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On the classification of almost null rings

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Abstract: An almost null ring is a ring R in which for all $a, b \in R$, $a^3 = 0$, $Ma^2 = 0$ for some square-free integer M that depends on a and $ab = ka^2 = lb^2$ for some integers k, l. This paper is devoted to the classification of the almost null rings.

Key words: Ideal, H-ring

1. Introduction and preliminaries

All considered rings are associative, but not necessarily with unity. If the additive group R^+ of the ring R is a p-group, then we say that R is a p-ring. A ring in which every subring is a two-sided ideal is called a **Hamiltonian ring**, or, more concisely, an **H-ring**. An element r in a ring R is said to be **nilpotent** if $r^n = 0$ for some $n \in \mathbb{N}$. A ring R is a **nil ring** if every element of R is nilpotent. If a nil ring R is both a p-ring and an H-ring, we shall say that R is a **nil-H-p-ring**. The class of H-rings have been studied by a number of authors and the most important results were obtained by Rédei [8, 9], Andrijanov [1] and Kruse [6, 7]. They reduced the description of nil-H-rings to the description of nil-p-H-rings. To describe the class of nil-p-H-rings they used many types of rings defined by complicated relations on generators. Unfortunately, the problem of classification of nil-p-H-rings (even rings from the same class), up to an isomorphism, is still open.

A very important subclass of the class of all H-rings is the class of so-called almost null rings, which were discovered by Kruse and independently by Andrijanov.

Definition 1.1 ([6], Definition 2.1) *A ring R is* **almost null** *if for all $a, b \in R$ the following conditions are satisfied:*

(i) $a^3 = 0$,

(ii) $Ma^2 = 0$ *for some square-free integer M that depends on a and*

(iii) $ab = ka^2 = lb^2$ *for some integers k, l.*

In [1], Kruse, reduced the problem of classification on nil-H-rings to the problem of classification of torsion nil-H-rings by proving that nontorsion nil-ring R is an H-ring if and only if R is an almost null ring.

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Moreover, the problem of describing torsion nil-H-rings is reduced to the problem of describing nil-H-p-rings (cf. Remark 1.8 of [1]), and even, as was shown by Kruse in Propositions 2.5 and 2.5, to the problem of describing nil-H-p-rings of bounded exponent (modulo the description of almost null rings).

Almost null rings play a central role in the classification of so-called filial rings (cf. [2, 3]).

Let p be any prime integer. By $\mathbb{Z}_{p^\infty}$ we denote the **quasicyclic p-group**, i.e. the group $\langle x_1, x_2, \dots \mid px_1 = 0;\ px_{i+1} = x_i \text{ for any } i \in \mathbb{N}\rangle$. Throughout the paper, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{P}$ stand for the set of all positive integers, the set of all integers, and the set of all primes, respectively. For $n \in \mathbb{N}$, let $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ be the residue class ring modulo n. Let $p \in \mathbb{P}$; an integer μ is called a quadratic nonresidue modulo p if the congruence $x^2 \equiv \mu \pmod p$ has no solution.

In the current paper for a ring R we will use the following notation: for a subset S of R, we denote by $\langle S\rangle$, $[S]$, $a_R(S) = \{x \in R : xS = Sx = 0\}$ the subgroup of R^+ generated by S, the subring of R generated by S, the two-sided annihilator of S in R, respectively. Instead of $a_R(R)$ we will write $a(R)$, for short. Moreover, $\mathbb{T}(R) = \{x \in R : nx = 0 \text{ for some } n \in \mathbb{N}\}$ and $R_p = \{x \in R : p^n x = 0 \text{ for some } n \in \mathbb{N}\}$ for any $p \in \mathbb{P}$. For an abelian group M, by M^0 we denote the ring with a zero multiplication and the additive group M.

2. Basic examples of almost null rings

Clearly, every almost null ring R is an H-ring such that $R^3 = 0$ and $a(R) = \{x \in R : x^2 = 0\}$. For a ring R and a prime integer p let $R[p] = \{a \in R : pa^2 = 0\}$.

Some characterizations of almost null rings were found by Kruse and Andrijanov, but are unsatisfactory due to lack of description up to an isomorphism. The following theorems were proved by Kruse, and independently by Andrijanov, via nontrivial methods such as Chevalley's theorem (cf. [5, pages 143–144]):

Theorem 2.1 ([6], Proposition 2.10) *Let S be a ring and let p be a prime integer. Then S is an almost null ring such that $S = S[p]$ if and only if one of the following conditions is satisfied:*

(1) $S^2 = 0$,

(2) there exists $x \in S$ such that $x^2 \neq 0$, $px^2 = 0$, $px, x^2 \in a(S)$, and $S = \langle x\rangle + a(S)$,

(3) there exist $x, y \in S$ such that $S = \langle x, y\rangle + a(S)$, $x^2 \neq 0$, $px^2 = 0$, $px, py, x^2 \in a(S)$, $y^2 = Ax^2$, $xy = F_1x^2$, $yx = F_2x^2$, where $A, F_1, F_2 \in \mathbb{Z}$ and the congruence

$$X^2 + (F_1 + F_2)X + A \equiv 0 \pmod p \tag{1}$$

has no integer solution.

