On Condition (PWP)$_w$ for $S$-posets

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Abstract: Golchin and Rezaei (Commun Algebra 2009; 37: 1995–2007) introduced the weak version of Condition (PWP) for $S$-posets, called Condition (PWP)$_w$. In this paper, we continue to study this condition. We first present a necessary and sufficient condition under which the $S$-poset $A(I)$ satisfies Condition (PWP)$_w$. Furthermore, we characterize pomonoids $S$ over which all cyclic (Rees factor) $S$-posets satisfy Condition (PWP)$_w$, and pomonoids $S$ over which all Rees factor $S$-posets satisfying Condition (PWP)$_w$ have a certain property. Finally, we consider direct products of $S$-posets satisfying Condition (PWP)$_w$.

Key words: Condition (PWP)$_w$, $S$-poset, Rees factor $S$-poset, direct product

1. Introduction and preliminaries

A partially ordered monoid, or briefly pomonoid, is a monoid $S$ together with a partial order $\leq$ on $S$ such that $s \leq s'$ implies $su \leq s'u$ and $us \leq us'$ for all $s, s', u \in S$. An ordered right ideal of a pomonoid $S$ is a nonempty subset $I$ of $S$ such that (1) $IS \subseteq I$ and (2) $s \leq t \in I$ implies $s \in I$, for all $s, t \in S$. In this paper, $S$ always denotes a pomonoid, and a right ideal of $S$ is simply a nonempty subset $I$ of $S$ for which $IS \subseteq I$ (not necessarily an ordered right ideal).

Let $S$ be a pomonoid. A right $S$-poset, usually denoted $A_S$, is a nonempty set $A$ equipped with a partial order $\leq$ and a right action $A \times S \to A$, $(a, s) \mapsto as$, which satisfies the conditions: (1) the action is monotonic in each variable, (2) $a(st) = (as)t$ and $a1 = a$ for all $a \in A$ and $s, t \in S$. Left $S$-posets $SB$ are defined analogously, and $\Theta_S = \{\theta\}$ is the one-element right $S$-poset. All left (resp., right) $S$-posets form a category, denoted $S$-POS (resp., $POS_S$), in which the morphisms are the functions preserving both the action and the order (see [3]).

Preliminary work on flatness properties of $S$-posets was done by Fakhruddin in the 1980s (see [4, 5]), and continued in recent papers (e.g., [1, 2, 6, 7, 10, 12, 13]).

An $S$-subposet $B_S$ of an $S$-poset $A_S$ is called convex if, for any $a \in A_S$ and $b, b' \in B_S$, $b' \leq a \leq b$ implies $a \in B$. A pomonoid $S$ is called weakly right reversible if, for any $s, s' \in S$, there exist $u, v \in S$ such that $us \leq vs'$. A pomonoid $S$ is called left collapsible if, for any $s, s' \in S$, there exists $u \in S$ such that $us = us'$. A pomonoid $S$ is called weakly left collapsible if, for any $s, s', z \in S$, $sz = s'z$ implies that there exists $u \in S$ such that $us = us'$.

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In [4], Fakhruddin introduced the concept of order congruence on $S$-posets. An order congruence on an $S$-poset $A_S$ is an $S$-act congruence $\theta$ such that the factor act $A/\theta$ can be equipped with a compatible order so that the natural map $A \to A/\theta$ is an $S$-poset morphism.

An $S$-poset $A_S$ is called cyclic if $A = aS = \{as \mid s \in S\}$ for some $a \in A_S$. An $S$-poset $A_S$ is cyclic if and only if it is isomorphic to the factor $S$-poset of $S_S$ by an $S$-poset congruence. If $K_S$ is a convex right ideal of a pomonoid $S$, then there exists an $S$-poset congruence such that one of its classes is $K$ and all the others are singletons. Moreover, the factor $S$-poset by this congruence is called the Rees factor $S$-poset of $S$ by $K$, and denoted $S/K$. For $s \in S$, the congruence class of $s$ in $S/K$ will be denoted by $[s]_{\rho_K}$, or briefly $[s]$.

The tensor product $A \otimes_S B$ of a right $S$-poset $A_S$ and a left $S$-poset $SB$ is a poset that can be constructed in a standard way (see [13] for details) so that the map $A \times B \to A \otimes_S B$ sending $(a, b)$ to $a \otimes b$ is balanced, monotonic in both variables, and universal among balanced, monotonic maps from $A \times B$ into posets. The order relation on $A \otimes_S B$ can be described as follows: $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$ if and only if there exist $a_1, a_2, \ldots, a_n \in A_S$, $b_2, \ldots, b_n \in SB$, and $s_1, t_1, \ldots, s_n, t_n \in S$ such that

$$a \leq a_1s_1$$
$$a_1t_1 \leq a_2s_2 \quad s_1b \leq t_1b_2$$
$$a_2t_2 \leq a_3s_3 \quad s_2b_2 \leq t_2b_3$$
$$\vdash$$
$$a_nt_n \leq a' \quad s_nb_n \leq t_nb'.$$

It is easily established, as for $S$-acts, that $A \otimes_S S$ can be equipped with a natural right $S$-action, and $A \otimes_S A \cong A$ for any $S$-poset $A_S$.

