Some properties of a class of analytic functions defined by generalized Struve functions

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Abstract: The aim of this paper is to define a new operator by using the generalized Struve functions
\[ \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(3/2)n!(k)} z^{n+1} \]
with \( k = p+ (b+2)/2 \neq 0, -1, -2, \ldots \) and \( b, c, k \in \mathbb{C} \). By using this operator we define a subclass of analytic functions. We discuss some properties of this class such as inclusion problems, radius problems, and some other interesting properties related to this operator.

Key words: Analytic functions, subordination, generalized Struve functions

1. Introduction

Let \( A \) be the class of functions \( f \) of the form
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \]
which are analytic in the open unit disk \( E = \{ z : |z| < 1 \} \). A function \( f \) is said to be subordinate to a function \( g \) written as \( f \prec g \), if there exists a Schwarz function \( w \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = g(w(z)) \). In particular, if \( g \) is univalent in \( E \), then \( f(0) = g(0) \) and \( f(E) \subset g(E) \).

For any two analytic functions \( f(z) \) and \( g(z) \) with
\[ f(z) = \sum_{n=0}^{\infty} b_n z^{n+1} \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} c_n z^{n+1}, \quad z \in E, \]
the convolution (Hadamard product) is given by
\[ (f \ast g)(z) = \sum_{n=0}^{\infty} b_n c_n z^{n+1}, \quad z \in E. \]

Consider the following second-order inhomogeneous differential equation and see [16] for more details:
\[ z^2 w''(z) + zw'(z) + (z^2 - p^2) w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi \Gamma(p+1/2)}}, \quad p \neq 0, -1, -2, \ldots \]

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Some geometric properties such as univalency, starlikeness, convexity, and close-to-convexity of the function $C$ in the whole complex plane and satisfies the differential equation

$$H_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+p+1}}{\Gamma(n+3/2) \Gamma(p+n+3/2)}.$$  \hfill (1.3)

Now we consider the differential equation

$$z^2 w''(z) + zw'(z) - (z^2 + p^2) w(z) = \frac{4 (z/2)^{p+1}}{\sqrt{\pi} \Gamma(p+1/2)}.$$  \hfill (1.4)

Equation (1.4) differs from equation (1.2) in the coefficients of $w(z)$. Its particular solution is called the modified Struve function of order $p$ and is given as

$$L_p(z) = -ie^{-ip\pi/2} H_p(iz) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+p+1}}{\Gamma(n+3/2) \Gamma(p+n+3/2)}.$$  \hfill (1.5)

Again consider the second-order inhomogeneous differential equation

$$z^2 w''(z) + bw'(z) + [cz^2 - p^2 + (1 - b)p] w(z) = \frac{4 (z/2)^{p+1}}{\sqrt{\pi} \Gamma(p+b/2)},$$  \hfill (1.6)

where $b,c,p \in \mathbb{C}$. Equation (1.5) generalizes equations (1.2) and (1.4). In particular for $b = 1, c = 1$, we obtain (1.2) and for $b = 1, c = -1$, we obtain (1.4). Its particular solution has the series form

$$M_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n (z/2)^{2n+p+1}}{\Gamma(n+3/2) \Gamma(p+n+(b+2)/2)}.$$  \hfill (1.6)

and is called the generalized Struve function of order $p$. This series is convergent everywhere but not univalent in the open unit disk $E$. We take the transformation

$$N_{p,b,c}(z) = 2^p \sqrt{\pi} \Gamma(p+(b+2)/2) z^{-(p-1)/2} M_{p,b,c}(\sqrt{z}) \sum_{n=0}^{\infty} \frac{(-c/4)^n z^n}{(3/2)_n (k)_n},$$  \hfill (1.7)

where $k = p+(b+2)/2 \neq 0, -1, -2, \ldots$ and $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} = \gamma (\gamma + 1) \ldots (\gamma + n - 1)$. This function is analytic in the whole complex plane and satisfies the differential equation

$$4z^2 w''(z) + 2(2p+b+3) zw'(z) + [cz+2p+b] w(z) = 2p+b.$$  \hfill (1.7)

Some geometric properties such as univalency, starlikeness, convexity, and close-to-convexity of the function $N_{p,b,c}(z)$ were studied recently by Orhan and Yağmur [10] and Yağmur and Orhan [14, 15].