Moreover, if S is an almost null ring, then $S/a(S)$ is a $\mathbb{Z}_p$-algebra and $\dim_{\mathbb{Z}_p} S/a(S) \leq 2$, wherein $\dim_{\mathbb{Z}_p} S/a(S) = k-1$ if and only if S satisfies the condition (k).

Theorem 2.2 ([6], Proposition 2.10) *A ring R is an almost null ring if and only if $R = \sum_{p\in\mathbb{P}} R[p]$, where for all distinct prime integers p, q we have $R[p] \cdot R[q] = 0$, $R[p] \lhd R$ and $R[p]$ satisfies one of the conditions (1), (2), or (3) of Theorem 2.1.*

Example 2.3 *Let p be any prime integer and let M be any, additively written, abelian group that possesses an element α of order p. Then $\langle\alpha\rangle$ is a vector space over a field $\mathbb{Z}_p$. In the abelian group $\mathbb{Z}_p^+ \times M$ we define a multiplication by the formula*

$$(k_1, m_1) \cdot (k_2, m_2) = (0, (k_1 k_2)\alpha), \tag{2}$$

for all $k_1, k_2 \in \mathbb{Z}$, $m_1, m_2 \in M$. A standard computation shows that this multiplication is well-defined, distributive over addition, and commutative. Moreover, $(ab)c = a(bc) = 0$ for all $a, b, c \in \mathbb{Z}_p \times M$. The ring constructed above will be denoted by

$$\mathbb{Z}_p \times_\alpha M.$$

Notice that $(\mathbb{Z}_p \times_\alpha M)^2 = \{0\} \times \langle\alpha\rangle$, $a(\mathbb{Z}_p \times_\alpha M) = \{0\} \times_\alpha M$ and if $x = (1, 0)$, then $\mathbb{Z}_p \times_\alpha M = \langle x\rangle + a(\mathbb{Z}_p \times_\alpha M)$ and $x^2 = (0, \alpha) \neq 0$, $px^2 = 0$.

From Theorem 2.1 it follows that $\mathbb{Z}_p \times_\alpha M$ is an almost null ring such that $\mathbb{Z}_p \times_\alpha M = (\mathbb{Z}_p \times_\alpha M)[p]$.

Proposition 2.4 *Let p be any prime integer and let M_1, M_2 be any additively written abelian p-groups. Let $\alpha \in M_1$, $\beta \in M_2$ be such that $o(\alpha) = o(\beta) = p$. Then the following conditions are equivalent:*

(i) $\mathbb{Z}_p \times_\alpha M_1 \cong \mathbb{Z}_p \times_\beta M_2$,

(ii) *there exists a group isomorphism $f\colon M_1 \to M_2$ such that $f(\alpha) = \beta$.*

Proof $(i) \Rightarrow (ii)$. Let $F\colon \mathbb{Z}_p \times_\alpha M_1 \to \mathbb{Z}_p \times_\beta M_2$ be a ring isomorphism. Then $F((\mathbb{Z}_p \times_\alpha M_1)^2) = (\mathbb{Z}_p \times_\beta M_2)^2$, and so $F(\{0\} \times \langle\alpha\rangle) = \{0\} \times \langle\beta\rangle$. Hence $F((0, \alpha)) = k(0, \beta)$ for some nonzero $k \in \mathbb{Z}_p$. Moreover, $F(a(\mathbb{Z}_p \times_\alpha M_1)) = a(\mathbb{Z}_p \times_\beta M_2)$; thus $F(\{0\} \times M_1) = \{0\} \times M_2$. Therefore, there exists a bijection $g\colon M_1 \to M_2$ such that $F((0, m)) = (0, g(m))$ for every $m \in M_1$. Hence g is a group isomorphism and $g(\alpha) = k\beta$. There exists $l \in \mathbb{Z}_p$ such that $kl \equiv 1 \pmod p$. However, M_2 is a p-group, and so the function $h\colon M_2 \to M_2$ given by the formula $h(x) = lx$ is a group isomorphism. Therefore, it is enough to set $f = h \circ g$.