In [1, 12], the properties of po-flatness, po-torsion freeness, and Conditions (P), (P_w), and (E) are introduced. An $S$-poset $A_S$ is called po-flat if, for all $a, a' \in A_S$ and $b, b' \in SB$, $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$ implies $a \otimes s \leq a' \otimes s'$ in $A \otimes_S (SB \cup SB')$. An $S$-poset $A_S$ is called (principally) weakly po-flat if, for all (principal) left ideals $I$ of a pomonoid $S$, and all $s, s' \in I$, $a, a' \in A$, $a \otimes s \leq a' \otimes s'$ in $A \otimes_S S$ implies $a \otimes s \leq a' \otimes s'$ in $A \otimes_I S$. An $S$-poset $A_S$ is said to satisfy Condition (P) if, for all $a, a' \in A_S$ and $s, s' \in S$, $as \leq as'$ implies $a = a'v$ for some $a'' \in A_S$ and $u, v \in S$ with $us \leq vs'$. An $S$-poset $A_S$ is said to satisfy Condition (E) if, for all $a, a' \in A_S$ and $s, s' \in S$, $as \leq as'$ implies $a = a'u$ for some $a' \in A_S$ and $u \in S$ with $us \leq us'$. An $S$-poset $A_S$ is called strongly flat if it satisfies Conditions (E) and (P). An $S$-poset $A_S$ is said to satisfy Condition (P_w) if, for all $a, a' \in A_S$ and $s, s' \in S$, $as \leq as'$ implies $a \leq a''u$, $a''v \leq a'$ for some $a'' \in A_S$ and $u, v \in S$ with $us \leq vs'$. An element $c \in S$ is called right po-cancellable if, for all $s, s' \in S$, $sc \leq sc' \leq s'$ implies $s \leq s'$. An $S$-poset $A_S$ is said to be po-torsion free if, for all $a, a' \in A_S$, and all right po-cancellable elements $c$ of $S$, $ac \leq ac'$ implies $a \leq a'$. Recall that an $S$-poset $A_S$ is said to satisfy Condition (E') if, for all $a \in A_S$ and $s, s', z \in S$, $as \leq as'$ and $sz = s'z$ imply $a = a'u$ for some $a' \in A_S$ and $u \in S$ with $us \leq us'$. An $S$-poset $A_S$ is called weakly subpullback flat if it satisfies Conditions (E') and (P).

In [6], Conditions (WP), (WP)_w, (PWP), and (PWP)_w were introduced. An $S$-poset $A_S$ is said to satisfy Condition (WP) if the corresponding $\phi$ is surjective for every subpullback diagram $P(I, I, f, f, S)$, where $I$ is a left ideal of $S$. An $S$-poset $A_S$ is said to satisfy Condition (WP)_w if, for all $a, a' \in A_S$, $s, t \in S$, and all homomorphisms $f: g(Ss \cup St) \to SB$, $af(s) \leq a'f(t)$ implies $a \otimes s \leq a'' \otimes us'$ and $a'' \otimes vt' \leq a' \otimes t$ in
A \otimes_S (Ss \cup St) for some a'' \in A_S, u, v \in S and s', t' \in \{s, t\} with f(us') \leq f(vs'). An S-poset $A_S$ is said to satisfy Condition (PWP) if the corresponding $\phi$ is surjective for every subpullback diagram $P(Ss, Ss, f, f, S)$, $s \in S$. An S-poset $A_S$ is said to satisfy Condition (PWP)$_w$ if, for all $a, a' \in A_S$ and $s \in S$,

$$a \leq a' \quad \text{implies} \quad a \leq a''u \text{ and } a''v \leq a' \quad \text{for some } a'' \in A_S, \ u, v \in S \text{ with } us \leq vs.$$  

Moreover, the authors in [6] gave equivalent descriptions of Conditions (PWP), (WP), and (WP)$_w$ for (cyclic, Rees factor) S-posets and obtained the relations between these conditions and properties already studied as follows:

$$\text{free } \Rightarrow \text{ projective } \Rightarrow \text{ strongly flat } \Rightarrow \ (P) \Rightarrow \ (WP) \Rightarrow \ (PWP)$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\quad (P_w) \Rightarrow \ (WP)_w \Rightarrow \ (PWP)_w$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\quad \text{po-flat } \Rightarrow \text{ w. po-f. } \Rightarrow \text{ p. w. po-f. } \Rightarrow \text{ po-t. f.}$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \text{(incomparable)}$$

$$\text{flat } \Rightarrow \text{ w. f. } \Rightarrow \text{ p. w. f. } \Rightarrow \text{ t. f.}$$

In this paper, we will continue the work of [6] to study Condition (PWP)$_w$. For S-posets, the definition of the S-poset $A(I)$ was introduced in [3]. Qiao et al. in [9] proved that $A(I)$ fails to satisfy Condition (P). Later, in [10], they further investigated some flatness properties of $A(I)$ and provided an equivalent description of $A(I)$ satisfying Condition (P)$_w$. In fact, observing the proof of [9, Lemma 2.4], we obtain that $A(I)$ also fails to satisfy Condition (PWP). However, the situation for Condition (PWP)$_w$ is markedly different. Thereby, in Section 2, we determine the condition under which $A(I)$ satisfies Condition (PWP)$_w$. In [1, 11], some flatness properties of Rees factor S-posets are discussed. Later, Golchin et al. in [6] gave an equivalent characterization of cyclic (Rees factor) S-posets satisfying Condition (PWP) (Conditions (WP)$_w$ and (WP)). In Section 3, we characterize monoids $S$ over which all cyclic (Rees factor) S-posets satisfy Condition (PWP)$_w$. In [7], Khosravi first studied direct products of S-posets satisfying some flatness properties. In Section 4, we investigate monoids $S$ over which S-posets satisfying Condition (PWP)$_w$ are preserved under direct products.

2. S-posets satisfying Condition (PWP)$_w$

In this section, we discuss S-posets satisfying Condition (PWP)$_w$ and provide a necessary and sufficient condition under which the S-poset $A(I)$ satisfies Condition (PWP)$_w$.

We first give an alternative description for Condition (PWP)$_w$.

**Proposition 2.1** A right S-poset $A_S$ satisfies Condition (PWP)$_w$ if and only if for all $a, a' \in A_S$, $x, y, s \in S$, and all homomorphisms $f : SsS \to SS$, $af(xs) \leq a'f(xs)$ implies that there exist $a'' \in A_S$ and $u, v \in S$ such that $f(us) \leq f(vs)$, $a \otimes x \leq a'' \otimes u$ and $a'' \otimes v \leq a' \otimes y$ in $A \otimes_S Ss$.

**Proof** It follows from the definition of Condition (PWP)$_w$.

The following proposition shows that right S-posets satisfying Condition (PWP)$_w$ are closed under directed colimits. For more information about directed colimits in the category POS-S the reader is referred to [2, 3].