Dziok and Srivastava [3, 4] defined the linear operator $H$ by using the generalized hypergeometric functions and it is given as $H(\alpha_1, \ldots, \alpha_s; \beta_1, \ldots, \beta_q) : A \to A$ with $\alpha_i \in \mathbb{C} (i=1,2,\ldots,s)$ and $\beta_i \in \mathbb{C}\setminus\mathbb{Z}_0^-$ (i = 1, 2, \ldots, q) such that

$$H(\alpha_1, \ldots, \alpha_s; \beta_1, \ldots, \beta_q) f(z) = z^s F_q(\alpha_1, \ldots, \alpha_s; \beta_1, \ldots, \beta_q; z) * f(z),$$  \hfill (1.7)
where
\[ _sF_q (\alpha_1, \ldots, \alpha_s; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \left( \frac{\alpha_1 \cdots \alpha_s}{\beta_1 \cdots \beta_q} \right)_n \frac{z^n}{n!}, \quad s \leq q + 1; \ s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \]
is the generalized hypergeometric function. Baricz et al. \cite{2} used a similar argument to define a convolution operator \( B_k^c : A \rightarrow A \) by using generalized Bessel functions and it is given as
\[
B_k^c f (z) = \varphi_{k,c} (z) \ast f (z) = z + \sum_{n=1}^{\infty} \frac{(-c/4)^n a_{n+1} z^{n+1}}{(k)_n n!}, \quad \left( k = p + \frac{b + 1}{2} \notin \mathbb{Z}_0^-, c \in \mathbb{C} \right),
\]
where
\[ \varphi_{k,c} (z) = z + \sum_{n=1}^{\infty} \frac{(-c/4)^n z^{n+1}}{(k)_n n!} . \]

For some references for convolution operators see \cite{11, 12, 13}.

Now using (1.7), we define the following convolution operator. Let
\[
\varphi_{p,b,c} (z) = 2^p \sqrt{\pi} \Gamma (p + (b + 2)/2) z^{(-p+1)/2} M_{p,b,c} (\sqrt{z}) = z + \sum_{n=1}^{\infty} \frac{(-c/4)^n z^{n+1}}{(3/2)_n (k)_n}.
\]
Then
\[
S_k^c f (z) = \varphi_{p,b,c} (z) \ast f (z) = z + \sum_{n=1}^{\infty} \frac{(-c/4)^n a_{n+1} z^{n+1}}{(3/2)_n (k)_n} \left( k = p + \frac{b + 2}{2} \notin \mathbb{Z}_0^-, b, c \in \mathbb{C} \right). \tag{1.8}
\]
It can easily be seen that
\[
z (S_k^c f (z))' = k S_k^c f (z) - (k - 1) S_{k+1}^c f (z). \tag{1.9}
\]
**Special cases**

(i) For \( b = 1, \ c = 1 \), we have the operator \( S_p : A \rightarrow A \) related with the Struve function of order \( p \). It is given as
\[
S_p f (z) = \varphi_{p,1,1} (z) \ast f (z) = \left[ 2^p \sqrt{\pi} \Gamma (p + 3/2) z^{(-p+1)/2} M_{p,1,1} (\sqrt{z}) \right] \ast f (z)
= z + \sum_{n=1}^{\infty} \frac{(-1/4)^n a_{n+1} z^{n+1}}{(3/2)_n (p + 3/2)_n}
\]
and the recursive relation
\[
z [S_{p+1} f (z)]' = (p + 3/2) S_p f (z) - (p + 1/2) S_{p+1} f (z)
\]
holds.

(ii) For \( b = 1, \ c = -1 \), we obtain the operator \( \mathfrak{S}_p : A \rightarrow A \) related with the modified Struve function of order \( p \). It is given as
\[
\mathfrak{S}_p f (z) = \varphi_{p,1,-1} (z) \ast f (z) = \left[ 2^p \sqrt{\pi} \Gamma (p + 3/2) z^{(-p+1)/2} M_{p,1,-1} (\sqrt{z}) \right] \ast f (z)
= z + \sum_{n=1}^{\infty} \frac{(1/4)^n a_{n+1} z^{n+1}}{(3/2)_n (p + 3/2)_n}
\]

and the recursive relation
\[ z \left[ \mathcal{G}_{p+1} f (z) \right]' = (p + 3/2) \mathcal{G}_p f (z) - (p + 1/2) \mathcal{G}_{p+1} f (z) \]
holds.

We define the following class of analytic functions by using the operator \( S^c_k f (z) \).

**Definition 1.1** Let \( f \in A \). Then \( f \in N^\alpha_{k,c} (\lambda, \mu, \phi) \) for \( 0 < \mu < 1, \lambda \in \mathbb{C}, k = p+(b+2)/2 \neq 0, -1, -2, \ldots, b, c, p \in \mathbb{C}, \) and \( |\alpha| < \frac{\pi}{2}, \) if and only if
\[
eq e^{i\alpha} \left\{ (1 + \lambda) \left( \frac{z}{S^c_{k+1} f (z)} \right)^\mu - \lambda \frac{S^c_k f (z)}{S^c_{k+1} f (z)} \left( \frac{z}{S^c_{k+1} f (z)} \right)^\mu \right\} < \cos \alpha \phi (z) + i \sin \alpha, \tag{1.10} \]
where \( \phi (z) \) is a convex univalent function with \( \phi (0) = 1. \)