$(ii) \Rightarrow (i)$. Let the function $F\colon \mathbb{Z}_p \times_\alpha M_1 \to \mathbb{Z}_p \times_\beta M_2$ be given by the formula

$$F((k, m)) = (k, f(m)) \text{ for any } k \in \mathbb{Z}_p,\ m \in M_1.$$

It is easy to check that F is an isomorphism of additive groups. Moreover, for any $k_1, k_2 \in \mathbb{Z}_p$, $m_1, m_2 \in M_1$ we have $F((k_1, m_1)(k_2, m_2)) = F((0, (k_1 k_2)\alpha)) = (0, k_1 k_2 f(m)) = (0, (k_1 k_2)\beta)$ and $F((k_1, m_1))F((k_2, m_2)) = (k_1, f(m_1))(k_2, f(m_2)) = (0, (k_1 k_2)\beta)$. It shows that F is a ring isomorphism. □

Proposition 2.5 *Let p be any prime integer. Then for a ring S the following conditions are equivalent:*

(i) *S is an almost null ring such that $S = S[p]$, $\dim_{\mathbb{Z}_p} S/a(S) = 1$ and there exists $x \in S \setminus a(S)$ such that $o(x) = p$,*

(ii) *$S \cong \mathbb{Z}_p \times_\alpha M$ for some abelian group M and an element $\alpha \in M$.*

Proof $(i) \Rightarrow (ii)$. It is easy to see that $S = \langle x\rangle + a(S)$. Since $o(x) \in \mathbb{P}$ and $x \in S \setminus a(S)$; so $\langle x\rangle \cap a(S) = 0$ and hence $S^+ = \langle x\rangle \oplus a(S)$. It is routine to check that the function $G\colon \mathbb{Z}_p \times_{x^2} a(S) \to S$ given by the formula $G((k, m)) = kx + m$ is a ring isomorphism.

$(ii) \Rightarrow (i)$ It follows directly from Example 2.3. □

Example 2.6 *Let p be any prime integer and let $F_1, F_2, A \in \mathbb{Z}$ be such that the congruence (1) has no solution. Let M be any, additively written, abelian group that possesses an element α of order p. In the abelian group $\mathbb{Z}_p^+ \times \mathbb{Z}_p^+ \times M$ we define a multiplication by the formula*

$$(k_1, l_1, m_1)(k_2, l_2, m_2) = (0, 0, (k_1 l_2 F_2 + l_1 k_2 F_1 + k_1 k_2 A + l_1 l_2)\alpha). \tag{3}$$

A standard computation shows that this multiplication is well-defined, distributive over addition, and commutative. Moreover, $(ab)c = a(bc) = 0$ for any $a, b, c \in \mathbb{Z}_p \times \mathbb{Z}_p \times M$. The ring constructed above will be denoted by

$$(\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M. \tag{4}$$

Note that $((\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M)^2 = \{0\} \times \{0\} \times \langle\alpha\rangle$, $a((\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M) = \{0\} \times \{0\} \times_\alpha M$ and if $y = (1,0,0)$ and $x = (0,1,0)$, then $(\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M = \langle x, y\rangle + a((\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M)$ and $x^2 = (0,0,\alpha) \neq 0$, $px^2 = 0$, $px, py, x^2 \in a(S)$, $y^2 = Ax^2$, $xy = F_1x^2$, $yx = F_2x^2$. From Theorem 2.1, it follows that $(\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M$ is an almost null ring such that $(\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M = ((\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M)[p]$.

Furthermore, if M is a p-group, then for any $W \in \mathbb{Z}$, $W \not\equiv 0 \pmod p$, the congruence $X^2 + (WF_1 + WF_2)X + W^2A \equiv 0 \pmod p$ has no solution and the function $f\colon (\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M \to (\mathbb{Z}_p \times \mathbb{Z}_p)_{WF_1,WF_2,W^2A} \times_\alpha M$, given by the formula $f((k,l,m)) = (k, Wl, W^2m)$ is a ring isomorphism. Moreover, for any $U \in \mathbb{Z}$, $U \not\equiv 0 \pmod p$ the function $g\colon (\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M \to (\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_{U\alpha} M$ given by the formula $g(k,l,m) = (k,l,Um)$ is a ring isomorphism.

Proposition 2.7 *Let p be any prime integer. Then for a ring S the following conditions are equivalent:*

(i) S is an almost null ring such that $S = S[p]$, $\dim_{\mathbb{Z}_p} S/a(S) = 2$ and there exist $x, y \in S \setminus a(S)$ such that $o(x) = o(y) = p$ and the cosets $x + a(S)$, $y + a(S)$ are linearly independent over $\mathbb{Z}_p$,

(ii) $S \cong (\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M$ for some abelian group M and an element $\alpha \in M$.

