(t)), \tau_0(0))
+ (-1)^{|y||x|+|z|}l_3([g_0(x), g_0(z)]_M, \tau_0(g_0(t)), \tau_0(g_0(y)))
+ (-1)^{|x||y|+|z|}l_3([g_0(t), g_0(y)]_M, \tau_0^2(g_0(x)), \tau_0(0))\]

we have

\[l'_3([t, x]_{M'}, \tau'_0(y), \tau'_0(z)) + (-1)^{|z||x|+|y|}l'_3([t, z]_{M'}, \tau'_0(x), \tau'_0(y))
+ (-1)^{|y||x|+|z|}l'_3([x, y]_{M'}, \tau'_0(t), \tau'_0(z)) + (-1)^{|x||y|+|z|}l'_3([x, t]_{M'}, \tau'_0(x), \tau'_0(y))
+ (-1)^{|x||y|+|z|}l'_3([y, z]_{M'}, \tau'_0(t), \tau'_0(x)) + (-1)^{|z||x|+|y|+|z|}l'_3([x, y, z]_{M'}, \tau'_0(t), \tau'_0(x), \tau'_0(y))
+ (-1)^{|z||x|+|y|}l'_3([t, y]_{M'}, \tau'_0(x), \tau'_0(z))
+ (-1)^{|x||y|+|z|}l'_3([x, z]_{M'}, \tau'_0(t), \tau'_0(y)) + (-1)^{|z||x|+|y|}l'_3([t, y, z]_{M'}, \tau'_0(x), \tau'_0(z)).\]
Let $V : V_1 \overset{d}{\to} V_0$ be a 2-term complex of super vector spaces with an even linear map $d$. In the following, we can construct a new 2-term complex of super vector spaces $\text{End}(V) : \text{End}^1(V) \overset{\delta}{\to} \text{End}^0_d(V)$. Define an even linear map $\delta$ by 

$$\delta(F) = d \circ F + F \circ d$$

for any $F \in \text{End}^1(V)$, where 

$$\text{End}^1(V) = \text{Hom}(V_0, V_1),$$

$$\text{End}^0_d(V) = \{G = (G_0, G_1) \in \text{End}(V_0, V_0) \oplus \text{End}(V_1, V_1) | G_0 \circ d = d \circ G_1 \},$$

$|G| = |G_0| = |G_1|$. Define an even bilinear map $l_2 : \text{End}(V) \times \text{End}(V) \to \text{End}(V)$ by setting:

$$\begin{cases}
l_2(G, G') = [G, G']_C, \\
l_2(G, F) = [G, F]_C, \\
l_2(F, F') = 0,
\end{cases}$$

for any $G, G' \in \text{h}(\text{End}^0_d(V)), F, F' \in \text{h}(\text{End}^1(V))$, where $[\cdot, \cdot]_C$ is the graded commutator. It is easy to show that:

**Theorem 2.7** $(\text{End}(V), \delta, l_2)$ is a strict Lie 2-superalgebra.

**Proof** It is a straightforward calculation. \hfill \Box

3. Derivations of Hom-Lie 2-superalgebras

In this section, we will give the notion of superderivations and obtain some properties of superderivations. A new 2-term complex of super vector spaces will be formed by the superdereration of Hom-Lie 2 superalgebras.

**Definition 3.1** Let $(M : M_1 \overset{d}{\to} M_0, [\cdot, \cdot]_M, l_3, \tau_0, \tau_1)$ be a Hom-Lie 2-superalgebra. A homogeneous superderivation of degree 0 of $M$ consists of

- a homogeneous element $D = (D_0, D_1) \in \text{h}(\text{End}^0_d(M))$ satisfying
  $$D_0 \circ \tau_0 = \tau_0 \circ D_0, \quad D_1 \circ \tau_1 = \tau_1 \circ D_1,$$

- a skew-supersymmetric bilinear map $l_D : M_0 \times M_0 \to M_1$ satisfying
  $$l_D(\tau_0(x), \tau_0(y)) = \tau_1(l_D(x, y))$$

such that the following equations hold for any $x, y, z \in \text{h}(M_0), a \in \text{h}(M_1)$:

1. $D[x, y]_M - [Dx, \tau_0(y)]_M - (-1)^{|D||x|}[\tau_0(x), Dy]_M = d_lD(x, y),$
2. $D[x, a]_M - [Dx, \tau_1(a)]_M - (-1)^{|D||x|}[\tau_0(x), Da]_M = l_D(x, da),$
3. $l_D(\tau_0(x), [y, z]_M) + (-1)^{|D||x|}[\tau_0^2(x), l_D(y, z)]_M + l_3(Dx, \tau_0(y), \tau_0(z)) + (-1)^{|D||x|}l_3(\tau_0(x), Dy, \tau_0(z)) + (-1)^{|D||x|}l_3(\tau_0(x), Dz, \tau_0(z))$ 
   $$= Dl_3(x, y, z) + l_D([x, y]_M, \tau_0(z)) + (-1)^{|x||y|}[\tau_0^2(y), l_D(x, y)]_M + [l_D(x, y), \tau_0^2(z)]_M + (-1)^{|y||z|}[\tau_0^2(y), l_D(x, z)]_M,$$

where $|D| = |l_D|$.  


A homogeneous superderivation of degree 0 of $\mathcal{M}$ is denoted by $(D, l_D)$ and the set of all homogeneous superderivations of degree 0 of $\mathcal{M}$ by $\text{Der}^0(\mathcal{M})$.

**Proposition 3.2** Let $(\mathcal{M} : M_1 \xrightarrow{d} M_0, [, , ]_\mathcal{M}, l_3, \tau_0, \tau_1)$ be a Hom-Lie 2-superalgebra. For any $x \in \text{hg}(M_0)$ satisfying $\tau_0(x) = x$, define a homogeneous linear map $\text{ad}_x$ by $\text{ad}_x(y + a) = [x, y + a]$ for any $y \in \text{hg}(M_0), a \in \text{hg}(M_1)$, and then $(\text{ad}_x, l_{\text{ad}_x} = l_3(x, \cdot, \cdot)) \in \text{Der}^0(\mathcal{L})$, where $|\text{ad}_x| = |l_{\text{ad}_x}| = |x|$, which is called an inner derivation.

**Proof** It is a straightforward calculation by Definition 2.1. $\square$

Let $(\mathcal{M} : M_1 \xrightarrow{d} M_0, [, , ]_\mathcal{M}, l_3, \tau_0, \tau_1)$ be an idempotent Hom-Lie 2-superalgebra. For any $(D, l_D), (D', l_{D'}) \in \text{hg}(\text{Der}^0(\mathcal{M})), x, y \in \text{hg}(M_0)$, we obtain

$$
[D, D']_C([x, y]_\mathcal{M}) = [[D, D']_C(x), \tau_0(y)]_\mathcal{M} - (-1)^{|x||D|+|D'|} \tau_0(x), [D, D']_C(y)]_\mathcal{M}
$$

$$
= d(l(D'(x), \tau_0(y))) + (-1)^{|D'||x|} l_D(\tau_0(x), D y) + D l_{D'}(x, y)
$$

$$
- (-1)^{|D||D'|} l_{D'}(D x, \tau_0(y)) - (-1)^{|D||D'|+|x|} l_{D'}(\tau_0(x), D y) - (-1)^{|D|} l_{D'}(l(D x, y)).
$$

Define

$$
l_{[D, D']_C}(x, y) \triangleq l_D(D'(x), \tau_0(y)) + (-1)^{|D'||x|} l_D(\tau_0(x), D y) + D l_{D'}(x, y) - (-1)^{|D||D'|} l_{D'}(D x, \tau_0(y))
$$

$$
- (-1)^{|D||D'|+|x|} l_{D'}(\tau_0(x), D y) - (-1)^{|D|} l_{D'}(l(D x, y)).
$$