**Proposition 2.2** Every directed colimit of a direct system of right S-posets that satisfy Condition (PWP)$_w$ satisfies Condition (PWP)$_w$.
Proof Let \((A_i, \phi_{i,j})\) be a direct system of right \(S\)-posets satisfying Condition \((PWP)_w\) over a directed index set \(I\) with directed colimit \((A, \alpha_i)\). Suppose that \(a < a'\) in \(A\). Then there exist \(a_i \in A_i\) and \(a_j \in A_j\) with \(a = \alpha_i(a_i)\), \(a' = \alpha_j(a_j)\). Since \(I\) is directed, by [2, Proposition 2.5] there exists \(k \geq i, j\) such that \(\phi_{i,k}(a_i) \leq \phi_{j,k}(a_j)\) in \(A_k\). Since \(A_k\) satisfies Condition \((PWP)_w\), there exist \(a'' \in A_k\) and \(u, v \in S\) such that \(\phi_{i,k}(a_i) \leq a''u\), \(a''v \leq \phi_{j,k}(a_j)\) and \(us \leq vs\), but then \(a = \alpha_i(a_i) = \alpha_k\phi_{i,k}(a_i) \leq \alpha_k(a'')u\). In a similar way \(\alpha_k(a'')v \leq a'\), and this implies that \(A\) satisfies Condition \((PWP)_w\).

It follows from [6] that Condition \((PWP)_w\) implies principally weakly po-flat but not conversely in general. However, for right po-cancellable pomonoids, we have the following corollary, which follows from [6] and [12, Theorems 3.21 and 3.22].

Corollary 2.3 Let \(S\) be a right po-cancellable pomonoid and \(A_S\) an \(S\)-poset. Then the following statements are equivalent:

1. \(A_S\) satisfies Condition \((P_w)\);
2. \(A_S\) satisfies Condition \((WP)_w\);
3. \(A_S\) satisfies Condition \((PWP)_w\);
4. \(A_S\) is weakly po-flat;
5. \(A_S\) is principally weakly po-flat;
6. \(A_S\) is po-torsion free.

Now we consider the right \(S\)-poset \(A(I)\) satisfying Condition \((PWP)_w\). The following definition of \(A(I)\) first appeared in [3].

Suppose \(I\) is a proper right ideal of a pomonoid \(S\). For any \(x, y, z \notin S\), let \(A(I) = (\{x\} \times (S - I)) \cup (\{y\} \times I)\). Define a right \(S\)-action on \(A(I)\) by

\[
(x, u)s = \begin{cases} (x, us), & \text{if } us \notin I, \\ (z, us), & \text{if } us \in I, \\ \end{cases}
\]

\[
y, u)s = \begin{cases} (y, us), & \text{if } us \notin I, \\ (z, us), & \text{if } us \in I, \\ \end{cases}
\]

\[
(z, u)s = (z, us).
\]

The order on \(A(I)\) is defined by

\[ (w_1, s) \leq (w_2, t) \iff (w_1 = w_2, s \leq t) \text{ or } (w_1 \neq w_2, s \leq i \leq t \text{ for some } i \in I). \]

Then \(A(I)\) is a right \(S\)-poset (see [10] for details).

Theorem 2.4 Let \(I\) be a proper right ideal of a pomonoid \(S\). Then the right \(S\)-poset \(A(I)\) satisfies Condition \((PWP)_w\) if and only if for any \(u, v, s \in S\) and \(i \in I\),

\[ us \leq i \leq vs \Rightarrow (\exists j \in I)(us \leq js \land j \leq v) \lor (js \leq vs \land u \leq j). \]
Proof Necessity: If $us \leq i \leq vs$ for any $u,v,s \in S$ and $i \in I$, then $(x,1)us \leq (y,1)vs$. There are four cases to be considered:

Case 1. $u,v \notin I$. Then $(x,u)s \leq (y,v)s$. Since $A(I)$ satisfies Condition $(PWP)_w$, there exist $u',v' \in S$ and $(w,p) \in A(I)$ such that

$$
(x,u) \leq (w,p)u', (w,p)v' \leq (y,v) \text{ and } u's \leq v's.
$$

There are three subcases:

Subcase 1. $w = x$. If $pv' \notin I$, then by (1) we have $(x,u) \leq (x,p)u'$, $(x,pv') \leq (y,v)$, and $u's \leq v's$. Hence, there exists $j \in I$ such that $u \leq pu'$ and $pv' \leq j \leq v$. Since $u's \leq v's$ implies $(pu')s \leq (pv')s$, we have $us \leq (pu')s \leq (pv')s \leq js$. If $pv' \in I$, then we can take $j = pv'$.

Subcase 2. $w = y$. If $pu' \notin I$, then by (1) we have $(x,u) \leq (y,pu')$, $(y,pv') \leq (y,v)$, and $u's \leq v's$. Hence, there exists $j \in I$ such that $u \leq j \leq pu'$ and $pu' \leq v$. Since $u's \leq v's$ implies $(pu')s \leq (pv')s$, we have $js \leq (pu')s \leq (pv')s \leq vs$. If $pu' \in I$, then we can take $j = pu'$.

Subcase 3. $w = z$. In the case, we may take $j = pu'$ or $j = pv'$.

Case 2. $u \notin I$, $v \in I$. This is analogous to case 1.

Case 3. $u \in I$, $v \notin I$. This is also analogous to case 1.

Case 4. $u \in I$, $v \in I$. Then $(z,u)s \leq (z,v)s$. Since $A(I)$ satisfies Condition $(PWP)_w$, there exist $u',v' \in S$ and $(w,p) \in A(I)$ such that $(z,u) \leq (w,p)u'$, $(w,p)v' \leq (z,v)$, and $u's \leq v's$. We have $(z,us) = (z,u)s \leq (w,p)u's \leq (w,p)v's \leq (z,v)s = (z,vs)$, so $us \leq vs$. Then we can take $j = u$ or $j = v$.

Sufficiency: Suppose that $(w_1,u),(w_2,v) \in A(I)$, and $s \in S$ are such that

$$
(w_1,u)s \leq (w_2,v)s.
$$

There are three cases to be considered:

Case 1. If $w_1 = w_2 = x$, then by (2) we have $(x,u)s \leq (x,v)s$. Hence $(x,u) \leq (x,1)u$, $(x,1)v \leq (x,v)$, and $us \leq vs$.