Proof $(i) \Rightarrow (ii)$. By the assumptions, $S = \langle x, y\rangle + a(S)$. Since $o(x) = o(y) = p \in \mathbb{P}$ and $\dim_{\mathbb{Z}_p} S/a(S) = 2$, and so $\langle x, y\rangle = \langle x\rangle \oplus \langle y\rangle$ and $S^+ = (\langle x\rangle \oplus \langle y\rangle) + a(S)$. Linear independence of cosets $x + a(S)$, $y + a(S)$ over $\mathbb{Z}_p$ implies that $S^+ = \langle x\rangle \oplus \langle y\rangle \oplus a(S)$. By Theorem 2.1, there exists $A, F_1, F_2 \in \mathbb{Z}$ such that $y^2 = Ax^2$, $xy = F_1x^2$, $yx = F_2x^2$ and the congruence (1) has no solutions. Direct computations show that the function $G\colon (\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M \to S$ given by the formula $G((k,l,m)) = lx + ky + m$ is a ring isomorphism.

$(ii) \Rightarrow (i)$ It follows directly from Example 2.6. □

Theorem 2.8 *Let p be any prime integer and if $p > 2$ then let μ_p be a fixed quadratic nonresidue modulo p. Let M be any additively written abelian p-group that possesses an element α of order p. Then:*

(i) If $p > 2$ and the ring $(\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M$ is commutative, then $F_1 \equiv F_2 \pmod p$ and $(\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M \cong (\mathbb{Z}_p \times \mathbb{Z}_p)_{0,0,-\mu_p} \times_\alpha M$.

(ii) If $p = 2$, then the ring $(\mathbb{Z}_2 \times \mathbb{Z}_2)_{F_1,F_2,A} \times_\alpha M$ is not commutative, $A \equiv 1 \pmod 2$ and $F_1 \equiv 0 \pmod 2$, $F_2 \equiv 1 \pmod 2$ or $F_1 \equiv 1 \pmod 2$, $F_2 \equiv 0 \pmod 2$. Moreover, $(\mathbb{Z}_2 \times \mathbb{Z}_2)_{F_1,F_2,A} \times_\alpha M \cong (\mathbb{Z}_2 \times \mathbb{Z}_2)_{1,0,1} \times_\alpha M$.

(iii) If $p > 2$ *and the ring* $(\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M$ *is not commutative, then* $F_1 \not\equiv F_2 \pmod p$ *and* $(\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M \cong (\mathbb{Z}_p \times \mathbb{Z}_p)_{-1,1,-V^2\mu_p} \times_\alpha M$*, where* $V = 1, 2, \dots, (p-1)/2$*. Moreover, the rings* $(\mathbb{Z}_p \times \mathbb{Z}_p)_{-1,1,-V^2\mu_p} \times_\alpha M$ *for* $V = 1, 2, \dots, (p-1)/2$ *are pairwise nonisomorphic.*

Proof

(i). By the assumptions $xy = yx$ and so $F_1 x^2 = F_2 x^2$ and since $o(x^2) = p$, $F_1 \equiv F_2 \pmod p$. The congruence (1) has no solutions and $p > 2$, and so $(F_1 + F_2)^2 - 4A$ is a quadratic nonresidue modulo p. Thus $F_1^2 - A$ is also a quadratic nonresidue modulo p. Hence, there exists $W \in \mathbb{Z}$ such that $F_1^2 - A \equiv W^2\mu_p \pmod p$ and $W \not\equiv 0 \pmod p$. The function $F\colon (\mathbb{Z}_p \times \mathbb{Z}_p)_{0,0,-W^2\mu_p} \times_\alpha M \to (\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M$ given by the formula $F((k,l,m)) = (k, (l - F_1 k) \cdot 1, m)$ is a ring isomorphism. Hence, and by Example 2.6, we get that $(\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M \cong (\mathbb{Z}_p \times \mathbb{Z}_p)_{0,0,-\mu_p} \times_\alpha M$.

(ii). A standard verification shows that for $p = 2$ the congruence (1) has no solutions only in the cases listed in the formulation of theorem. The function $F\colon (\mathbb{Z}_2 \times \mathbb{Z}_2)_{1,0,1} \times_\alpha M \to (\mathbb{Z}_2 \times \mathbb{Z}_2)_{0,1,1} \times_\alpha M$ given by the formula $F((k,l,m)) = (k, (k + l) \cdot 1, m)$ is a ring isomorphism.