For any $a \in \text{hg}(M_1)$, we have

$$
[D, D']_C([x, a]_\mathcal{M}) = [[D, D']_C(x), \tau_1(a)]_\mathcal{M} - (-1)^{|x||D|+|D'|} \tau_0(x), [D, D']_C(a)]_\mathcal{M} = l_{[D, D']_C}(x, da).
$$

Since $\mathcal{M}$ is idempotent and $l_D, l_{D'}$ satisfy equation (3) in Definition 3.1, we obtain that $l_{[D, D']}$ satisfies equation (3) in Definition 3.1. Define an even skew-supersymmetric bilinear map on $\text{Der}^0(\mathcal{M})$ by

$$
[\cdot, \cdot]_{\text{Der}} : \text{Der}^0(\mathcal{M}) \times \text{Der}^0(\mathcal{M}) \rightarrow \text{Der}^0(\mathcal{M})
$$

$$
[(D, l_D), (D', l_{D'})]_{\text{Der}} \triangleq ([D, D']_C, l_{[D, D']_C}).
$$

(1)

We obtain the following theorem:

**Theorem 3.3** Let $(\mathcal{M} : M_1 \xrightarrow{d} M_0, [, , ]_\mathcal{M}, l_3, \tau_0, \tau_1)$ be an idempotent Hom-Lie 2-superalgebra. Then $(\text{Der}^0(\mathcal{M}), [\cdot, \cdot]_{\text{Der}})$ is a Lie superalgebra.

**Proof** We only need to verify

$$
\circ_{D_1, D_2, D_3} (-1)^{|D_1||D_3|} l_{[[D_1, D_2]_C, D_3]} = 0.
$$
For any \((D_1, l_{D_1}), (D_2, l_{D_2}), (D_3, l_{D_3}) \in \text{Der}^0(M)\), \(x, y \in hg(M_0)\), we have
\[
\circ_{D_1, D_2, D_3} -1|D_1||D_3|l_{[D_1, D_2]}|c_{D_3}|c(x, y) \\
= (-1)^{|D_1||D_3|}l_{D_1}([D_2]D_3x, \tau_0(x, D_3y)) + (-1)^{|D_1||D_3|+|D_2|}|D_3|l_{D_2}([D_1]D_3x, \tau_0(x, D_3y)) \\
- (-1)^{|D_1||D_3|+|D_2||D_3|}l_{D_1}([D_2]D_3x, \tau_0(x, D_3y)) - (-1)^{|D_1||D_3|+|D_2||D_3|}l_{D_2}([D_1]D_3x, \tau_0(x, D_3y)) \\
+ (-1)^{|D_1||D_3|+|D_2||D_3|}l_{D_1}([D_2]D_3x, \tau_0(x, D_3y)) + (-1)^{|D_1||D_3|+|D_2||D_3|}l_{D_2}([D_1]D_3x, \tau_0(x, D_3y)) \\
- (-1)^{|D_1||D_3|+|D_2||D_3|}l_{D_1}([D_2]D_3x, \tau_0(x, D_3y)) - (-1)^{|D_1||D_3|+|D_2||D_3|}l_{D_2}([D_1]D_3x, \tau_0(x, D_3y)) \\
+ (-1)^{|D_1||D_3|+|D_2||D_3|}l_{D_1}([D_2]D_3x, \tau_0(x, D_3y)) + (-1)^{|D_1||D_3|+|D_2||D_3|}l_{D_2}([D_1]D_3x, \tau_0(x, D_3y)) \\
- (-1)^{|D_1||D_3|+|D_2||D_3|}l_{D_1}([D_2]D_3x, \tau_0(x, D_3y)) - (-1)^{|D_1||D_3|+|D_2||D_3|}l_{D_2}([D_1]D_3x, \tau_0(x, D_3y))) \\
= 0,
\]
where \(\circ_{D_1, D_2, D_3}\) denotes summation over the cyclic permutation on \(D_1, D_2, D_3\). 

Let \((M : M_1 \xrightarrow{d} M_0, [\cdot, \cdot], l_3, \tau_0, \tau_1)\) be a Hom-Lie 2-superalgebra. We consider the complex \(\text{End}^1(M) \xrightarrow{\delta} \text{End}^0(M) \oplus \text{Hom}(M_0 \times M_0, M_1)\), where \(\delta\) is given by
\[
\delta(G) = (\delta(G), l_{\delta(G)}),
\]
in which \(l_{\delta(G)} : M_0 \times M_0 \to M_1\) is given by
\[
l_{\delta(G)}(x, y) = G([x, y]|_{M}) - (-1)^{|G||x|}[\tau_0(x), G(y)]|_{M} - [G(x), \tau_0(y)]|_{M}.
\]

Lemma 3.4 Let \((M : M_1 \xrightarrow{d} M_0, [\cdot, \cdot]|_M, l_3, \tau_0, \tau_1)\) be a Hom-Lie 2-superalgebra. Then \(\delta(G) \in \text{Der}^0(M)\).

Proof For any \(x, y, z \in hg(M_0)\), \(a \in hg(M_1)\), we have
\[
\delta(G)(x, y) + (-1)^{|G||x|}d([\tau_0(x), G(y)]|_{M}) + d([G(x), \tau_0(y)]|_{M}) \\
- d(G(x), \tau_0(y))|_{M} - (-1)^{|G||x|}[\tau_0(x), d(G(y))]|_{M} \\
d\delta(G)(x, y).
\]
Similarly, we have

$$\delta(G)[x, a]_M - [\delta(G)(x), \tau_1(a)]_M - (-1)^{|x||a|}[\tau_0(x), \delta(G)(a)]_M = l_{\delta(G)}(x, da).$$

Finally, we obtain

$$l_{\delta(G)}(\tau_0(x), [y, z]_M) + (-1)^{|x||y|}[\tau_0^2(y), l_{\delta(G)}(y, z)]_M + l_3(\delta(G)(x), \tau_0(y), \tau_0(z))$$

$$+ (-1)^{|x||y|}l_3(\tau_0(x), \delta(G)(y), \tau_0(z)) + (-1)^{|x||y|+|y|}l_3(\tau_0(y), \tau_0(x), \delta(G)(z))$$

$$- \delta(G)(l_3(x, y, z)) - l_{\delta(G)}([x, y]_M, \tau_0(z)) - (-1)^{|x||y|}l_{\delta(G)}(\tau_0(y), [x, z]_M)$$

$$- [l_{\delta(G)}(x, y), \tau_0^2(z)]_M - (-1)^{|y|(|G|+|x|)}[\tau_0^2(y), l_{\delta(G)}(x, z)]_M$$

$$= G[\tau_0(x), [y, z]_M] - (-1)^{|x||y|}[\tau_0^2(x), G[y, z]_M] - [G(\tau_0(x)), \tau_0([y, z]_M)]_M$$

$$+ (-1)^{|x||y|}[\tau_0^2(x), G[y, z]_M] - (-1)^{|x||y|+|y|}G[\tau_0^2(x), [\tau_0(y), G(z)]_M]_M$$

$$- (-1)^{|x||y|}[\tau_0^2(x), G[y, \tau_0(z)]_M] + l_3(\delta(G)(x), \tau_0(y), \tau_0(z))$$

$$+ (-1)^{|x||y|}l_3(\tau_0(x), \delta(G)(y), \tau_0(z)) + (-1)^{|x||y|+|y|}l_3(\tau_0(y), \tau_0(x), \delta(G)(z))$$

$$- \delta(G)(l_3(x, y, z)) - G([x, y]_M, \tau_0(z)) + (-1)^{|x||y|+|y|}[\tau_0^2(y), G([x, y]_M)]_M$$

$$+ [G([x, y]_M), \tau_0^2(z)]_M - (-1)^{|x||y|}G([\tau_0(y), [x, z]_M]) + (-1)^{|y|(|x|+|G|)}[\tau_0^2(y), G([x, z]_M)]_M$$

$$+ (-1)^{|x||y|}G([\tau_0(y), [y, z]_M]) - G([x, y]_M), \tau_0(z)]_M - (-1)^{|x||y|+|y|}[\tau_0(x), G(y)]_M, \tau_0^2(z)]_M$$

$$+ [G(x), \tau_0(y)]_M, \tau_0^2(z)]_M - (-1)^{|y|(|x|+|G|))}[\tau_0^2(y), G([x, z]_M)]_M$$

$$+ (-1)^{|y|(|x|+|G|)+|G|}[\tau_0^2(y), [\tau_0(x), G(z)]_M] + (-1)^{|y|(|x|+|G|)}[\tau_0^2(y), [G(x), \tau_0(z)]_M]_M$$

$$= 0.$$

From Lemma 3.4, there exists a complex

$$\text{Der}(M) : \text{Der}^1(M) \cong \text{End}^1(M) \xrightarrow{\mathbb{Z}} \text{Der}^0(M),$$