(i) For \( \phi (z) = 1 + \frac{Az}{1+Bz}, -1 \leq B < A \leq 1, \) we have the class \( N^\alpha_{k,c} \left( \lambda, \mu, 1+\frac{Az}{1+Bz} \right) \), which consists of functions \( f \) such that
\[
J (\alpha, c, k, f (z)) < \frac{1+Az}{1+Bz},
\]
where
\[
J (\alpha, c, k, f (z)) = \frac{1}{\cos \alpha} \left[ e^{i\alpha} \left\{ (1 + \lambda) \left( \frac{z}{S^c_{k+1} f (z)} \right)^\mu - \lambda \frac{S^c_k f (z)}{S^c_{k+1} f (z)} \left( \frac{z}{S^c_{k+1} f (z)} \right)^\mu \right\} - i \sin \alpha \right].
\]

(ii) For \( \phi (z) = \frac{1+z}{1-z}, \) we have the class \( N^\alpha_{k,c} \left( \lambda, \mu, \frac{1+z}{1-z} \right) \). That is, \( f \in N^\alpha_{k,c} \left( \lambda, \mu, \frac{1+z}{1-z} \right) \) if
\[
J (\alpha, c, k, f (z)) < \frac{1+z}{1-z}.
\]

Since it is well known that for a function \( p (z) < \frac{1+z}{1-z} \), then \( \text{Re} p (z) > 0. \) This implies that \( f \in N^\alpha_{k,c} \left( \lambda, \mu, \frac{1+z}{1-z} \right) \) if
\[
\text{Re} J (\alpha, c, k, f (z)) > 0.
\]

**Lemma 1.2** [8] Let \( F \) be analytic and convex in \( E \). If \( f, g \in A \) and \( f, g \prec F, \) then
\[
\sigma f + (1-\sigma) g \prec F, \quad 0 \leq \sigma \leq 1.
\]

**Lemma 1.3** [6] Let \( h \) be convex in \( E \) with \( h(0) = a \) and \( \beta \in \mathbb{C} \) such that \( \text{Re} \beta \geq 0. \) If \( p \in H [a, n] \) and
\[
p(z) + \frac{zp(z)}{\beta} \prec h (z),
\]
then \( p(z) \prec q (z) \prec h (z), \) where
\[
q (z) = \frac{\beta}{n z^{n-1}} \int_0^z h(t) t^{\beta/n-1} dt
\]
and \( q (z) \) is the best dominant.
Lemma 1.4 [1]. Let \( a, b, \) and \( c \neq 0, -1, -2 \ldots \) be complex numbers. Then, for \( \Re c > \Re b > 0, \)

\[
(i) \quad _2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt,
\]

\[
(ii) \quad _2F_1(a, b, c; z) = _2F_1(b, a, c; z),
\]

\[
(iii) \quad _2F_1(a, b, c; z) = (1-z)^{-a} _2F_1\left(a, c-b, c; \frac{z}{z-1}\right).
\]

Lemma 1.5 [7] Let \(-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1\). Then

\[
\frac{1 + A_2z}{1 + B_2z} < \frac{1 + A_1z}{1 + B_1z}.
\]

Lemma 1.6 [9] Let the function \( g(z) \) be analytic and univalent in \( E \) and let the functions \( \theta(w) \) and \( \varphi(w) \)
be analytic in a domain \( D \) containing \( g(E) \), with \( \theta(w) \neq 0 \) \((w \in g(E))\). Set \( Q(z) = zg'(z)\varphi(g(z)) \) and \( h(z) = \theta(g(z)) + Q(z) \) and suppose that

(i) \( Q(z) \) is univalently starlike in \( E \);

(ii) \( Re^{\frac{g'(z)}{Q(z)}} = Re\left\{\frac{\theta'(g(z))}{\theta(g(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0 \ (z \in E) \). If \( q(z) \) is analytic in \( E \) with \( q(0) = g(0), q(E) \subset D \) and

\[
\theta(q(z)) + zq'(z)\varphi(q(z)) < \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z) \ (z \in E),
\]

then \( q(z) < g(z) \ (z \in E) \) and \( g(z) \) is the best dominant.

2. Main results

Theorem 2.1 Let \( f \in N^\mu_{k,c} (\lambda, \mu, \phi) \). Then for \( Re^\mu k > 0 \),

\[
e^{i\alpha} \left( \frac{z}{S^k_{k+1} f(z)} \right)^\mu < \frac{\mu k}{\lambda} \cos \alpha z^{-\frac{\mu k}{\lambda}} \int_0^z \phi(t) t^{-\mu k-1} dt + i \sin \alpha < (\cos \alpha) \phi(z) + i \sin \alpha.
\]

This result is the best possible.