Case 2. If $w_1 = x$, $w_2 = y$, then by (2) we have $(x,u)s \leq (y,v)s$. By the definition of $A(I)$, there exists $i \in I$ such that $us \leq i \leq vs$. By assumption, there exists $j \in I$ such that $us \leq js$ and $j \leq v$, or $js \leq vs$ and $u \leq j$. If $us \leq js$, $j \leq v$, then $(x,1)j = (z,j) = (y,1)j \leq (y,1)v = (y,v)$ and $(x,u) \leq (x,1)u$, and if $js \leq vs$, $u \leq j$, then $(x,u) = (x,1)u \leq (x,1)j = (z,j) = (y,1)j$ and $(y,1)v \leq (y,v)$. Therefore, $A(I)$ satisfies Condition $(PWP)_w$.

Case 3. If $w_1 = x$, $w_2 = z$, then (2) means $(x,u)s \leq (z,v)s$. Hence, $(x,u) \leq (x,1)u$, $(x,1)v \leq (z,v)$, and $us \leq vs$.

The other cases can be discussed similarly and we obtain that $A(I)$ satisfies Condition $(PWP)_w$. \(\square\)

Corollary 2.5 Let $S$ be a pomonoid and $1$ the identity of $S$, in which $1$ is incomparable with every other element of $S$. Then the following conditions on pomonoids are equivalent:

1. All right $S$-posets satisfy Condition $(PWP)_w$;
2. All right $S$-posets satisfying Condition $(E)$ satisfy Condition $(PWP)_w$;
3. All finitely generated right $S$-posets satisfy Condition $(PWP)_w$.
(4) All finitely generated right $S$-posets satisfying Condition (E) satisfy Condition $(PWP)_w$;

(5) $S$ is a pogroup.

**Proof** The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are all clear.

(4) $\Rightarrow$ (5). Suppose that $I$ is a proper right ideal of a pomonoid $S$. It follows from [10, Lemma 2.2] that the right $S$-poset $A(I)$ satisfies Condition (E). By assumption, $A(I)$ satisfies Condition $(PWP)_w$. Since $i \leq i$ for every $i \in I$, we take $u = v = 1$, and by Theorem 2.4, there exists $j \in I$ such that $j \leq 1$ or $1 \leq j$. However, 1 is isolated and we obtain $j = 1$, a contradiction. Hence, $S$ has no proper right ideals, and so $S$ is a pogroup.

(5) $\Rightarrow$ (1). It is straightforward to verify. □

At the end of this section, we present an example from [12, Example] that $A(I)$ satisfies Condition $(PWP)_w$. Let $S = \{1, 0\}$ be a monoid with the usual order. We consider the ideal $I = \{0\}$. It follows from Theorem 2.4 that $A(I)$ satisfies Condition $(PWP)_w$. However, if $S = \{1, 0\}$ with the discrete order, and the ideal $I = \{0\}$, then $A(I)$ does not satisfy Condition $(PWP)_w$. This is because, taking $u = v = 1$ and $s = i = 0$ in Theorem 2.4, we have $us \leq i \leq vs$, and there does not exist $j \in I$ such that $j \leq v$, or $u \leq j$.

### 3. Cyclic (Rees factor) $S$-posets satisfying $(PWP)_w$

In this section, we will give a description of pomonoids $S$ by Condition $(PWP)_w$ of cyclic (Rees factor) $S$-posets.

A relation $\sigma$ on an $S$-poset $A_S$ is called a pseudo-order on $A_S$ if it is transitive, compatible with the $S$-action, and contains the relation $\leq$ on $A_S$. The relationship between order congruences and pseudo-orders on $A_S$ was given in [14].

Suppose that $\rho$ is a right order congruence on a pomonoid $S$. Define a relation $\widehat{\rho}$ by

$$s \widehat{\rho} t \iff [s]_{\rho} \leq [t]_{\rho} \text{ in } S/\rho.$$  

It is clear that $\widehat{\rho}$ is a pseudo-order on $A_S$. Below we will describe cyclic $S$-posets satisfying Condition $(PWP)_w$.

**Proposition 3.1** Let $\rho$ be a right order congruence on a pomonoid $S$. Then the cyclic right $S$-poset $S/\rho$ satisfies Condition $(PWP)_w$ if and only if

$$(\forall x, y, t \in S)((x)_{\rho} t \leq (y)_{\rho} t \Rightarrow (\exists u, v \in S)(ut \leq vt \land x\widehat{\rho}u \land v\widehat{\rho})).$$

**Proof** It is a routine matter. □

The following is a direct corollary of Proposition 3.1.

**Corollary 3.2** Let $S$ be any pomonoid. Then $\Theta_S$ satisfies Condition $(PWP)_w$.

To get the results for Rees factor $S$-posets we need some more preliminary material.

**Lemma 3.3** ([1, Lemma 3]) Let $K$ be a convex, proper right ideal of a pomonoid $S$. Then for $x, y \in S$,

$$[x]_{\rho_K} \leq [y]_{\rho_K} \text{ in } S/K \iff (x \leq y) \text{ or } (x \leq k \text{ and } k' \leq y \text{ for some } k, k' \in K).$$

Moreover, $[x]_{\rho_K} = [y]_{\rho_K}$ in $S/K$ if and only if either $x = y$ or else $x, y \in K$.
Recall from [1, 11] that a convex, proper right ideal $K$ of a pomonoid $S$ is strongly left stabilizing, if

$$(\forall k \in K)(\forall s \in S)(k \leq s \Rightarrow (\exists k' \in K)(k' \leq s), \text{ and } s \leq k \Rightarrow (\exists k'' \in K)(s \leq k''))).$$

The following two concepts first appeared in [6]. For convenience, we will define them as follows.

**Definition 3.4** A convex, proper right ideal $K$ of a pomonoid $S$ is called strongly left annihilating, if

$$(\forall t \in S)(\forall x, y \in S\backslash K)([x]_{\rho_K} \leq [y]_{\rho_K} \Rightarrow xt \leq yt).$$

**Definition 3.5** A convex, proper right ideal $K$ of a pomonoid $S$ is called double-strongly left annihilating (briefly, $D$-strongly left annihilating), if for every $s, t \in S\backslash K$ and homomorphism $f : s(Ss \cup St) \rightarrow sS$,

$$[f(s)]_{\rho_K} \leq [f(t)]_{\rho_K} \Rightarrow f(s) \leq f(t).$$

Every $D$-strongly left annihilating convex, proper right ideal of a pomonoid $S$ is strongly left annihilating. Indeed, if $[x]_{\rho_K} \leq [y]_{\rho_K}$ for $t \in S$ and $x, y \in S\backslash K$, then $[\rho_t(x)]_{\rho_K} \leq [\rho_t(y)]_{\rho_K}$. (If $S$ is a pomonoid and $t \in S$, then $\rho_t : S \rightarrow S$ will denote the right translation by $t$, that is, $\rho_t(s) = st$ for any $s \in S$.) This implies that if $K$ is $D$-strongly left annihilating, then $\rho_t(x) \leq \rho_t(y)$, that is, $xt \leq yt$. Hence, $K$ is strongly left annihilating. The next example from [8, Example 2] shows that not all strongly left annihilating convex, proper right ideals are $D$-strongly left annihilating.