(4)

where $\text{End}^1(M) = \{ G \in \text{Hom}(M_0, M_1) | G \circ \tau_0 = \tau_1 \circ G \}.$

Define an even skew-supersymmetric bilinear map $[\cdot, \cdot]_{\text{Der}} : \text{Der}^0(M) \times \text{Der}^1(M) \to \text{Der}^1(M)$ by

$$[(D, l_D), G]_{\text{Der}} \triangleq [D, G]_{\text{C}}.$$

(5)

**Theorem 3.5** Let $(M : M_1 \rightarrow M_0, [\cdot, \cdot]_M, l_3, \tau_0, \tau_1)$ be an idempotent Hom-Lie 2-superalgebra. Then $(\text{Der}(M) : \text{Der}^1(M) \xrightarrow{\mathbb{Z}} \text{Der}^0(M), [\cdot, \cdot]_{\text{Der}})$ is a strict Lie 2-superalgebra, where the complex $\text{Der}(M)$ is given by (4), the differential $\mathbb{Z}$ is given by (2), and the bracket is given by (1) and (5).
Proof. We only need to show that \( l_{\delta[D,G]c} = l_{[D,\delta(G)]c} \). For any \( x, y \in hg(M_0) \), we have

\[
\begin{align*}
l_{\delta[D,G]c}(x, y) &= DL_{\delta(G)}(x, y) + (-1)^{|G||x|} l_D(q_0(x), d(G(y))) \\
&\quad + (-1)^{|G||x|+|D||x|} [q_0^2(x), DG(y)]_M + (-1)^{|G||x|} [Dq_0(x), q_1G(y)]_M \\
&\quad + l_D(d(G(x)), q_0(y)) + [Dg(x), q_0^2(y)]_M \\
&\quad + (-1)^{|D|(|G|+|x|)|} [q_1(G(x)), Dq_0(y)]_M - (-1)^{|D||G|} G(d(l_D(x, y))) \\
&\quad - (-1)^{|D|(|G|+|x|)} [q_1q_0(G(x)), Dq_0(y)]_M + (-1)^{|x|(|D|+|G|)+|D||G|} [q_0^2(x), G(Dy)]_M \\
&\quad - [DG(x), q_0(y)]_M + (-1)^{|D||G|} [G(Dx), q_0(y)]_M.
\end{align*}
\]

Similarly,

\[
\begin{align*}
l_{[D,\delta(G)]c}(x, y) &= l_D(d(G(x)), q_0(y)) + (-1)^{|G||x|} l_D(q_0(x), d(G(y))) \\
&\quad + DL_{\delta(G)}(x, y) - (-1)^{|D||G|} G[Dx, q_0(y)]_M \\
&\quad + (-1)^{|G||x|} [Dq_0(x), q_1G(y)]_M + (-1)^{|D||G|} [G(Dx), q_0^2(y)]_M \\
&\quad - (-1)^{|D|(|G|+|x|)} [q_1q_0(G(x)), Dq_0(y)]_M + (-1)^{|x|(|D|+|G|)+|D||G|} [q_0^2(x), G(Dy)]_M \\
&\quad + (-1)^{|D|(|G|+|x|)} [q_1(G(x)), Dq_0(y)]_M - (-1)^{|D||G|} G(d(l_D(x, y)))).
\end{align*}
\]

\[\square\]

4. 2-cocycles of Hom-Lie 2-superalgebras

In this section, we will give notions of representations and 2-cocycles of Hom-Lie 2-superalgebras and show the relation between 1-parameter infinitesimal deformations and 2-cocycles of Hom-Lie 2-superalgebras.

**Definition 4.1** A representation \( \rho = (\rho_0, \rho_1, \rho_2) \) of a Hom-Lie 2-superalgebra \( (M : M_1 \xrightarrow{d} M_0, [\cdot, \cdot], M_3, q_0, q_1) \) on 2-term complex \( \mathbb{V} \) with respect to an even linear map \( \varphi_V = (\varphi_{V_0}, \varphi_{V_1}) : \mathbb{V} \to \mathbb{V} \), where \( \varphi_{V_0} : V_0 \to V_0 \), \( \varphi_{V_1} : V_1 \to V_1 \), consists of:

- an even linear map \( \rho_0 : M_0 \to \text{End}^2_0(\mathbb{V}) \) satisfying \( \rho_0(q_0(x))\varphi_V = \varphi_{V_0}\rho_0(x) \),
- an even linear map \( \rho_1 : M_1 \to \text{End}^1(\mathbb{V}) \) satisfying \( \rho_1(q_1(a))\varphi_{V_0} = \varphi_{V_1}\rho_1(a) \),
- an even bilinear map \( \rho_2 : M_0 \times M_0 \to \text{End}^1(\mathbb{V}) \) satisfying \( \rho_2(q_0(x), q_0(y))\varphi_{V_0} = \varphi_{V_1}\rho_2(x, y) \) such that for any \( x, y, z \in hg(M_0) \), \( a \in hg(M_1) \), the following equations are satisfied:

1. \( \rho_0 \circ d = d \circ \rho_1 \),
2. \( \rho_0([x, y], M_3)\varphi_V - \rho_0(q_0(x))\rho_0(y) + (-1)^{|x||y|}\rho_0(q_0(y))\rho_0(x) = \delta(q_2(x, y)) \),
3. \( \rho_1([x, a], M_3)\varphi_{V_0} - \rho_0(q_0(x))\rho_1(a) + (-1)^{|x||a|}\rho_0(q_1(a))\rho_0(x) = \rho_2(x, da) \),
4. \( (-1)^{|x||z|}\rho_2([x, y], q_0(z))\varphi_{V_0} + (-1)^{|x||y|}\rho_2([y, z], q_0(t_0(x)))\varphi_{V_0} \)
\begin{align*}
&\quad + (-1)^{|y||z|}\rho_2([z, x], q_0(y))\varphi_{V_0} + (-1)^{|z||x|}\rho_1(t_3(x, y, z))\varphi_{V_0}^2 \\
&\quad = (-1)^{|x||z|}\rho_0(q_0(t_0(x)))\rho_2(y, z) - (-1)^{|z||y|}\rho_2(q_0(y), q_0(t_0(x)))\rho_0(x)
\end{align*}
such that the following equations hold for any $x, y, z, t$
respectively, and $\tau_0, \tau_1$
and
an even skew-supersymmetric bilinear map $\chi_0 : M_0 \times M_0 \to M_0$ satisfying $\tau_0(\chi_0(x, y)) = \chi_0(\tau_0(x), \tau_0(y))$,

an even skew-supersymmetric trilinear map $\chi_3 : M_0 \times M_0 \times M_0 \to M_1$ satisfying $\tau_1(\chi_3(x, a)) = \chi_3(\tau_0(x), \tau_1(a))$,

an even skew-supersymmetric bilinear map $\chi_2 : M_0 \times M_1 \to M_1$ satisfying $\tau_1(\chi_2(x, a)) = \chi_2(\tau_0(x), \tau_1(a))$,

such that the following equations hold for any $x, y, z, t, \in hg(M_0)$, $a, b \in hg(M_1)$:

(1) $\rho_0(x)\chi_1(a) + \chi_0(x, da) - \chi_1([x, a]_M) - d\chi_2(x, a) = 0$,

(2) $\rho_1(a)\chi_1(b) + \chi_1(a, db) + (-1)^{|a||b|}\rho_1(b)(\chi_1(a)) - \chi_1(da, b) = 0$,