Proof Consider

\[
p(z) = \frac{1}{\cos \alpha} \left\{ e^{i\alpha} \left( \frac{z}{S^k_{k+1} f(z)} \right)^\mu - i \sin \alpha \right\}. \tag{2.1}
\]

Then \( p \) is analytic in \( E \) with \( p(0) = 1 \). Therefore, we have

\[
e^{i\alpha} \left( \frac{z}{S^k_{k+1} f(z)} \right)^\mu = (\cos \alpha) p(z) + i \sin \alpha.
\]

Differentiating both sides and using (1.9) and simplifying, we obtain

\[
\frac{\lambda (\cos \alpha) z p'(z)}{\mu k} = \lambda e^{i\alpha} \left\{ \left( \frac{z}{S^k_{k+1} f(z)} \right)^\mu - \frac{S^k_{k+1} f(z) z}{S^k_{k+1} f(z)} \left( \frac{z}{S^k_{k+1} f(z)} \right)^\mu \right\}.
\]
It follows from the above equation and (2.1) that

\[ p(z) + \frac{\lambda}{\mu k} z p'(z) = \frac{1}{ \cos \alpha } \left[ e^{i\alpha} \left\{ (1 + \lambda) \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu - \lambda \frac{S_{k}^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu \right\} - i \sin \alpha \right]. \]

Since \( f \in N_{k,c}^\alpha (\lambda, \mu, \phi) \), therefore

\[ p(z) + \frac{\lambda}{\mu k} z p'(z) \prec \phi(z). \]

Now using Lemma 1.3 for \( \beta = \frac{\mu k}{\lambda} \) with \( \text{Re} \frac{\mu k}{\lambda} \geq 0 \), we obtain the required result. \( \square \)

**Corollary 2.2** Let \( f \in N_{k,c}^\alpha \left( \lambda, \mu, \frac{1 + Az}{1 + Bz} \right) \). Then for \( k, \lambda \in \mathbb{R} \) and \( \frac{\mu k}{\lambda} \geq 0 \),

\[ e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu \prec h(z) \cos \alpha + i \sin \alpha, \]

where

\[ h(z) = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B}) \left( 1 + Bz \right)^{-1} \binom{2}{1,1} \binom{1}{1,1} \binom{k+1}{1+Bz}, & \text{if } B \neq 0, \\ 1 + \frac{\mu k}{\lambda + \lambda} Az, & \text{if } B = 0. \end{cases} \]

Furthermore,

\[ \text{Re} \left[ e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu \right] > (\cos \alpha) h(-1). \]

**Proof** Since \( f \in N_{k,c}^\alpha \left( \lambda, \mu, \frac{1 + Az}{1 + Bz} \right) \), therefore from Theorem 2.1, we have

\[ e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu \prec \frac{\mu k}{\lambda} (\cos \alpha) z^{-\frac{\mu k}{\lambda}} \int_0^1 \frac{1 + At}{1 + Bt} \left( \frac{z}{u} \right)^{k-1} dt + i \sin \alpha. \]  

(2.2)

Putting \( t = zu \) and after simple calculations, one can get

\[ e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu \prec \left\{ \frac{A}{B} + \frac{\mu k}{\lambda} \left( 1 - \frac{A}{B} \right) \int_0^1 \left( 1 + Buz \right)^{-1} u^{k-1} \left( \frac{z}{u} \right)^{-1} dt \right\} \cos \alpha + i \sin \alpha. \]

Now using Lemma 1.4 for \( a = 1, b = \frac{\mu k}{\lambda}, c = b + 1, \) and \( B \neq 0 \), we obtain

\[ e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu \prec \left( \frac{A}{B} + (1 - \frac{A}{B}) \left( 1 + Bz \right)^{-1} \binom{2}{1,1} \binom{1}{1,1} \binom{k+1}{1+Bz} \right) \cos \alpha + i \sin \alpha. \]
For the case of $B = 0$, it can easily be followed from (2.2) that

$$e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} < \left( \frac{\mu k}{\lambda} \int_0^1 (1 + Atz) t^\frac{\mu k}{\lambda - 1} dt \right) \cos \alpha + i \sin \alpha.$$

$$= \frac{\mu k}{\lambda} \left\{ \left( \int_0^1 t^\frac{\mu k}{\lambda - 1} dt \right) + \int Azt^\frac{\mu k}{\lambda - 1} dt \right\} \cos \alpha + i \sin \alpha.$$

$$= \left\{ 1 + \frac{\mu k}{\mu k + \lambda} A \right\} \cos \alpha + i \sin \alpha.$$