**Example 3.6** (strongly left annihilating $\neq D$-strongly left annihilating) Let $S$ be an annihilating chain of semigroup $S_1 = \{1\}$, a right zero semigroup $S_2 = \{s, t\}$, a left zero semigroup $S_3 = \{x, y\}$, and a semigroup $S_4 = \{0\} \ (1 > 2 > 3 > 4)$. The order of $S$ is discrete. (A chain of semigroups $S_\gamma, \gamma \in \Gamma$ is called an annihilating chain if $x \in S_\alpha$ and $y \in S_\beta$, $\alpha > \beta$ implies $xy = yx = y$.) Consider the right ideal $K = \{x, y, 0\}$. If $[u]_{\rho_K} z \leq [v]_{\rho_K} z$ for $z \in S$ and $u, v \in S\backslash K$, then $uz \leq vz$, proving that $K$ is strongly left annihilating. Define a mapping $f : Ss \cup St \rightarrow S$ by $f(us) = ux$ and $f(ut) = uy$ for all $u \in S$. It is straightforward to check that $f$ is a homomorphism of left $S$-posets. Now $[f(s)]_{\rho_K} \leq [f(t)]_{\rho_K}$, but it does not imply $f(s) \leq f(t)$, so $K$ is not $D$-strongly left annihilating.

**Lemma 3.7** ([1, Propositions 10 and 13]) Let $K$ be a convex, proper right ideal of a pomonoid $S$. Then:

1. $S/K$ is principally weakly po-flat if and only if $K$ is strongly left stabilizing.
2. $S/K$ is weakly po-flat if and only if $S$ is weakly right reversible and $K$ is strongly left stabilizing.

**Lemma 3.8** ([6, Theorem 4.5, Corollary 5.7]) Let $K$ be a convex, proper right ideal of a pomonoid $S$. Then:

1. $S/K$ satisfies Condition $(PWP)$ if and only if $K$ is strongly left stabilizing and strongly left annihilating.
2. $S/K$ satisfies Condition $(WP)$ if and only if $S$ is weakly right reversible, and $K$ is strongly left stabilizing and $D$-strongly left annihilating.

For Rees factor $S$-posets satisfying Condition $(PWP)_w$, we can give the following description.
Definition 3.9 A convex, proper right ideal $K$ of a pomonoid $S$ is called $w$-strongly left annihilating, if \([x]_{\rho_K}t \leq [y]_{\rho_K}t\) for any \(x, y \in S\backslash K\) and \(t \in S\), there exist \(u, v \in S\), and \(k, k', l, l' \in K\) such that one of the following four conditions is satisfied:

(a) \(x \leq u, v \leq y\), and \(ut \leq vt\);
(b) \(x \leq u, v \leq l, l' \leq y\), and \(ut \leq vt\);
(c) \(x \leq k, k' \leq u, v \leq y\), and \(ut \leq vt\);
(d) \(x \leq k, k' \leq u, v \leq l, l' \leq y\), and \(ut \leq vt\).

By the definition, every strongly left annihilating convex, proper right ideal of a pomonoid $S$ is $w$-strongly left annihilating, but the converse is not true by the following Example 3.11.

Theorem 3.10 Let $K$ be a convex, proper right ideal of a pomonoid $S$. Then $S/K$ satisfies Condition $(PWP)_w$ if and only if

1. $K$ is strongly left stabilizing, and
2. $K$ is $w$-strongly left annihilating.

Proof Necessity: Suppose that $S/K$ satisfies Condition $(PWP)_w$. Then $S/K$ is principally weakly po-flat, so by Lemma 3.7, we have (1).

To prove (2) we suppose that \([x]_{\rho_K}t \leq [y]_{\rho_K}t\) for \(x, y \in S\backslash K\) and \(t \in S\). Since $S/K$ satisfies Condition $(PWP)_w$, by Proposition 3.1, there exist \(u, v \in S\) such that \(x\hat{\rho}_K u, v\hat{\rho}_K y\) and \(ut \leq vt\). Thus, we have \([x]_{\rho_K} \leq [u]_{\rho_K}\) and \([v]_{\rho_K} \leq [y]_{\rho_K}\). By Lemma 3.3, \([x]_{\rho_K} \leq [u]_{\rho_K}\) implies \(x \leq u\), or \(x \leq k\) and \(k' \leq u\) for \(k, k' \in K\). Similarly, \([v]_{\rho_K} \leq [y]_{\rho_K}\) implies \(v \leq y\), or \(v \leq l\) and \(l' \leq y\) for \(l, l' \in K\). Hence, we get the four possible cases of Definition 3.9, and this implies that $K$ is $w$-strongly left annihilating.

Sufficiency: Assume (1) and (2) hold. To show that $S/K$ satisfies Condition $(PWP)_w$, where $K$ is a convex, proper right ideal of the pomonoid $S$, it suffices to show that $S/K$ satisfies the conditions of Proposition 3.1. Now we suppose that \([x]_{\rho_K}t \leq [y]_{\rho_K}t\) for \(x, y, t \in S\). Then \([x]_{\rho_K} \leq [y]_{\rho_K}\). By Lemma 3.3, we have \(xt \leq yt\), or \(xt \leq k\) and \(k' \leq yt\) for \(k, k' \in K\). If \(xt \leq yt\), then it suffices in Proposition 3.1 to take \(u = x, y = v\). Otherwise, there are the following four cases:

**Case 1.** \(x, y \in K\). We can take \(u = v = x\).

**Case 2.** \(x \in K, y \notin K\). Since \(k' \leq yt\), by assumption (1) there exists \(k'' \in K\) such that \(k''yt \leq yt\), and so it suffices in Proposition 3.1 to take \(u = k''y\) and \(v = y\).