(3) $\rho_0(\tau_0(x))\chi_2(x, y, z) + (-1)^{|x||y|+|a|}\rho_0(\tau_0(y))\chi_2(x, y, z) + (-1)^{|x||y|+|b|}\rho_0(\tau_0(z))\chi_2(x, y, z) + \chi_2(x, y, z) + (-1)^{|x||y|+|z|}\chi_2^0(x, y, z) + \chi_2(x, y, z) - d\chi_3(x, y, z) = 0$,

(4) $\chi_3(x, y, da) - \rho_2(x, y)\chi_1(a) - \chi_1([x, a]_M) - (-1)^{|a|}[x, a]_M -(-1)^{|a||x||y|}\rho_1(b)(\chi_1(a)) = 0$,

(5) $\chi_3([t, x]_M, \tau_0(y), \tau_0(z)) + (-1)^{|t||x|+|y|}\rho_2(\tau_0(y), \tau_0(z))\chi_2^0(t, x) + (-1)^{|t||x|+|y|}\rho_2(\tau_0(x), \tau_0(y))\chi_2^0(t, x) + (-1)^{|t||x|+|y|}\rho_1(a)(\chi_2(x, y, z) - (-1)^{|t||x|+|y|}\rho_0(\tau_0(y))\chi_2(x, y, z) - (-1)^{|t||x|+|y|}\rho_0(\tau_0(z))\chi_2(x, y, z) - \chi_2(x, y, z) = 0$.

Let $(M : M_1 \to M_0, [\cdot, \cdot]_M, l_3, \tau_0, \tau_1)$ be a Hom-Lie 2-superalgebra, $\chi_1 : M_1 \to M_0$ satisfying $\tau_0 \circ \chi_1 = \chi_1 \circ \tau_1$ be an even linear map, $\chi_0 : M_0 \times M_0 \to M_0$ satisfying $\tau_0(\chi_0(x, y)) = \chi_0(\tau_0(x), \tau_0(y))$ and $\chi_3 : M_0 \times M_0 \times M_0 \to M_1$ satisfying $\tau_1(\chi_3(x, a)) = \tau_1(\chi_3(x, a))$ be two even skew-supersymmetric bilinear maps respectively, and $\chi_3 : M_0 \times M_0 \times M_0 \to M_1$ satisfying $\tau_3 \circ \tau_3 = \tau_1 \circ \chi_3$ be an even skew-supersymmetric trilinear map. In the following, we consider a $\lambda$-parameterized family of even linear maps:
(1) $d\chi^0(a) \triangleq da + \lambda_1(a), \\
(2) [x,y]_\lambda \triangleq [x,y]_M + \lambda_2^0(x,y), \\
(3) [x,a]_\lambda \triangleq [x,a]_M + \lambda_2^1(x,a), \\
(4) [a,b]_\lambda \triangleq [a,b]_M = 0, \\
(5) l_3^0(x,y,z) \triangleq l_3(x,y,z) + \lambda_3(x,y,z).

With the above notations, if $(\mathbb{M} : M_1 \overset{\phi}{\rightarrow} M_0, [\cdot, \cdot]_\lambda, l_3^0, \tau_0, \tau_1)$ is a Hom-Lie 2-superalgebra, then $(\chi_1, \chi_2^0, \chi_2^1, \chi_3)$ generates a 1-parameter infinitesimal deformation of the Hom-Lie 2 superalgebra $\mathbb{M}$.

**Theorem 4.3** Let $(\mathbb{M} : M_1 \overset{\phi}{\rightarrow} M_0, [\cdot, \cdot]_\lambda, l_3^0, \tau_0, \tau_1)$ be a Hom-Lie 2-superalgebra. $(\chi_1, \chi_2^0, \chi_2^1, \chi_3)$ generates a 1-parameter infinitesimal deformation of the Lie 2-superalgebra $\mathbb{M}$ if and only if the following conditions hold:

1. $(\chi_1, \chi_2^0, \chi_2^1, \chi_3)$ is a 2-cocycle of $\mathbb{M}$ with coefficients in the adjoint representation,

2. $(\mathbb{M} = M_0 \oplus M_1, \chi_1, \chi_2^0, \chi_2^1, \chi_3, \tau_0, \tau_1)$ is a Hom-Lie 2-superalgebra.

**Proof** It is clear that $[\cdot, \cdot]_\lambda$ is skew-supersymmetric.

For all $x, y, z, t \in hg(M_0), a, b \in hg(M_1)$, equation (4) in Definition 4.1 holds if and only if

$$d\chi_2^1(x,a) + \chi_1([x,a]_M) - \chi_2^0(x, da) - [x, \chi_1(a)]_M = 0, \tag{6}$$

and

$$\chi_1(\chi_2^1(x,a)) - \chi_2^0(x, \chi_1(a)) = 0. \tag{7}$$

Equation (5) in Definition 4.1 holds if and only if

$$\chi_2^1(da, b) + [\chi_1(a), b]_M - \chi_2^1(a, db) - [a, \chi_1(b)]_M = 0, \tag{8}$$

and

$$\chi_2^1(\chi_1(a), b) - \chi_2^1(a, \chi_1(b)) = 0. \tag{9}$$

Equation (6) in Definition 4.1 holds if and only if

$$\tau_0 \chi_2^0(x,y) - \chi_2^0(\tau_0(x), \tau_0(y)) = 0. \tag{10}$$

Equation (7) in Definition 4.1 holds if and only if

$$\tau_1 \chi_3^1(x,a) - \chi_1(\chi_2^1(x,a), \tau_1(a)) = 0. \tag{11}$$

Equation (8) in Definition 4.1 holds if and only if

$$d(\chi_3(x,y,z)) + \chi_1(l_3(x,y,z)) - \chi_2(\tau_0, [y,z]_M)
- (-1)^{|y|(|y|+|z|)} \chi_2^0(\tau_0(y), [z,x]_M)
- (-1)^{|z|(|y|+|z|)} \chi_2^0(\tau_0(z), [x,y]_M)
- [\tau_0, \chi_2^1(y,z)]_M
- (-1)^{|y|(|y|+|z|)} \tau_0(y), \chi_2^0(z,x)]_M
- (-1)^{|z|(|y|+|z|)} [\tau_0(z), \chi_2^0(x,y)]_M = 0, \tag{12}$$

and
\[
\chi_1(\chi_3(x, y, z)) - \chi_0^2(\tau_0(x), \chi_0^2(y, z)) \\
- (-1)^{|x|(|y|+|z|)} \chi_2^0(\tau_0(y), \chi_2^0(z, x)) - (-1)^{|x|(|y|+|z|)} \chi_2^0(\phi_0(z), \chi_2^0(x, y)) \\
= 0.
\] (13)

Equation (9) in Definition 4.1 holds if and only if
\[
\chi_3(x, y, da) - l_3(x, y, \chi_1(a)) - \chi_1^1(\tau_0(x), [y, a]_M) - (-1)^{|x|(|y|+|a|)} \chi_2^1(\tau_0(y), [a, x]_M) \\
- (-1)^{|a|(|x|+|y|)} \chi_1^1(\tau_1(a), [x, y]_M) - [\tau_0(x), \chi_1^1(y, a), a]_M \\
- (-1)^{|x|(|y|+|a|)} [\tau_0(y), \chi_2^1(a, x)]_M - (-1)^{|a|(|x|+|y|)} [\tau_1(a), \chi_2^1(x, y)]_M \\
= 0,
\] (14)

and
\[
\chi_3(x, y, \chi_1(a)) - \chi_1^1(\tau_0(x), \chi_2^1(y, a)) \\
- (-1)^{|x|(|y|+|a|)} \chi_2^1(\tau_0(y), \chi_1^1(a, x)) - (-1)^{|a|(|x|+|y|)} \chi_2^1(\tau_1(a), \chi_2^1(x, y)) \\
= 0.
\] (15)