Now we have to prove that $\Re \left[ e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} \right] > (\cos \alpha) h(-1)$. From (2.2), we can have this relation by using subordination

$$\frac{1}{\cos \alpha} \left\{ e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} - i \sin \alpha \right\} = h(w(z)),$$

where $h(z) = \frac{\mu k}{\lambda} z^{-\frac{\mu k}{\lambda}} \int_0^1 \frac{1 + At}{1 + Bt} t^\frac{\mu k}{\lambda - 1} dt$. Therefore,

$$\Re \left[ \frac{1}{\cos \alpha} \left\{ e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} \right\} \right] = \Re \frac{\mu k}{\lambda} \int_0^1 \frac{1 + Atw(z)}{1 + Btw(z)} t^\frac{\mu k}{\lambda - 1} dt$$

$$> \frac{\mu k}{\lambda} \int_0^1 \frac{1 - At}{1 - Bt} t^\frac{\mu k}{\lambda - 1} dt$$

$$= h(-1).$$

To show that this result is sharp, we have to prove that $\inf_{|z|<1} \{ \Re h(z) \} = h(-1)$. Now

$$\Re h(z) \geq \frac{\mu k}{\lambda} \int_0^1 t^\frac{\mu k}{\lambda - 1} \frac{1 - Atr}{1 - Btr} dt = h(-r).$$

Therefore, $h(-r) \rightarrow h(-1)$ as $r \rightarrow 1^-$. \hfill $\square$

**Theorem 2.3** Let $e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} < \phi(z) \cos \alpha + i \sin \alpha$ with $\phi(z) = \frac{1 + z}{1 - z}$. Then $f \in N_{k,c}^\alpha(\lambda, \mu, \phi(z))$ for $|z| = r < -c + \sqrt{c^2 + 1}$, where $c = \left| \frac{1}{\mu k} \right|$.

**Proof** Let

$$e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} = p(z) \cos \alpha + i \sin \alpha,$$
where \( p(z) < \frac{1 + z}{1 - z} \). Then from Theorem 2.1, we have

\[
p(z) + \frac{\lambda}{\mu k} z p'(z) = \frac{1}{\cos \alpha} \left[ e^{i\alpha} \left\{ (1 + \lambda) \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu - \lambda \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu \right] - i \sin \alpha \right].
\]

Since \( p(z) < \frac{1 + z}{1 - z} \), then it is well known (see [5]) that:

\[
\frac{1 - r}{1 + r} \leq \text{Re} p(z) \leq |p(z)| \leq \frac{1 + r}{1 - r} \quad \text{and} \quad |zp'(z)| \leq \frac{2r \text{Re} p(z)}{1 - r^2}.
\]

Thus, we have

\[
\begin{align*}
\text{Re} \frac{1}{\cos \alpha} \left[ e^{i\alpha} \left\{ (1 + \lambda) \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu - \lambda \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu \right] - i \sin \alpha \right] & \geq \text{Re} p(z) - \frac{\lambda}{\mu k} |zp'(z)|. \\
\end{align*}
\]

Using (2.3), we obtain

\[
\begin{align*}
\text{Re} \frac{1}{\cos \alpha} \left[ e^{i\alpha} \left\{ (1 + \lambda) \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu - \lambda \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu \right] - i \sin \alpha \right] & \geq \text{Re} p(z) - \frac{2r \text{Re} p(z)}{1 - r^2} \\
& = \text{Re} p(z) \frac{1 - r^2 - 2cr}{1 - r^2}.
\end{align*}
\]

Since \( p(z) < \frac{1 + z}{1 - z} \), therefore \( \text{Re} p(z) > 0 \). This implies that \( f \in N_{k,c}^\alpha (\lambda, \mu, \phi(z)) \) for \( r < -c + \sqrt{c^2 + 1} \). This result is sharp for the function \( p(z) = \frac{1 + z}{1 - z} \). \qed

**Theorem 2.4** Let \( 0 < \mu < 1, \ k = p + (b + 2)/2 \neq 0, -1, -2, \ldots, b, c, p \in \mathbb{C} \). Then

\[
N_{k,c}^\alpha (\lambda_2, \mu, \phi) \subset N_{k,c}^\alpha (\lambda_1, \mu, \phi), \ 0 \leq \lambda_1 < \lambda_2.
\]

**Proof** Since \( f \in N_{k,c}^\alpha (\lambda_2, \mu, \phi) \), therefore we have

\[
h_1(z) = (1 + \lambda_2) \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu - \lambda_2 \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu \prec \phi(z).
\]

From Theorem 2.1 for \( \alpha = 0 \), we write

\[
h_2(z) = \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu \prec \phi(z), \ z \in E.
\]
Now for \( \lambda_1 \geq 0 \), we obtain

\[
(1 + \lambda_1) \left( \frac{z}{S_{k+1}^c f(z)} \right) - \lambda_1 \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right) = (1 - \frac{\lambda_1}{\lambda_2}) \left( \frac{z}{S_{k+1}^c f(z)} \right) + \frac{\lambda_1}{\lambda_2} f(z) + \lambda_2 \left( \frac{z}{S_{k+1}^c f(z)} \right).
\]