(iii). Since the ring $(\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M$ is not commutative, $xy \neq yx$. Hence $F_1 x^2 \neq F_2 x^2$ and since $o(x^2) = p$, $F_1 \not\equiv F_2 \pmod p$. Thus, there exist $u, v, W \in \mathbb{Z}$ such that $W(F_1 - F_2) \equiv 1 \pmod p$, $u(F_1 - F_2) \equiv F_1 + F_2 \pmod p$ and $v(F_1 - F_2) \equiv -2 \pmod p$. Moreover, the congruence (1) has no solutions and $p > 2$, so $(F_1 + F_2)^2 - 4A$ is a quadratic non-residue modulo p. Hence, there exists $V \in \{1, 2, \dots, (p-1)/2\}$ such that $(4A - (F_1 + F_2)^2)W^2 \equiv -V^2\mu_p \pmod p$. A standard verification shows that the function $F\colon (\mathbb{Z}_p \times \mathbb{Z}_p)_{-1,1,(4A-(F_1+F_2)^2)W^2} \times_\alpha M \to (\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M$ given by the formula $F((k,l,m)) = (vk, (uk + l) \cdot 1, m)$ is a ring isomorphism. Therefore, $(\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1,F_2,A} \times_\alpha M \cong (\mathbb{Z}_p \times \mathbb{Z}_p)_{-1,1,-V^2\mu_p} \times_\alpha M$. By Remark 1 of [4] we get that the rings $(\mathbb{Z}_p \times \mathbb{Z}_p)_{-1,1,-V^2\mu_p} \times_\alpha M$ for $V = 1, 2, \dots, (p-1)/2$ are pairwise nonisomorphic. □

3. Embeddings of almost null rings

Theorem 3.1 *Let* R *be an almost null ring. There exists an almost null ring* S *in which* R *is an essential ideal and such that the group* $a(S)^+$ *is divisible and* $pS = p^2 S$ *for every prime* p*.*

Proof Let $A = a(R)$ and denote by B^+ a divisible group in which A^+ is an essential subgroup. Denote by B the ring with zero multiplication with additive group B^+. It is a simple matter to check that the ring $R \oplus B$ is almost null. Moreover, the ring $R \oplus B$ is commutative if and only if R is commutative. Obviously $I = \{(x, x) : x \in A\} \lhd R \oplus B$ and $I \subseteq a(R \oplus B)$. Let $S = (R \oplus B)/I$. Then $(R + I)/I \cong R/(R \cap I) \cong R$ and $(R+I)/I \lhd S$, and so one can identify R with $(R+I)/I$. Moreover, S is an almost null ring as a homomorphic image of the almost null ring $R \oplus B$.

Let $J \lhd R \oplus B$ be such that $I \subsetneq J$. Take any $(a, b) \in J \setminus I$. If $a \notin a(R)$, then there exists $x \in R$ such that $xa \neq 0$ or $ax \neq 0$. Then $(x, 0)(a, b) = (xa, 0) \notin I$ or $(a, b)(x, 0) = (ax, 0) \notin I$; hence $(xa, 0) + I \in (R+I)/I \cap J/I$ or $(ax, 0) + I \in (R + I)/I \cap J/I$. If $a \in a(R)$, then $(a, a) \in I$ and $(a, b) = (a, 0) + (0, b - a)$, and so $(a, b) + I = (0, b - a) + I$ wherein $(0, b - a) \in J \setminus I$. Therefore, $b - a \neq 0$ and $\langle b - a \rangle \cap a(R) \neq 0$ by the essentiality of the subgroup A^+ in B^+. Thus $0 \neq k(b-a) \in a(R)$ for some $k \in \mathbb{Z}$ and hence $(0, k(b-a)) \in J \setminus I$. Next $(0, k(b - a)) + I = (-k(b - a), 0) + I \in (R + I)/I \cap J/I$. This shows that $(R + I)/I$ is an essential ideal in the ring S.

Note that $(0,b)+I \in a(S)$ for any $b \in B$. If $(r,b)+I \in a(S)$ for some $r \in R, b \in B$, then for every $y \in R$, $[(r,b)+I]\cdot[(y,0)+I] = (0,0)+I$. Hence $(ry,0) \in I$, and so $ry = 0$. This shows that $rR = 0$, and similarly $Rr = 0$. Thus $r \in a(R) = A$ and $(r,b)+I = [(r,r)+(0,b-r)]+I = (0,b-r)+I$. Hence $a(S) = \{(0,b)+I \ : \ b \in B\}$. Moreover, the function $b \mapsto (0,b)+I$ is a ring isomorphism from B onto $a(S)$. Hence the group $a(S)^+$ is divisible.