Equation (10) in Definition 4.1 holds if and only if
\[
\chi_3([t, x]_M, \tau_0(y), \tau_0(z)) + l_3(\chi_2^0(t, x), \tau_0(y), \tau_0(z)) \\
+ (-1)^{|x|(|y|+|y|)} \chi_3([t, z]_M, \tau_0(x), \tau_0(y)) + (-1)^{|x|(|x|+|y|)} l_3(\chi_2^0(t, z), \tau_0(x), \tau_0(y)) \\
+ (-1)^{|t|(|x|+|y|)} \chi_3([x, y]_M, \tau_0(t), \tau_0(z)) + (-1)^{|t|(|x|+|y|)} l_3(\chi_2^0(x, y), \tau_0(t), \tau_0(z)) \\
+ (-1)^{|x|(|y|+|z|)} \chi_3([y, z]_M, \tau_0(t), \tau_0(x)) + (-1)^{|x|(|x|+|z|+|y|)} l_3(\chi_2^0(y, z), \tau_0(t), \tau_0(x)) \\
+ (-1)^{|y|(|y|+|z|)} \chi_2^1(l_3(t, x, z), \tau_0^2(y)) + (-1)^{|y|(|x|+|y|)} [\chi_3(t, x, z), \tau_0^2(t)]_M \\
- \chi_2^1(l_3(t, x, y), \tau_0^2(z)) - [\chi_3(t, x, y), \tau_0^2(z)]_M \\
- (-1)^{|x|(|y|+|y|)} \chi_3([t, y]_M, \tau_0(x), \tau_0(z)) - (-1)^{|x|(|x|+|y|)} l_3(\chi_2^0(t, y), \tau_0(x), \tau_0(z)) \\
- (-1)^{|y|(|x|+|x|+|z|)} \chi_3([x, z]_M, \tau_0(t), \tau_0(y)) - (-1)^{|y|(|x|+|x|+|z|)} l_3(\chi_2^0(x, z), \tau_0(t), \tau_0(y)) \\
- (-1)^{|y|(|y|+|z|)} \chi_2^1(l_3(t, y, z), \tau_0^2(x)) - (-1)^{|y|(|y|+|z|)} [\chi_3(t, y, z), \tau_0^2(x)]_M \\
= 0,
\] (16)

and
\[
\chi_3(\chi_2^0(t, x), \tau_0(y), \tau_0(z)) + (-1)^{|x|(|y|+|y|)} \chi_3(\chi_2^0(t, z), \tau_0(x), \tau_0(y)) \\
+ (-1)^{|t|(|x|+|y|)} \chi_3(\chi_2^0(x, y), \tau_0(t), \tau_0(z)) + (-1)^{|t|(|x|+|y|)} \chi_3(\chi_2^0(x, y), \tau_0(t), \tau_0(z))
\]
From equations (6), (8), (12), (14), and (16), we show that $(\chi_1, \chi_2, \chi_3)$ is a 2-cocycle of $\mathbb{M}$ with the coefficients in the adjoint representation. Moreover, by equations (7), (9), (10), (11), (13), (15), and (17), $(\mathbb{M} = \mathbb{M}_0 \oplus \mathbb{M}_1, \chi_1, \chi_2, \chi_3, \tau_0, \tau_1)$ is a Hom-Lie 2-superalgebra.

5. Hom-Nijenhuis operators on Hom-Lie 2-superalgebras

In this section, we introduce the notion of Hom-Nijenhuis operators and study trivial deformations of Hom-Lie 2-superalgebras.

Let $(\mathbb{M} : M_1 \overset{d}{\to} M_0, [\cdot, \cdot]_\mathbb{M}, l_3, \tau_0, \tau_1)$ be a Hom-Lie 2-superalgebra, and $N_0 : M_0 \to M_0$ and $N_1 : M_1 \to M_1$ be two even linear maps satisfying $N_0 \circ \tau_0 = \tau_0 \circ N_0$ and $N_1 \circ \tau_1 = \tau_1 \circ N_1$. For any $x, y, z \in h\mathfrak{g}(M_0)$, $a \in h\mathfrak{g}(M_1)$, define

$$
\begin{align*}
  d_N &= d \circ N_1 - N_0 \circ d = 0, \\
  [x, y]_N &= [N_0 x, y]_\mathbb{M} + [x, N_0 y]_\mathbb{M} - N_0 [x, y]_\mathbb{M}, \\
  [x, a]_N &= [N_0 x, a]_\mathbb{M} + [x, N_1 a]_\mathbb{M} - N_1 [x, a]_\mathbb{M}, \\
  l^N_3(x, y, z) &= l_3(N_0 x, y, z) + l_3(x, N_0 y, z) + l_3(x, y, N_0 z) - N_1^2 l_3(x, y, z).
\end{align*}
$$

**Definition 5.1** An even linear map $N = (N_0, N_1)$ is called a Hom-Nijenhuis operator on Hom-Lie 2-superalgebras if for any $x, y, z \in h\mathfrak{g}(M_0)$, $a \in h\mathfrak{g}(M_1)$, the following conditions are satisfied:

1. $d \circ N_1 = N_0 \circ d = 0$,
2. $N_0 [x, y]_N = [N_0 x, N_0 y]_\mathbb{M}$,
3. $N_1 [x, a]_N = [N_0 x, N_1 a]_\mathbb{M}$,
4. $N_1^2 l^N_3(x, y, z) = 0$,
5. $l_3(N_0 x, N_0 y, N_0 z) = 0$,
6. $l_3(N_0 x, N_0 y, z) + l_3(N_0 x, y, N_0 z) + l_3(x, N_0 y, N_0 z) = 0$.

**Proposition 5.2** Let $N = (N_0, N_1)$ be a Hom-Nijenhuis operator, then for any $\lambda \in \mathbb{R}$, $\lambda N = (\lambda N_0, \lambda N_1)$ is also a Hom-Nijenhuis operator. Furthermore, $(\mathbb{M} : M_1 \overset{d_N}{\to} M_0, [\cdot, \cdot]_\mathbb{M}, l^N_3, \tau_0, \tau_1)$ is a skeletal Hom-Lie 2-superalgebra and

$$
\lambda N : (\mathbb{M} : M_1 \overset{d_N}{\to} M_0, [\cdot, \cdot]_\mathbb{M}, l^N_3, \tau_0, \tau_1) \to (\mathbb{M} : M_1 \overset{d}{\to} M_0, [\cdot, \cdot]_\mathbb{M}, l_3, \tau_0, \tau_1)
$$

is a homomorphism of Hom-Lie 2-superalgebras.

**Proof** It is a straightforward calculation. □

13
Let \((M \oplus R : R \xrightarrow{d=0} M, [\cdot, \cdot], l_3, \beta, I_R)\) be a Hom-Lie 2-superalgebra in Example 2.2. We define even operators \(N_0 : M \to M\) and \(N_1 = 0 : R \to R\). We can see that \(N = (N_0, 0)\) is a Hom-Nijenhuis operator if and only if
\[
N_0 \circ \beta - \beta \circ N_0 = 0, \quad (18)
\]
\[
N_0[N_0x, y]_M + N_0[x, N_0y]_M - N_0^2[x, y]_M - [N_0x, N_0y]_M = 0, \quad (19)
\]
\[
B([N_0x, N_0y]_M, N_0z) = 0, \quad (20)
\]
\[
B([N_0x, N_0y]_M, z) + B([N_0x, y]_M, N_0z) + B([x, N_0y]_M, N_0z) = 0. \quad (21)
\]

**Proposition 5.3** Let \((M \oplus R : R \xrightarrow{d=0} M, [\cdot, \cdot], l_3, \beta, I_R)\) be a Hom-Lie 2-superalgebra in Example 2.2. If the even linear map \(N_0 : M \to M\) satisfies equations (18) and (19), bilinear form \(B\) satisfies \(B(G\lambda x, G\lambda y) = B(x, y)\), where \(G\lambda \triangleq I_M + \lambda N_0\), \(\lambda \in R\) is a parameter, and then \(N = (N_0, 0)\) is a Hom-Nijenhuis operator on the Hom-Lie 2-superalgebra \((M \oplus R : R \xrightarrow{d=0} M, [\cdot, \cdot], l_3, \beta, I_R)\).