Using the convexity of the class of the functions \( \phi(z) \) and Lemma 1.2, we write

\[
\frac{\lambda_1}{\lambda_2} f(z) + \left( 1 - \frac{\lambda_1}{\lambda_2} \right) f(z) < \phi(z), \quad z \in E,
\]

and this implies that \( f \in N_{k,c}^0 (\lambda_1, \mu, \phi) \). Hence, the proof of the theorem is complete.

\[\square\]

**Corollary 2.5** Let \( 0 < \mu < 1, \ k = p + (b + 2)/2 \neq 0, -1, -2, \ldots, b, c, p \in \mathbb{C} \). Then for \(-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1,\)

\[
N_{k,c}^0 \left( \lambda_2, \mu, \frac{1 + A_2 z}{1 + B_2 z} \right) \subset N_{k,c}^0 \left( \lambda_1, \mu, \frac{1 + A_1 z}{1 + B_1 z} \right), \quad 0 \leq \lambda_1 < \lambda_2, \quad z \in E.
\]

**Proof** Let \( f \in N_{k,c}^0 \left( \lambda_2, \mu, \frac{1 + A_2 z}{1 + B_2 z} \right) \). Then

\[
h_1(z) = (1 + \lambda_2) \left( \frac{z}{S_{k+1}^c f(z)} \right) - \lambda_2 \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right) < \frac{1 + A_2 z}{1 + B_2 z}.
\]

Since \(-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1,\) therefore by Lemma 1.5, we have

\[
h_1(z) = (1 + \lambda_2) \left( \frac{z}{S_{k+1}^c f(z)} \right) - \lambda_2 \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right) < \frac{1 + A_1 z}{1 + B_1 z}.
\]

Theorem 2.1 implies for \( \phi(z) = \frac{1 + A_1 z}{1 + B_1 z} \) that

\[
h_2(z) = \left( \frac{z}{S_{k+1}^c f(z)} \right) < \frac{1 + A_1 z}{1 + B_1 z}.
\]
Now for $\lambda_2 > \lambda_1 \geq 0$,

\[
(1 + \lambda_1) \left( \frac{z}{S_{k+1}^c(z)} \right)^{\mu} - \lambda_1 \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} = (1 - \frac{\lambda_1}{\lambda_2}) \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} + \frac{\lambda_1}{\lambda_2} \left\{ (1 + \lambda_2) \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} - \lambda_2 \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} \right\} = \frac{\lambda_1}{\lambda_2} h_1(z) + (1 - \frac{\lambda_1}{\lambda_2}) h_2(z).
\]

Using the convexity of the function $\frac{1 + 4z}{1 + B_1 z}$ with Lemma 1.2, we write

\[
\frac{\lambda_1}{\lambda_2} h_1(z) + (1 - \frac{\lambda_1}{\lambda_2}) h_2(z) < \frac{1 + A_1 z}{1 + B_1 z}, \quad z \in E,
\]

and this implies that $f \in N^0_k,c \left( \lambda_1, \mu; \frac{1 + A_1 z}{1 + B_1 z} \right)$. \qed

**Theorem 2.6** Let $f \in N^0_k,c (\lambda, \mu, \phi), \quad 0 < \mu < 1, \quad k = p + (b + 2)/2 \neq 0, -1, -2, \ldots, b, c, p \in \mathbb{C}$ and $\lambda \leq -1$. Then

\[
\frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} \prec \phi(z).
\]

**Proof** Since $f \in N^0_k,c (\lambda, \mu, \phi)$, therefore we have

\[
(1 + \lambda) \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} - \lambda \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} \prec \phi(z).
\]

Now consider

\[
\lambda \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} = (1 + \lambda) \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} + \lambda \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} - (1 + \lambda) \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu}.
\]

This implies that

\[
\frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} = \left( 1 + \frac{1}{\lambda} \right) \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} - \frac{1}{\lambda} \left\{ (1 + \lambda) \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} + \lambda \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^{\mu} \right\}.
\]

Using Theorem 2.1, Lemma 1.2, and the convexity of $\phi(z)$ with $\lambda \leq -1$, we have the required result. \qed
Theorem 2.7 Let $f \in N_{k,c}^\alpha (\lambda, \mu, h), h(z) = \frac{1+Az}{1+Bz} + \frac{\lambda \mu (A-B)z}{k (1+Bz)^2}$. Then for $Re \frac{\lambda}{\mu} k > 0$,

$$e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu < (\cos \alpha \phi(z) + i \sin \alpha,$$

where $\phi(z) = \frac{1+Az}{1+Bz}$. This result is the best possible.