**Case 3.** \(x \notin K, y \in K\). This is analogous to Case 2.

**Case 4.** \(x, y \notin K\). By (2) of the assumption, there exist \(u, v \in S\) and \(k, k', l, l' \in K\) such that one of the conditions of Definition 3.9 holds. However, in any condition, we always have \(x\hat{\rho}_K u, v\hat{\rho}_K y\), and \(xt \leq yt\). □

The following example illustrates that Condition $(PWP)_w$ does not imply Condition $(PWP)$.

Example 3.11 $(PWP)_w \nRightarrow (PWP)$ Let $S = \{1, e, f, 0\}$ denote the monoid with the Cayley table
and suppose that the only nontrivial order relations are $e < 1$ and $0 < f$. We consider the ideal $K_S = \{e, 0\}$. Then $(S, \leq)$ is a pomonoid, and $K$ is a strongly left stabilizing and $w$-strongly left annihilating convex, proper right ideal. It follows from Theorem 3.10 that $S/K$ satisfies Condition $(PWP)_w$. On the other hand, since $1, f \in S \setminus K$ and $[1]e \leq [f]e$, but $1e \nleq fe$. Hence, $K$ is not strongly left annihilating. It follows from Lemma 3.8 that $S/K$ does not satisfy Condition $(PWP)$.

In what follows, we give the homological classification of pomonoids $S$ over which all Rees factor $S$-posets satisfying Condition $(PWP)_w$ have a certain flatness property. To do this, we require the following results.

**Lemma 3.12** ([1], Theorem 1) Let $S$ be any pomonoid. Then:

1. $\Theta_S$ satisfies Condition $(E)$ if and only if $S$ is left collapsible.
2. $\Theta_S$ satisfies Condition $(E')$ if and only if $S$ is weakly left collapsible.
3. The following statements are equivalent:
   a. $\Theta_S$ satisfies Condition $(P)$;
   b. $\Theta_S$ satisfies Condition $(WP)$ (see [6, Corollary 5.4]);
   c. $\Theta_S$ is weakly (po-)flat;
   d. $S$ is weakly right reversible.
4. $\Theta_S$ is (always) principally weakly (po-) flat and (po-)torsion free.

**Lemma 3.13** ([11, Lemma 1.8]) Let $K$ be a convex, proper right ideal of a pomonoid $S$. Then the following statements are equivalent:

1. $S/K$ is strongly flat;
2. $S/K$ satisfies Condition $(P)$;

**Theorem 3.14** For any pomonoid $S$, the following statements are equivalent:

1. $S/K$ satisfying Condition $(PWP)_w$ is weakly po-flat;
2. $S/K$ satisfying Condition $(PWP)_w$ is weakly flat;
3. $S$ is weakly right reversible.
Proof (1) ⇒ (2). It is obvious.

(2) ⇒ (3). Since $\Theta_S$ always satisfies Condition $(PWP)_w$ and, by assumption, $\Theta_S$ is weakly flat, it follows from Lemma 3.12 that $S$ is weakly right reversible.

(3) ⇒ (1). Suppose that $K$ is a convex right ideal of a pomonoid $S$ and $S/K$ satisfies Condition $(PWP)_w$. If $K$ is a proper, convex right ideal, using Theorem 3.10, $K$ is a strongly left stabilizing convex, proper right ideal, since $S$ is weakly right reversible and by Lemma 3.7, $S/K$ is weakly po-flat. However, if $K = S$ and $S$ is weakly right reversible, then by Lemma 3.12, $S/K \cong \Theta_S$ is weakly po-flat.

Note that Condition $(PWP)_w$ and weakly po-flat are independent notions. Indeed, on the one hand, if a pomonoid $S$ is not weakly right reversible, then by Theorem 3.14, there exists a Rees factor $S$-poset $S/K$ satisfying Condition $(PWP)_w$ that is not weakly po-flat. Therefore, Condition $(PWP)_w$ does not imply weakly po-flat in general. On the other hand, by [6, Example 6.3], there exists a weakly po-flat Rees factor $S$-poset that fails to satisfy Condition $(PWP)_w$.

Theorem 3.15 For any pomonoid $S$, the following statements are equivalent:

1. $S/K$ satisfying Condition $(PWP)_w$ satisfies Condition $(WP)$;
2. $S$ is weakly right reversible, and every strongly left stabilizing and $w$-strongly left annihilating convex, proper right ideal $K$ of $S$ is $D$-strongly left annihilating.

Proof (1) ⇒ (2). Since $\Theta_S$ satisfies Condition $(PWP)_w$ and by assumption, $\Theta_S$ satisfies Condition $(WP)$, from Lemma 3.8, it follows that $S$ is weakly right reversible. Let $K$ be a strongly left stabilizing and $w$-strongly left annihilating convex, proper right ideal. From Theorem 3.10, it follows that $S/K$ satisfies Condition $(PWP)_w$. By assumption, $S/K$ satisfies Condition $(WP)$ and so by Lemma 3.8, $K$ is $D$-strongly left annihilating.

(2) ⇒ (1). Let $K$ be a convex right ideal of the pomonoid $S$ and $S/K$ satisfies Condition $(PWP)_w$. If $K$ is a convex, proper right ideal, then by Theorem 3.10, $K$ is a strongly left stabilizing and $w$-strongly left annihilating convex, proper right ideal, and so by assumption, $K$ is a $D$-strongly left annihilating right ideal. Since $S$ is weakly right reversible, from Lemma 3.8, it follows that $S/K$ satisfies Condition $(WP)$. However, if $K = S$ and $S$ is weakly right reversible, then by Lemma 3.12, $S/K \cong \Theta_S$ satisfies Condition $(WP)$.

Applying Lemma 3.8 and Theorem 3.10, we can get:

Theorem 3.16 For any pomonoid $S$, the following statements are equivalent:

1. $S/K$ satisfying Condition $(PWP)_w$ satisfies Condition $(PWP)$;
2. Every strongly left stabilizing and $w$-strongly left annihilating convex, proper right ideal $K$ of $S$ is strongly left annihilating.