**Proof** We only need to show that \(N = (N_0, 0)\) satisfies equations (20) and (21). By
\[
B(G\lambda x, G\lambda y) = B(x, y),
\]
we have
\[
B(x, N_0y) = -B(N_0x, y), \quad B(N_0x, N_0y) = 0.
\]
Since \(B\) is nondegenerate, we obtain \(N_0^2 = 0\) and
\[
B([N_0x, N_0y]_M, N_0z)
= B(N_0[N_0x, y]_M, N_0z) + B(N_0[x, N_0y]_M, N_0z) - B(N_0^2[x, y]_M, N_0z)
= -B([N_0x, y]_M, N_0^2z) - B([x, N_0y], N_0^2z) = 0,
\]
and
\[
B([N_0x, N_0y]_M, z) + B([N_0x, y]_M, N_0z) + B([x, N_0y]_M, N_0z)
= B([N_0x, N_0y]_M, z) - B(N_0[N_0x, y], z) - B(N_0[x, N_0y]_M, z)
= -B(N_0^2[x, y]_M, z) = 0.
\]

**Definition 5.4** Let \((M : M_1 \xrightarrow{d} M_0, [\cdot, \cdot], l_3, \tau_0, \tau_1)\) be a Hom-Lie 2-superalgebra. A deformation of \(M\) is called trivial if there exist even linear maps \(N_0 : M_0 \to M_0\), \(N_1 : M_1 \to M_1\) and an even bilinear map \(N_2 : M_0 \times M_0 \to M_1\) such that \(G = (G_0, G_1, G_2)\) is a homomorphism from the Hom-Lie 2-superalgebra \((M^\lambda : M_1 \xrightarrow{d^\lambda} M_0, [\cdot, \cdot], l_3, \tau_0, \tau_1)\) to the Hom-Lie 2-superalgebra \((M : M_1 \xrightarrow{d} M_0, [\cdot, \cdot], l_3, \tau_0, \tau_1)\), where \(G_0 = I_{M_0} + \lambda N_0\), \(G_1 = I_{M_1} + \lambda N_1\), \(G_2 = \lambda N_2\).
Theorem 5.5  A deformation of the Hom-Lie 2-superalgebra $(M : M_1 \xrightarrow{d} M_0, [\cdot, \cdot]|_M, l_3, \tau_0, \tau_1)$ is trivial if and only if there exist even linear maps $N_0 : M_0 \to M_0$, $N_1 : M_1 \to M_1$ and an even bilinear map $N_2 : M_0 \times M_0 \to M_1$ such that for any $x, y, z, t \in hg(M_0)$, $a \in hg(M_1)$, the following equalities are satisfied:

1. $N_0 \circ \tau_0 = \tau_0 \circ N_0$,
2. $N_1 \circ \tau_1 = \tau_1 \circ N_1$,
3. $N_2(\tau_0(x), \tau_0(y)) = \tau_1(N_2(x, y))$,
4. $N_0(d(N_1a) - N_0(da)) = 0$,
5. $N_0(dN_2(x, y)) + N_0[N_0x, y]|_M + N_0[x, N_0y]|_M - N_0^2[x, y]|_M = [N_0x, N_0y]|_M$,
6. $N_1N_2(x, da) + N_1[N_0x, a]|_M + N_1[x, N_1a]|_M - N_1^2[x, a]|_M - [N_0x, N_1a]|_M = N_2(x, \chi_1(a))$,
7. $(-1)^{|x||z|}N_1l_3(N_0x, y, z) + (-1)^{|x||z|}N_1l_3(x, N_0y, z) + (-1)^{|x||z|}N_1l_3(x, y, N_0z)$
   $+ (-1)^{|y||z|}N_1[\tau_0(z), N_2(x, y)]|_M + (-1)^{|y||z|}N_1[\tau_0(y), N_2(x, z)]|_M + (-1)^{|y||z|}N_1[\tau_0(x), N_2(y, z)]|_M$
   $- (1)^{|x||z|}l_2N_1(x, y, z) - (1)^{|y||z|}N_1N_2([x, z]|_M, \tau_0(y)) - (1)^{|y||z|}N_1N_2([y, z]|_M, \tau_0(x))$
   $- (1)^{|y||z|}N_1N_2(x, y, \tau_0(z)) + (-1)^{|y||z|}N_2(x_0, y, \tau_0(z)) + (-1)^{|y||z|}N_2(x_0, \tau_0(y), \tau_0(x))$
   $+ (-1)^{|y||z|}N_2(x_0, \tau_0(y), \tau_0(z)) - (-1)^{|x||z|}[N_0\tau_0(x), N_2(y, z)]|_M - (-1)^{|x||y|}[N_0\tau_0(y), N_2(x, z)]|_M$
   $- (-1)^{|x||y|}[N_0\tau_0(y), N_2(x, y)]|_M - (-1)^{|x||y|}l_3(x, N_0y, N_0z) - (-1)^{|x||y|}l_3(N_0x, y, N_0z)$
   $- (-1)^{|x||y|}l_3(N_0x, N_0y, z) = 0$,
8. $l_3(N_0x, N_0y, N_0z) = 0$.

Proof  We only need to show that $G = (G_0, G_1, G_2)$ is a homomorphism of Hom-Lie 2-superalgebras. Since $G_0d^\lambda(a) = dG_1(a)$, $d^\lambda(a) = da + \lambda_1(a)$, we have

$$da + \lambda_1(a) + \lambda N_0 da + \lambda^2 N_0 \chi_1(a) = da + \lambda d(N_1a),$$

which implies that

$$\chi_1(a) + N_0(da) = d(N_1a), \quad N_0(\chi_1(a)) = 0.$$
From equation (4) in Definition 2.3, we have

\[
\begin{align*}
(-1)^{[x][z]} & N_2([x, y]_M, \tau_0(z)) + (-1)^{[y][z]} N_2([y, z]_M, \tau_0(x)) + (-1)^{[y][z]} N_2([z, x]_M, \tau_0(y)) \\
+ (-1)^{[x][z]} & \chi(x, y, z) + (-1)^{[x][z]} N_1 l_3(x, y, z) - (-1)^{[x][z]} [\tau_0(x), N_2(y, z)]_M \\
- (-1)^{[y][z]} & [\tau_0(y), N_2(z, x)]_M - (-1)^{[y][z]} [\tau_0(z), N_2(x, y)]_M - (-1)^{[x][z]} l_3(x, y, N_0z) \\
- (-1)^{[x][z]} & l_3(x, N_0y, z) - (-1)^{[x][z]} l_3(N_0x, y, z) = 0,
\end{align*}
\]

and

\[
\begin{align*}
(-1)^{[x][z]} & N_2(\chi(x, y, z), \tau_0(x)) + (-1)^{[y][z]} N_2(\chi(y, z, \tau_0(x)) + (-1)^{[y][z]} N_2(\chi(z, x, \tau_0(y)) \\
+ (-1)^{[x][z]} & N_1 \chi(x, y, z) - (-1)^{[x][z]} [\tau_0(x), N_2(y, z)]_M - (-1)^{[x][z]} [\tau_0(z), N_2(x, y)]_M \\
- (-1)^{[y][z]} & [\tau_0(y), N_2(x, y)]_M - (-1)^{[x][z]} l_3(x, N_0y, N_0z) - (-1)^{[x][z]} l_3(N_0x, y, N_0z) \\
- (-1)^{[x][z]} & l_3(N_0x, N_0y, N_0z) = 0,
\end{align*}
\]

and

\[
l_3(N_0x, N_0y, N_0z) = 0.
\]

Thus, \( G = (G_0, G_1, G_2) \) is a homomorphism of Hom-Lie 2-superalgebra if and only if equations (1)–(8) in Theorem 5.5 hold.

\[\square\]

**Remark 5.6** \( N = (N_0, N_1, N_2) \) is not a Hom-Nijenhuis operator in Theorem 5.5.

### 6. Abelian extensions of Hom-Lie 2-superalgebras

In this section, we will study abelian extensions of Hom-Lie 2-superalgebras and show that there exists a representation and a 2-cocycle by means of abelian extensions.