Proof Consider

$$p(z) = \frac{1}{\cos \alpha} \left\{ e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu - i \sin \alpha \right\}.$$

Then $p$ is analytic in $E$ with $p(0) = 1$. Therefore, we have

$$e^{i\alpha} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu = (\cos \alpha) p(z) + i \sin \alpha.$$

Differentiating both sides, using (1.9), and simplifying, we obtain

$$\frac{\lambda (\cos \alpha) z p'(z)}{\mu k} = \lambda e^{i\alpha} \left\{ \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu - \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu \right\}.$$

It follows from the above equation and (2.1) that

$$p(z) + \frac{\lambda}{\mu k} z p'(z) = \frac{1}{\cos \alpha} \left\{ e^{i\alpha} \left( (1+\lambda) \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu - \frac{S_k^c f(z)}{S_{k+1}^c f(z)} \left( \frac{z}{S_{k+1}^c f(z)} \right)^\mu \right) \right\} - i \sin \alpha.$$

Since $f \in N_{k,c}^\alpha (\lambda, \mu, h)$, therefore

$$p(z) + \frac{\lambda}{\mu k} z p'(z) < h(z).$$

Now we choose $g(z) = \frac{1+Az}{1+Bz}$, and then $\theta(w) = w$ and $\varphi(w) = \frac{\mu k}{\lambda}$. It is clear that $g(z)$ is analytic in $E$ with $g(0) = 1$. Also, $\theta(w)$ and $\varphi(w)$ are analytic with $\theta(w) \neq 0$.

We see that

$$Q(z) = zg'(z) \varphi(g(z)) = \frac{\mu k (A-B)z}{\lambda (1+Bz)^2}.$$

(2.4)

We have to prove that $Q(z)$ is starlike. In other words, we show that $Re \frac{zQ'(z)}{Q(z)} > 0$. From (2.4), we have

$$Re \frac{zQ'(z)}{Q(z)} = Re \left\{ 1 - \frac{2Bz}{1+Bz} \right\}$$

$$= 1 - 2BRe \frac{re^{i\psi}}{1 + Bre^{i\psi}} \quad (z = re^{i\psi})$$

$$= \frac{1 - B^2r^2}{(1 + Br \cos \psi)^2 + B^2r^2 \sin^2 \psi}.$$
Since \(-1 \leq B < 1, r < 1\). This implies that \(\text{Re} \frac{zQ'(z)}{Q(z)} > 0\). Consider

\[
\begin{align*}
\text{Re} \frac{z^2 h'(z)}{Q(z)} &= \text{Re} \left\{ \sum_{n=0}^{N} \frac{a_n}{Q(z)} \right\} \\
&= \text{Re} \sum_{n=0}^{N} \frac{a_n}{Q(z)} \geq 0.
\end{align*}
\]

Using Lemma 1.6, we have 

\[
e^{i\alpha} \left( \frac{z}{S_{k+1}^n f(z)} \right)^\mu < (\cos \alpha) \phi(z) + i \sin \alpha.
\]

The function \(\phi(z) = \frac{1+Az}{1+Bz}\) is the best possible.

**Theorem 2.8** Let \(f \in N_{k,c}^\mu \left( \lambda, \mu, \frac{1+Az}{1+Bz} \right)\). Then for \(k, \lambda \in \mathbb{R}\) and \(\frac{\mu k}{\lambda} \geq 0\),

\[
\begin{align*}
\frac{A}{B} + (1 - \frac{A}{B}) \ 2F_1 \left( 1, \frac{\mu k}{\lambda}, \frac{\mu k}{\lambda} + 1; B \right), & \quad B \neq 0, \\
1 + \frac{\mu k}{\mu k + \lambda} A, & \quad B = 0.
\end{align*}
\]

**Proof** Since \(f \in N_{k,c}^\mu \left( \lambda, \mu, \frac{1+Az}{1+Bz} \right)\), therefore, by using (2.2), we have

\[
\begin{align*}
\frac{1}{\cos \alpha} \text{Re} \left\{ e^{i\alpha} \left( \frac{z}{S_{k+1}^n f(z)} \right)^\mu \right\} < \text{Re} \left\{ \frac{\mu k}{\lambda} \int_0^1 \frac{1 + Atz}{1 + Btz} t^{\frac{\mu k}{\lambda} - 1} dt \right\}.
\end{align*}
\]