Theorem 3.17 For any pomonoid $S$, the following statements are equivalent:

1. $S/K$ satisfying Condition $(PWP)_w$ satisfies Condition $(P)$;
2. $S$ is weakly right reversible, and $S$ has no strongly left stabilizing and $w$-strongly left annihilating convex, proper right ideal $K$ with $|K| > 1$.
**Proof** (1) ⇒ (2). Since \( \Theta_S \) satisfies Condition \((PWP)_w\), by assumption, \( \Theta_S \) satisfies Condition \((P)\). From Lemma 3.12, we obtain that \( S \) is weakly right reversible. Assume \( S \) has a strongly left stabilizing and \( w \)-strongly left annihilating convex, proper right ideal \( K \) with \(|K| > 1\). From Theorem 3.10 it follows that \( S/K \) satisfies Condition \((PWP)_w\). By assumption, \( S/K \) satisfies Condition \((P)\), and so by Lemma 3.13, \(|K| = 1\), a contradiction is obtained.

(2) ⇒ (1). Let \( K \) be a convex right ideal of the pomonoid \( S \). Suppose that \( S/K \) satisfies Condition \((PWP)_w\). If \( K \) is a convex, proper right ideal of \( S \), it follows from Theorem 3.10 that \( K \) is a strongly left stabilizing and \( w \)-strongly left annihilating convex, proper right ideal. By assumption, \(|K| = 1\), and so \( S/K \) satisfies Condition \((P)\). However, if \( K = S \), \( S/K \cong \Theta_S \) satisfies Condition \((PWP)_w\). Since \( S \) is weakly right reversible, by Lemma 3.12, \( \Theta_S \) satisfies Condition \((P)\). □

The following example shows that Condition \((PWP)_w\) does not imply Condition \((P)\).

**Example 3.18** ([11, Example 3.22]) Let \( S \) be a left zero semigroup \( K \) with 1 adjoined and \(|K| > 1\). The order of \( S \) is discrete. It is easy to verify that \( K \) is strongly left stabilizing and \( w \)-strongly left annihilating.

It follows from Theorem 3.10 that \( S/K \) satisfies Condition \((PWP)_w\). However, by Theorem 3.17, \( S/K \) does not satisfy Condition \((P)\).

In what follows we will use the following.

**Theorem 3.19** Let \( K \) be a convex right ideal of a pomonoid \( S \). The right Rees factor \( S \)-poset \( S/K \) is weakly subpullback flat if and only if \( K = S \) is weakly right reversible and weakly left collapsible, or \(|K| = 1\).

**Proof**  
**Necessity:** If \( K = S \) and \( S/K \cong \Theta_S \) is weakly subpullback flat, then \( \Theta_S \) satisfies Conditions \((P)\) and \((E')\). From Lemma 3.12, we obtain that \( S \) is weakly right reversible and weakly left collapsible. Assume \( S \) has a convex, proper right ideal \( K \) and \( S/K \) is weakly subpullback flat. Then \( S/K \) satisfies Condition \((P)\), and so by lemma 3.13, \(|K| = 1\).

**Sufficiency:** If \( K \) is a convex, proper right ideal of the pomonoid \( S \), then by assumption, we have \(|K| = 1\) and \( S/K \cong S \) is strongly flat, and it is clear that \( S/K \) is weakly subpullback flat. However, if \( K = S \) is weakly right reversible and weakly left collapsible, then by Lemma 3.12, \( S/K \cong \Theta_S \) satisfies Conditions \((P)\) and \((E')\). Hence, \( \Theta_S \) is weakly subpullback flat. □

**Theorem 3.20** For any pomonoid \( S \), the following statements are equivalent:

1. \( S/K \) satisfying Condition \((PWP)_w\) is weakly subpullback flat;

2. \( S \) is weakly right reversible and weakly left collapsible, and \( S \) has no strongly left stabilizing and \( w \)-strongly left annihilating convex, proper right ideal \( K \) with \(|K| > 1\).

**Proof** (1) ⇒ (2). Since \( \Theta_S \) satisfies Condition \((PWP)_w\), by assumption, \( \Theta_S \) is weakly subpullback flat. Applying Theorem 3.19, we obtain that \( S \) is weakly right reversible and weakly left collapsible. Assume \( S \) has a strongly left stabilizing and \( w \)-strongly left annihilating convex, proper right ideal \( K \) with \(|K| > 1\). From Theorem 3.10, it follows that \( S/K \) satisfies Condition \((PWP)_w\). By assumption, \( S/K \) is weakly subpullback flat, and so by Theorem 3.19, \(|K| = 1\), a contradiction.
(2) ⇒ (1). Suppose that $K$ is a convex right ideal of the pomonoid $S$ and $S/K$ satisfies Condition $(PWP)_w$. If $K$ is a convex, proper right ideal of $S$, by Theorem 3.10, $K$ is a strongly left stabilizing and $w$-strongly left annihilating convex, proper right ideal. By assumption, $|K| = 1$, and so $S/K \cong S$ is strongly flat. Clearly, $S/K$ is weakly subpullback flat. However, if $K = S$, $S/K \cong \Theta_S$ satisfies Condition $(PWP)_w$. Since $S$ is weakly right reversible and weakly left collapsible, by Lemma 3.12, $\Theta_S$ satisfies Conditions $(P)$ and $(E')$. Hence, $\Theta_S$ is weakly subpullback flat.

**Theorem 3.21** For any pomonoid $S$, the following statements are equivalent:

1. $S/K$ satisfying Condition $(PWP)_w$ is strongly flat;
2. $S$ is left collapsible, and $S$ has no strongly left stabilizing and $w$-strongly left annihilating convex, proper right ideal $K$ with $|K| > 1$.

**Proof** (1) ⇒ (2). Since $\Theta_S$ satisfies Condition $(PWP)_w$ and by assumption, $\Theta_S$ is strongly flat, thus $\Theta_S$ satisfies Condition $(E)$. Using Lemma 3.12, $S$ is left collapsible. Assume $S$ has a strongly left stabilizing and $w$-strongly left annihilating convex, proper right ideal $K$ with $|K| > 1$. By Theorem 3.10, $S/K$ satisfies Condition $(PWP)_w$, so by assumption, $S/K$ is strongly flat, and by Lemma 3.13, $|K| = 1$, a contradiction is obtained.