Take any $s \in S$ and any prime number p. Then there exists a square-free integer M such that $Ms^2 = 0$. Hence $(Ms)^2 = 0$, and directly by the definition of an almost null ring, $Ms \in a(S)$. Since $GCD(p^2, M) \mid p$, there exist $k, l \in \mathbb{Z}$ such that $p = kM + lp^2$. Thus $ps = k(Ms) + p^2(ls)$, and by divisibility of $a(S)^+$, $ps \in p^2S$. Finally, $pS = p^2S$. □

Lemma 3.2 *Let R be an almost null ring with divisible annihilator. Then $R = \mathbb{T}(R) \oplus C$, where $C \lhd R$, $C^2 = 0$ and the group C^+ is divisible.*

Proof Since the group $a(R)^+$ is divisible, $a(R)^+ = \mathbb{T}(a(R))^+ \oplus C^+$ for some torsion-free subgroup $C^+ \leq a(R)^+$. The subgroups $\mathbb{T}(a(R))^+$ and C^+ are divisible as direct summands of divisible group and obviously $\mathbb{T}(R) \cap C = \{0\}$. We claim that $R = \mathbb{T}(R) + C$. Take any $a \in R$. Since R is an almost null ring, there exists a square-free integer $m \in \mathbb{N}$ such that $ma^2 = 0$. Hence $(ma)^2 = 0$ and $ma \in a(R)$. Therefore, $ma = t + c$, where $t \in \mathbb{T}(a(R))$, $c \in C$. Moreover, $c = mc_1$, $t = mt_1$ for some $c_1 \in C$, $t_1 \in \mathbb{T}(a(R))$. Thus $m(a - t_1 - c_1) = 0$, $a - t_1 - c_1 \in \mathbb{T}(R)$, and $a = (a - t_1 - c_1) + t_1 + c_1 \in \mathbb{T}(R) + C$. □

Remark 3.3 *Let p be any prime integer and let R be an almost null p-ring such that the group $a(R)^+$ is divisible and $R^2 \neq 0$. By Theorem 2.1, it follows that there exists $x \in R$ (or exist $x, y \in R$) such that $x^2 \neq 0$ and $R = \langle x \rangle + a(R)$ (x, y are as in the item (3) of Theorem 2.1). Then $px \in a(R)$ and $px = px_1$ for some $x_1 \in a(R)$. Hence $p(x - x_1) = 0$ and $x - x_1 \notin a(R)$, and so $R = \langle x - x_1 \rangle + a(R)$ and without loss of generality we may assume that $o(x) = p$ (similarly, we may assume that $o(x) = o(y) = p$). Since $o(x^2) = p$, there exists a subgroup $M \leq a(R)$, $M \cong \mathbb{Z}_{p^\infty}$ such that $x^2 \in M$ and $a(R)^+ = M \oplus N$ for some divisible subgroup $N \leq a(R)$. Hence, N and $\langle x \rangle + M$, ($\langle x, y \rangle + M$) are subrings of an H-ring R, and so $N \lhd R$ and $\langle x \rangle + M \lhd R$, ($\langle x, y \rangle + M \lhd R$). Moreover, $R = (\langle x \rangle + M) + N$, ($R = (\langle x, y \rangle + M) + N$). Let $k \in \mathbb{Z}$, $m \in M$, $n \in N$ be such that $kx + m = n$. Then $kx = n - m \in a(R)$, and so $p \mid k$, $kx = 0$, and $m = n \in M \cap N = 0$. This shows that $R = (\langle x \rangle + M) \oplus N^0$ (similar arguments to those above show that $R = (\langle x, y \rangle + M) \oplus N^0$).*

By Proposition 2.5 (Proposition 2.7), $\langle x \rangle + M \cong \mathbb{Z}_p \times_{x_1} \mathbb{Z}_{p^\infty}$ (and $\langle x, y \rangle + M \cong (\mathbb{Z}_p \times \mathbb{Z}_p)_{F_1, F_2, A} \times_{x_1} \mathbb{Z}_{p^\infty}$, where $F_1, F_2, A \in \mathbb{Z}$ are such that the congruence (1) has no solutions).

4. Main Theorem

Lemma 4.1 *Let S be a ring described in Theorem 2.1 in the item (2) or (3). If the subgroup $\langle x^2 \rangle$ is essential in the group $a(S)^+$, then the ring S cannot be expressed as a direct sum of two nonzero ideals.*

Proof Assume that $I \oplus J = S$ for some nonzero ideals I, J of a ring S. Then $I^2 \oplus J^2 = S^2 = \langle x^2 \rangle$. Since $o(x^2) = p$, $I^2 = 0$ or $J^2 = 0$. Assume that $J^2 = 0$. Then $J \subseteq a(S)$. Because $J \neq 0$ and the subgroup $\langle x^2 \rangle^+$ is essential in the group $a(S)^+$, $J \cap \langle x^2 \rangle \neq 0$. However, $I^2 = \langle x^2 \rangle$, and so $I \cap J \neq 0$, a contradiction. □

Proposition 4.2 ([10], Proposition 4.2.2) *Suppose that an abelian group G can be expressed in two ways as a direct sum of quasicyclic groups, cyclic groups of prime-power order, and infinite cyclic groups. Then the sets of direct summands of each isomorphism type in the two decompositions have the same cardinality.*

The next theorem classifies all almost null rings with a divisible annihilator.