It follows from the definition of subordination that

\[
\begin{align*}
\frac{1}{\cos \alpha} \text{Re} \left\{ e^{i\alpha} \left( \frac{z}{S_{k+1}^n f(z)} \right)^\mu \right\} < \sup_{|z| < 1} \left\{ \frac{\mu k}{\lambda} \int_0^1 \frac{1 + Atz}{1 + Btz} t^{\frac{\mu k}{\lambda} - 1} dt \right\}.
\end{align*}
\]

\[
\begin{align*}
&\leq \left\{ \frac{\mu k}{\lambda} \int_0^1 \sup_{|z| < 1} \left\{ \frac{1 + Atz}{1 + Btz} \right\} t^{\frac{\mu k}{\lambda} - 1} dt \right\} \\
&\leq \frac{\mu k}{\lambda} \int_0^1 \left\{ \frac{1 + Az}{1 + Bt} \right\} t^{\frac{\mu k}{\lambda} - 1} dt \\
&= \frac{\mu k}{\lambda} \int_0^1 \left\{ A/B + (1 - A/B) \right\} t^{\frac{\mu k}{\lambda} - 1} dt.
\end{align*}
\]

Now using Lemma 1.4 for the case \(B \neq 0\), we have

\[
\begin{align*}
\frac{1}{\cos \alpha} \text{Re} \left\{ e^{i\alpha} \left( \frac{z}{S_{k+1}^n f(z)} \right)^\mu \right\} < \frac{A}{B} + (1 - \frac{A}{B}) \ 2F_1 \left( 1, \frac{\mu k}{\lambda}, \frac{\mu k}{\lambda} + 1; -B \right).
\end{align*}
\]
When $B = 0$, it can be easily seen that

$$
\frac{1}{\cos \alpha} \Re \left\{ e^{i\alpha} \left( \frac{z}{S_{k+1}^{c} f(z)} \right)^{\mu} \right\} < \frac{\mu k}{\lambda} \int_{0}^{1} (1 + At) t^{\frac{\mu k}{\lambda} - 1} dt
$$

$$
= 1 + \frac{\mu k}{\mu k + \lambda} \cdot A.
$$

We also have

$$
\frac{1}{\cos \alpha} \Re \left\{ e^{i\alpha} \left( \frac{z}{S_{k+1}^{c} f(z)} \right)^{\mu} \right\} > \inf_{|z| < 1} \Re \left\{ \frac{\mu k}{\lambda} \int_{0}^{1} \frac{1 + At}{1 + Bt} t^{\frac{\mu k}{\lambda} - 1} dt \right\}
$$

$$
\geq \left\{ \frac{\mu k}{\lambda} \int_{0}^{1} \inf_{|z| < 1} \Re \left\{ \frac{1 + At}{1 + Bt} t^{\frac{\mu k}{\lambda} - 1} dt \right\} \right\}
$$

$$
= \frac{\mu k}{\lambda} \int_{0}^{1} \left\{ A/B + \left( \frac{1 - A/B}{1 - B} \right) \right\} t^{\frac{\mu k}{\lambda} - 1} dt.
$$

Using again Lemma 1.4, we have the required result. \qed

**Theorem 2.9** Let $f \in N^{a}_{k,c} \left( \lambda, \mu, \frac{1 + A}{1 + B} \right)$. Then for $k, \lambda \in \mathbb{R}$ and $\frac{\mu k}{\lambda} \geq 0$,

$$
\frac{\mu k}{\lambda} \int_{0}^{1} F_{1} \left( 1, -\frac{A}{B} \right) \left( \frac{1 + A/\lambda}{1 + B/\lambda} \right) \frac{1 + At}{1 + Bt} t^{\frac{\mu k}{\lambda} - 1} dt
$$

Proof Since $f \in N^{a}_{k,c} \left( \lambda, \mu, \frac{1 + A}{1 + B} \right)$, therefore, by using (2.2), we have

$$
\frac{1}{\cos \alpha} \left\{ e^{i\alpha} \left( \frac{z}{S_{k+1}^{c} f(z)} \right)^{\mu} - i \sin \alpha \right\} = \frac{\mu k}{\lambda} \int_{0}^{1} \frac{1 + At}{1 + Bt} t^{\frac{\mu k}{\lambda} - 1} dt.
$$

It follows from the definition of subordination that

$$
\frac{1}{\cos \alpha} \left\{ e^{i\alpha} \left( \frac{z}{S_{k+1}^{c} f(z)} \right)^{\mu} - i \sin \alpha \right\} = \left\{ \frac{\mu k}{\lambda} \int_{0}^{1} \frac{1 + At}{1 + Bt} t^{\frac{\mu k}{\lambda} - 1} dt \right\}.
$$
where \( w(z) = c_1z + c_2z^2 + \ldots \) is analytic and \( |w(z)| \leq |z| \). Therefore,

\[
\left| \frac{1}{\cos \alpha} \left\{ e^{i\alpha} \left( \frac{z}{S_{k+1} f(z)} \right)^\mu - i \sin \alpha \right\} \right| \leq \frac{\mu k}{\lambda} \int_0^1 \frac{1 + Atr}{1 + Btr} t^{\frac{\mu k}{\lambda} - 1} \, dt.
\]

Now using the same process as in the theorem above, we get the required result. 

\[ \square \]

References