**Definition 6.1** Let \((\mathbb{M} : M_1 \xrightarrow{d} M_0, [,]_M, l_3, \tau_0, \tau_1)\), \((\mathbb{M}' : M'_1 \xrightarrow{d'} M'_0, [,]_{M'}, l'_3, \tau'_0, \tau'_1)\) and \((\mathbb{M} : \tilde{M}_1 \xrightarrow{d} \tilde{M}_0, [,]_{\tilde{M}}, \tilde{l}_3, \tilde{\tau}_0, \tilde{\tau}_1)\) be Hom-Lie 2-superalgebras, and \(i = (i_0, i_1) : \mathbb{M}' \rightarrow \mathbb{M}\), \(p = (p_0, p_1) : \tilde{M} \rightarrow \mathbb{M}\) be strict homomorphisms. The following sequence is called a short exact sequence if \(\text{Im}(i) = \text{Ker}(p)\).

\[
\begin{array}{cccccc}
0 & \rightarrow & M'_1 & \xrightarrow{i'_3} & \tilde{M}_1 & \xrightarrow{p_1} & M_1 & \rightarrow & 0 \\
& & d' \downarrow & & \tilde{d} \downarrow & & d \downarrow & & \\
0 & \rightarrow & M'_0 & \xrightarrow{i_0} & \tilde{M}_0 & \xrightarrow{p_0} & M_0 & \rightarrow & 0
\end{array}
\] (22)

\(\tilde{M}\) is called an extension of \(M\) by \(M'\), denoted by \(E_{M'}\). The extension \(E_{\tilde{M}}\) is called an abelian extension if \([,]_{M'} = 0\) and \(l'_3(\cdot, \cdot, \cdot) = 0\).

A splitting of an extension is an even linear map \(\varphi = (\varphi_0, \varphi_1) : \mathbb{M} \rightarrow \tilde{M}\) such that \(p_0 \circ \varphi_0 = I_{M_0}\) and \(p_1 \circ \varphi_1 = I_{M_1}\), where \(\varphi_0 : M_0 \rightarrow \tilde{M}_0\) and \(\varphi_1 : M_1 \rightarrow \tilde{M}_1\).

**Theorem 6.2** Let \(\tilde{M}\) be an abelian extension of \(M\) by \(M'\) given by (22), and let \(\varphi = (\varphi_0, \varphi_1) : \mathbb{M} \rightarrow \tilde{M}\)
be a splitting. For any \( x, y \in \text{hg}(M_0), \) \( a, b \in \text{hg}(M_1), \) \( s \in \text{hg}(M'_0), \) \( t \in \text{hg}(M'_1), \) define an even linear map \( \rho = (\rho_0, \rho_1, \rho_2) \) by

\[
\begin{align*}
\rho_0 : M_0 &\rightarrow \text{End}^0(M'^0), \quad \rho_0(x)(s + t) = [\varphi(x), s + t]_{M'^0}, \\
\rho_1 : M_1 &\rightarrow \text{End}^1(M'^0), \quad \rho_1(a)(s) = [\varphi(a), s]_{M'^0}, \\
\rho_2 : M_0 \times M_0 &\rightarrow \text{End}^1(M'^0), \quad \rho_2(x, y)(s) = \overline{l_3}(\varphi(x), \varphi(y), s),
\end{align*}
\]

and then \( \rho = (\rho_0, \rho_1, \rho_2) \) is a representation of \( M \) on \( M' \) with respect to \( \tau'_0, \tau'_1. \)

**Proof** It is a straightforward calculation by Definition 4.1.

**Theorem 6.3** Let \( \tilde{M} \) be an abelian extension of \( M \) by \( M' \) given by (22) and \( \varphi = (\varphi_0, \varphi_1) : \tilde{M} \rightarrow \tilde{M} \) be a splitting. For any \( x, y, z \in \text{hg}(M_0), \) \( a, b \in \text{hg}(M_1), \) \( s \in \text{hg}(M'_0), \) \( t \in \text{hg}(M'_1), \) define an even linear map \( \chi = (\chi_1, \chi_2^0, \chi_2^1, \chi_3) \) by

\[
\begin{align*}
\chi_1 : M_1 &\rightarrow M'_0, \quad \chi_1(a) = \tilde{d}\varphi_1(a) - \varphi_0(da), \\
\chi_2^0 : M_0 \times M_0 &\rightarrow M'_0, \quad \chi_2^0(x, y) = [\varphi_0(x), \varphi_0(y)]_{M'} - \varphi_0(x, y)_{M',} \\
\chi_2^1 : M_0 \times M_1 &\rightarrow M'_1, \quad \chi_2^1(x, a) = [\varphi_0(x), \varphi_1(a)]_{M'} - \varphi_0(x, a)_{M'}, \\
\chi_3 : M_0 \times M_0 \times M_0 &\rightarrow M'_1, \quad \chi_3(x, y, z) = \tilde{l}_3(\varphi_0(x), \varphi_0(y), \varphi_0(z)) - \varphi_1(l_3(x, y, z)),
\end{align*}
\]

and then \( \chi = (\chi_1, \chi_2^0, \chi_2^1, \chi_3) \) is a 2-cocycle of \( \tilde{M} \) with coefficients in \( M', \) where \( \rho = (\rho_0, \rho_1, \rho_2) \) is a representation of \( \tilde{M} \) on \( M'. \)

**Proof** It is easy to show that

\[
\begin{align*}
\rho_0(x)\chi_1(a) + \chi_2^0(x, da) - \chi_1([x, a]_M) - \tilde{d}\chi_2^1(x, a) &= 0, \\
\rho_1(a)\chi_1(b) + \chi_2^1(da, db) + (-1)^{|a||b|}\rho_1(b)(\chi_1(a)) - \chi_2^1(da, b) &= 0.
\end{align*}
\]

Since \( \tilde{M} \) is a Hom-Lie 2-superalgebra, we have

\[
\begin{align*}
\rho_0(\tau_0(x))\chi_2^0(\tau_0(y), z) + (-1)^{|x|(\text{pr} + |z|)}\rho_0(\tau_0(y))\chi_2^0(\tau_0(z), x) + (-1)^{|z|(\text{pr} + |y|)}\rho_0(\tau_0(z))\chi_2^0(\tau_0(y), x) \\
+ \chi_2^0(\tau_0(x), [y, z]_M) + (-1)^{|x|(\text{pr} + |z|)}\chi_2^0(\tau_0(y), [z, x]_M) + (-1)^{|z|(\text{pr} + |y|)}\chi_2^0(\tau_0(z), [x, y]_M) \\
- \tilde{d}\chi_3(x, y, z) - \chi_1l_3(x, y, z) \\
= [\varphi_0(\tau_0(x)), [\varphi_0(y), \varphi_0(z)]_{M'}]_{M'} - [\varphi_0(\varphi_0(x)), \varphi_0(y, z)_{M}]_{M'} \\
+ (-1)^{|x|(\text{pr} + |z|)}[\varphi_0(\tau_0(y)), [\varphi_0(z), \varphi_0(x)]_{M'}]_{M'} - (-1)^{|z|(\text{pr} + |y|)}[\varphi_0(\tau_0(y)), \varphi_0(z, x)_{M}]_{M'} \\
+ (-1)^{|x|(\text{pr} + |z|)}[\varphi_0(\tau_0(z)), [\varphi_0(x), \varphi_0(y)]_{M'}]_{M'} - (-1)^{|z|(\text{pr} + |y|)}[\varphi_0(\tau_0(z)), \varphi_0(x, y)_{M}]_{M'} \\
+ [\varphi_0(\tau_0(x)), \varphi_0(y, z)_{M}]_{M'} - \varphi_0(\tau_0(x), [y, z]_M)_{M} \\
+ (-1)^{|x|(\text{pr} + |z|)}[\varphi_0(\tau_0(y)), \varphi_0(z, x)_{M}]_{M'} - (-1)^{|z|(\text{pr} + |y|)}[\varphi_0(\tau_0(y)), [z, x]_M]_M \\
+ (-1)^{|x|(\text{pr} + |z|)}[\varphi_0(\tau_0(z)), \varphi_0(x, y)_{M}]_{M'} - (-1)^{|z|(\text{pr} + |y|)}[\varphi_0(\tau_0(z)), [x, y]_M]_M \\
- \tilde{d}\chi_3(\varphi_0(x), \varphi_0(y), \varphi_0(z)) + \tilde{d}\varphi_1l_3(x, y, z) - \tilde{d}\varphi_1l_3(x, y, z) + \varphi_0dl_3(x, y, z) \\
= 0.
\]
Similar to the above proof, equations (4) and (5) in Definition 4.2 can be obtained. Thus, $\chi = (\chi_1, \chi_2, \chi_3)$ is a 2-cocycle of $M$ with coefficients in $M'$.