Theorem 4.3 *For every odd prime integer p let μ_p be a fixed quadratic nonresidue modulo p. All, up to isomorphism, almost null rings with a divisible annihilator are rings of the form:*

$$\bigoplus_{p\in\Pi} R_{(p)} \oplus C, \tag{5}$$

where $\Pi \subseteq \mathbb{P}$ and $R_{(p)}$ is one of the following rings:

(i) $\mathbb{Z}_p \times_{x_1} \mathbb{Z}_{p^\infty}$,

(ii) $(\mathbb{Z}_p \times \mathbb{Z}_p)_{0,0,-\mu_p} \times_{x_1} \mathbb{Z}_{p^\infty}$, *for* $p > 2$,

(iii) $(\mathbb{Z}_p \times \mathbb{Z}_p)_{-1,1,-V^2\mu_p} \times_{x_1} \mathbb{Z}_{p^\infty}$, *for* $p > 2$, $V = 1, 2, ..., \frac{p-1}{2}$,

(iv) $(\mathbb{Z}_2 \times \mathbb{Z}_2)_{1,0,1} \times_{x_1} \mathbb{Z}_{2^\infty}$,

and C is any ring such that $C^2 = 0$ and the group $a(C)^+$ is divisible. Moreover, the rings described in items $(i) - (iv)$ cannot be expressed as a direct sum of two nonzero ideals.

Proof Let R be an almost null ring with a divisible annihilator. By Lemma 3.2, $R = \mathbb{T}(R) \oplus C_1$ for some $C_1 \lhd R$ such that $C_1^2 = 0$ and the group C_1^+ is divisible. Hence, $a(R) = a(\mathbb{T}(R)) \oplus C_1$ and the group $a(\mathbb{T}(R))^+$ is divisible. Next, $\mathbb{T}(R) = \bigoplus_{p\in\mathbb{P}} \mathbb{T}(R)_p$, and so the group $\mathbb{T}(R)_p^+$ is divisible for all $p \in \mathbb{P}$. Let $\Pi = \{p \in \mathbb{P} : \mathbb{T}(R)_p^2 \neq 0\}$. If $\Pi = \emptyset$, then $R^2 = 0$ and the group R^+ is divisible. Otherwise, fix a $p \in \Pi$. By Remark 3.3 and Theorem 2.8, $\mathbb{T}(R)_p \cong R_{(p)} \oplus N_p$ where $R_{(p)}$ is one of the rings described in items $(i) - (iv)$ and N_p is a p-ring with a zero multiplication and divisible additive group. Therefore, it is enough to assume that $C = (\bigoplus_{p\in\Pi} N_p) \oplus C_1$. Moreover, Lemma 4.1 implies that the rings described in items $(i) - (iv)$ cannot be expressed as a direct sum of two nonzero ideals.

Let $\Pi, \Pi' \subseteq \mathbb{P}$. Assume that $R \cong \bigoplus_{p\in\Pi} R_{(p)} \oplus C$ and $R' \cong \bigoplus_{p\in\Pi'} R'_{(p)} \oplus C'$, where $R_{(p)}$ for $p \in \Pi$ and $R'_{(p)}$ for $p \in \Pi'$ are rings described in items $(i) - (iv)$, while C and C' are arbitrary rings with a zero multiplication and divisible additive groups. Assume that $R \cong R'$. Then

$$\mathbb{T}(R) \cong \mathbb{T}(R'),\ R/\mathbb{T}(R) \cong R'/\mathbb{T}(R'),\ R_p \cong R'_p \text{ for every } p \in \mathbb{P}. \tag{6}$$

Note that $\Pi = \{p \in \mathbb{P} : (R_p)^2 \neq 0\}$ and $\Pi' = \{p \in \mathbb{P} : (R'_p)^2 \neq 0\}$, and so $\Pi = \Pi'$. Fix any $p \in \Pi$. Then $R_p \cong R_{(p)} \oplus C_p$ and $R'_p \cong R'_{(p)} \oplus C'_p$, and so (6) implies that $R_{(p)} \oplus C_p \cong R'_{(p)} \oplus C'_p$. Thus $\dim_{\mathbb{Z}_p}(R_{(p)} \oplus C_p)/a(R_{(p)} \oplus C_p) = \dim_{\mathbb{Z}_p}(R'_{(p)} \oplus C'_p)/a(R'_{(p)} \oplus C'_p)$; hence $\dim_{\mathbb{Z}_p}(R_{(p)})/a(R_{(p)}) = \dim_{\mathbb{Z}_p}(R'_{(p)})/a(R'_{(p)})$. Therefore, if $\dim_{\mathbb{Z}_p}(R_{(p)})/a(R_{(p)}) = 1$, then $R_{(p)} = R'_{(p)} = \mathbb{Z}_p \times_{x_1} \mathbb{Z}_{p^\infty}$ and by Proposition 4.2, $C_p \cong C'_p$. Next, let $\dim_{\mathbb{Z}_p}(R_{(p)})/a(R_{(p)}) = 2$. If

$p=2$, then $R_{(p)} = R'_{(p)} = (\mathbb{Z}_2 \times \mathbb{Z}_2)_{1,0,1} \times_{x_1} \mathbb{Z}_{2^\infty}$ and by Proposition 4.2, $C_p \cong C'_p$. Now assume that $p>2$. If a ring $R_{(p)}$ is commutative, then $R_{(p)} = R'_{(p)} = (\mathbb{Z}_p \times \mathbb{Z}_p)_{0,0,-\mu_p} \times_{x_1} \mathbb{Z}_{p^\infty}$ and by Proposition 4.2, $C_p \cong C'_p$. If a ring $R_{(p)}$ is not commutative, then $R_{(p)} = (\mathbb{Z}_p \times \mathbb{Z}_p)_{-1,1,-V^2\mu_p} \times_{x_1} \mathbb{Z}_{p^\infty}$ and $R'_{(p)} = (\mathbb{Z}_p \times \mathbb{Z}_p)_{-1,1,-V'^2\mu_p} \times_{x_1} \mathbb{Z}_{p^\infty}$ for all $V, V' \in \{1,2,...,(p-1)/2\}$. By Remark 1 of [4] we get that $V=V'$, and so $R_{(p)} = R'_{(p)}$ and by Proposition 4.2, $C_p \cong C'_p$. Moreover, for any $p \in \mathbb{P} \setminus \Pi$, by (6), $C_p \cong C'_p$. Hence, $C_p \cong C'_p$ for every $p \in \mathbb{P}$ and thus $\mathbb{T}(C) \cong \mathbb{T}(C')$. Since $C/\mathbb{T}(C) \cong R/\mathbb{T}(R)$ and $C'/\mathbb{T}(C') \cong R'/\mathbb{T}(R')$, and so by (6), $C/\mathbb{T}(C) \cong C'/\mathbb{T}(C')$. However, divisibility C and C' implies that $C \cong C/\mathbb{T}(C) \times \mathbb{T}(C)$ and $C' \cong C'/\mathbb{T}(C') \times \mathbb{T}(C')$, and so $C \cong C'$. $\square$

From Theorems 3.1 and 4.3 it follows at once the following theorem, which classifies all almost null rings.

Theorem 4.4 *A ring R is an almost null ring if and only if R is isomorphic to an essential subring of the ring $\bigoplus_{p\in\mathbb{P}} R_{(p)} \oplus C$, where the rings C and $R_{(p)}$ for every $p \in \mathbb{P}$ are the same as in Theorem 4.3.*

Note that the problem of isomorphism of the rings described in the previous theorem remains. It seems to be very complicated, especially for nontorsion rings. This problem requires separate, extensive research.

From Theorems 2.1 and 2.2 it follows that an almost null ring R is noetherian i.e satisfies the ascending chain condition on subrings if and only if the group R^+ is finitely generated. Hence, and by description of subgroups of the group $\mathbb{Z}_{p^\infty}$ and Theorem 4.3 we have the following theorem, which classifies all noetherian almost null rings.

Theorem 4.5 *For every odd prime integer p let μ_p be a fixed quadratic nonresidue modulo p. The ring R is a noetherian almost null ring if and only if R is a subring of the ring of the form:*

$$\bigoplus_{p\in\Pi} R_{(p)} \oplus C, \tag{7}$$

where Π is a finite subset of $\mathbb{P}$ and $R_{(p)}$ is one of the following rings:

(i) $\mathbb{Z}_p \times_{p^{s-1}} \mathbb{Z}_{p^s}$, $s \in \mathbb{N}$,

(ii) $(\mathbb{Z}_p \times \mathbb{Z}_p)_{0,0,-\mu_p} \times_{p^{s-1}} \mathbb{Z}_{p^s}$, *for* $p>2$, $s \in \mathbb{N}$,

(iii) $(\mathbb{Z}_p \times \mathbb{Z}_p)_{-1,1,-V^2\mu_p} \times_{p^{s-1}} \mathbb{Z}_{p^s}$, *for* $p>2$, $s \in \mathbb{N}$, $V = 1,2,...,(p-1)/2$,

(iv) $(\mathbb{Z}_2 \times \mathbb{Z}_2)_{1,0,1} \times_{p^{s-1}} \mathbb{Z}_{2^s}$, $s \in \mathbb{N}$,

and C is any ring such that $C^2 = 0$ and the group C^+ is finitely generated.

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