7. The construction of Hom-Lie 2-superalgebras

In this section, we will construct a strict Hom-Lie 2-superalgebra and a skeletal Hom-Lie 2-superalgebra from Hom-associative Rota-Baxter superalgebras.

**Definition 7.1** [1] A Hom-associative superalgebra is a triple $(A, \cdot, \tau)$ consisting of a super vector space $A$, an even bilinear map $\cdot : A \times A \to A$, and an even homomorphism $\tau : A \to A$ satisfying

\[(x \circ y) \circ \phi(z) = \phi(x) \circ (y \circ z).\]

**Definition 7.2** A Hom-associative Rota–Baxter superalgebra $(M, \cdot, \tau, R)$ is a Hom-associative superalgebra $(M, \cdot, \tau)$ with an even linear map $R : M \to M$ satisfying

\[R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y) + \theta x \cdot y),\]  

(24)

where $\theta \in \mathbb{R}$. The even linear map $R$ is called a Rota–Baxter operator of weight $\theta$, and the identity (24) is called a Rota–Baxter identity.

A Hom-associative Rota–Baxter superalgebra $(M, \cdot, \tau, R)$ is called multiplicative if $\tau(x \cdot y) = \tau(x) \cdot \tau(y)$.

**Theorem 7.3** Let $(M, \cdot, \tau, R)$ be a multiplicative Hom-associative Rota–Baxter superalgebra with a Rota-Baxter operator of weight 0. Assume that even linear maps $\phi_0 = \tau$, $\phi_1 = \tau$, and even linear map $d : M = M_1 \to M_0 = M$ satisfies

\[
\begin{align*}
    d \circ \tau &= \tau \circ d, \\
    d(R(x) \cdot a) &= R(x) \cdot da + x \cdot R(da) & x \in hg(M_0), a \in hg(M_1), \\
    d(a \cdot R(x)) &= da \cdot R(x) + R(da) \cdot x & x \in hg(M_0), a \in hg(M_1), \\
    R(da) \cdot b &= a \cdot R(db) & a, b \in hg(M_1), \\
    b \cdot R(da) &= R(db) \cdot a & a, b \in hg(M_1).
\end{align*}
\]

Define an even bilinear map $l_2 : M_i \times M_j \to M_{i+j}$ $(0 \leq i + j \leq 1)$ by

\[
\begin{align*}
    l_2(x, y) &= R(x) \cdot y + x \cdot R(y) - (-1)^{|x||y|} R(y) \cdot R(x) \cdot x & x, y \in hg(M_0), \\
    l_2(x, a) &= -(-1)^{|x||a|} l_2(a, x) = R(x) \cdot a - (-1)^{|x||a|} a \cdot R(x) & x \in hg(M_0), a \in hg(M_1), \\
    l_2(a, b) &= 0 & a, b \in hg(M_1).
\end{align*}
\]

If $R \circ \tau = \tau \circ R$, then $(M : M_1 \overset{d}{\to} M_0, l_2, \phi_0, \phi_1)$ is a strict Hom-Lie 2-superalgebra.

**Proof** For any $x, y \in hg(M_0)$, we have

\[
    \phi_0(l_2(x, y)) = R(\tau(x)) \cdot \tau(y) + \tau(x) \cdot R(\tau(y)) - (-1)^{|x||y|} \tau(y) \cdot R(\tau(x)) - (-1)^{|x||y|} R(\tau(y)) \cdot \tau(x)
    = \phi_0(l_2(x, y)).
\]

Similarly, we obtain $\phi_1(l_2(x, a)) = l_2(\phi_0(x), \phi_1(a))$. By the Rota-Baxter identity (24), we deduce that equations (8) and (9) in Definition 2.1 hold. \qed
Definition 7.4 Let \((M, \cdot, \tau, R)\) be a Hom-associative Rota–Baxter superalgebra and \(B : M \times M \to \mathbb{R}\) be a bilinear form on \(M\). For any \(x, y, z \in \text{hg}(M)\), \(B\) is called super-symmetric if \(B(x, y) = (-1)^{|x||y|} B(y, x)\). \(B\) is called invariant if \(B(x \cdot y, z) = B(x, y \cdot z)\). \(B\) is called even if \(B(L_\tau^0, L_\tau^0) = B(L_\tau^1, L_\tau^1) = 0\).

Definition 7.5 A Hom-associative Rota–Baxter superalgebra \((M, \cdot, \tau, R)\) with a Rota–Baxter operator of weight 0 is called a quadratic Hom-associative Rota–Baxter superalgebra if there exists a nondegenerate, supersymmetric, and even invariant bilinear form \(B\) on \((M, \cdot, \tau, R)\) such that \(\tau\) satisfies \(B(\tau(x), y) = B(x, \tau(y))\). It is denoted by \((M, \cdot, \tau, R, B)\). A quadratic Hom-associative Rota–Baxter superalgebra is called involutive if \(\tau^2 = I_M\).

Theorem 7.6 Let \((M, \cdot, \tau, R, B)\) be an involutive multiplicative quadratic Hom-associative Rota–Baxter superalgebra with a Rota–Baxter operator of weight 0. Assume that even linear maps \(d = 0 : R = M_1 \to M_0 = M\), \(\phi_0 = \tau, \phi_1 = \tau\). Define an even bilinear map \(l_2 : M_i \times M_j \to M_{i+j}\) \((0 \leq i + j \leq 1)\) by
\[
\begin{aligned}
l_2(x, y) &= R(x) \cdot y + x \cdot R(y) - (-1)^{|x||y|} (y \cdot R(x) + R(y) \cdot x) & &\text{if } x, y \in \text{hg}(M_0), \\
l_2(x, a) &= -(-1)^{|x||a|} l_2(a, x) = 0 & &\text{if } x \in \text{hg}(M_0), a \in \text{hg}(M_1), \\
l_2(a, b) &= 0 & &\text{if } a, b \in \text{hg}(M_1),
\end{aligned}
\]
and an even trilinear map \(l_3 : M_0 \times M_0 \times M_0 \to M_1\) by
\[
l_3(x, y, z) = B(l_2(x, y), z).
\]
If \(R \circ \tau = \tau \circ R\) and \(R(x) \cdot y = x \cdot R(y)\), then \((\mathbb{M} : M_1 \xrightarrow{d=0} M_0, l_2, l_3, \phi_0, \phi_1)\) is a skeletal Hom-Lie 2-superalgebra.

Proof It is obvious that even linear maps \(l_2\) and \(l_3\) are skew-supersymmetric. By the Rota–Baxter identity (24), we deduce that equations (8) and (10) in Definition 2.1 hold. \(\square\)

Acknowledgments

We would like to thank Professor Yunhe Sheng for his useful comments. This work was supported by the National Natural Science Foundation of China (No. 11471090) and the Natural Science Foundation of Jilin Province (No. 20130101068JC).

References