(2) ⇒ (1). Let $K$ be a convex right ideal of the pomonoid $S$ and $S/K$ satisfies Condition $(PWP)_w$. If $K$ is a convex, proper right ideal, by Theorem 3.10, $K$ is a strongly left stabilizing and $w$-strongly left annihilating convex, proper right ideal. By assumption, $|K| = 1$, and so $S/K \cong S$ is strongly flat. However, if $K = S$, $S/K \cong \Theta_S$ satisfies Condition $(PWP)_w$. Since $S$ is left collapsible, by Lemma 3.12, $\Theta_S$ satisfies Condition $(E)$. Hence, $\Theta_S$ is strongly flat.

**Theorem 3.22** For any pomonoid $S$, the following statements are equivalent:

1. $S/K$ satisfying Condition $(PWP)_w$ is projective;
2. $S$ has a left zero element, and $S$ has no strongly left stabilizing and $w$-strongly left annihilating convex, proper right ideal $K$ with $|K| > 1$.

**Proof** It is similar to that of Theorem 3.21.

**Theorem 3.23** For any pomonoid $S$, the following statements are equivalent:

1. $S/K$ satisfying Condition $(PWP)_w$ is free;
2. $|S| = 1$.

**Proof** It can be easily proved.

**Example 3.24** Let $S$ be a pogroup and $|S| > 1$. Then the Rees factor $S$-poset $\Theta_S$ satisfies Condition $(PWP)_w$, but, by Theorem 3.23, $\Theta_S$ is not free.
4. Direct products of $S$-posets satisfying Condition $(PWP)_w$

In this section, we are going to discuss direct products of any arbitrary nonempty family of $S$-posets satisfying Condition $(PWP)_w$.

If $S$ is a pomonoid, the Cartesian product $S^I$ is a right and left $S$-poset equipped with the order and the action componentwise where $I$ is a nonempty set. Moreover, $(s_i)_{i \in I} \in S^I$ is denoted simply by $(s_i)$, and the right $S$-poset $S \times S$ is called the diagonal right $S$-poset of $S$, usually denoted $D(S)$. (For more information the reader is referred to [7]).

According to [7], the set $L(s,s) := \{(u,v) \in D(S) | us \leq vs\}$ is a left $S$-subposet of $D(S)$. Moreover, for each $(p,q) \in D(S)$, the set $\overline{S(p,q)} := \{(u,v) \in D(S) | u \leq wp \text{ and } wq \leq v \text{ for some } w \in S\}$ is a left $S$-poset. Clearly, $\overline{S(p,q)}$ contains the cyclic $S$-poset $S(p,q)$.

**Theorem 4.1** Let $S$ be a pomonoid. Then the following statements are equivalent:

1. Any finite product of right $S$-posets satisfying Condition $(PWP)_w$ satisfies Condition $(PWP)_w$;
2. The diagonal right $S$-poset $D(S)$ satisfies Condition $(PWP)_w$;
3. For every $s \in S$, the set $L(s,s)$ is either empty or for each 2 elements $(u,v), (u',v') \in L(s,s)$, there exists $(p,q) \in L(s,s)$ such that $(u,v), (u',v') \in \overline{S(p,q)}$.

**Proof** (1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3). Suppose that $D(S)$ satisfies Condition $(PWP)_w$. Let $(u,v), (u',v') \in L(s,s)$ for any $s \in S$. From the inequalities $us \leq vs$ and $u's \leq v's$ we obtain $(u,u')s \leq (v,v')s$. Since $D(S)$ satisfies Condition $(PWP)_w$, there exist $(w,w') \in D(S)$ and $p,q \in S$ such that $(u,u') \leq (w,w')p$, $(w,w')q \leq (v,v')$, and $ps \leq qs$. Thus, we have $(p,q) \in L(s,s)$ and we are done.

(3) $\Rightarrow$ (1). Suppose that $A_1, \ldots, A_n$ are right $S$-posets satisfying Condition $(PWP)_w$. Suppose $a_i, a'_i \in A_i$ for each $1 \leq i \leq n$, and let $s \in S$ be such that $(a_1, \ldots, a_n)S \leq (a'_1, \ldots, a'_n)S$ in $A = \prod_{i=1}^n A_i$. For every $A_i$, applying Condition $(PWP)_w$ to the inequalities $a_iS \leq a'_iS$ ($1 \leq i \leq n$), we get $a''_i \in A_i$ and $p_i, q_i \in S$ such that $a_i \leq a''_ip_i$, $a'_i q_i \leq a'_i$ and $p_iS \leq q_iS$. Then $(p_i, q_i) \in L(s,s)$ for each $i$, and so by assumption, there exists $(p,q) \in L(s,s)$ such that $(p_i,q_i) \in L(p,q)$ for each $i$. Thus, $p_i \leq w_i p$ and $w_i q \leq q_i$ for some $w_i \in S$ ($1 \leq i \leq n$). Thus, we calculate that $(a_1, \ldots, a_n) \leq (a''_1 w_1, \ldots, a''_n w_n)p$, $(a''_1 w_1, \ldots, a''_n w_n)q \leq (a'_1, \ldots, a'_n)$, and $ps \leq qs$, proving that $A = \prod_{i=1}^n A_i$ satisfies Condition $(PWP)_w$. \[\square\]

For a right po-cancellative pomonoid, Theorem 4.1 yields the following.

**Corollary 4.2** If the pomonoid $S$ is right po-cancellative, then the diagonal right $S$-poset $D(S)$ satisfies Condition $(PWP)_w$.

As an extension of Theorem 4.1, the following result is obtained.

**Theorem 4.3** Let $S$ be a pomonoid. Then the following statements are equivalent:

1. The direct product of every nonempty family of right $S$-posets satisfying Condition $(PWP)_w$ satisfies Condition $(PWP)_w$;

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(2) \((S^I)_S\) satisfies Condition \((PWP)_w\) for every nonempty set \(I\);

(3) For every \(s \in S\), the set \(L(s, s)\) is either empty or there exists \((p, q) \in L(s, s)\) such that \(L(s, s) = \overline{S(p, q)}\).

**Proof**  
(1) \(\Rightarrow\) (2) It is obvious.

(2) \(\Rightarrow\) (3). Let \(s \in S\) and \(L(s, s) \neq \emptyset\). Write \(L(s, s) = \{(u_i, v_i) \mid i \in I\}\). Let \(u\) and \(v\) be the elements of \(S^I\) whose \(i\)th components are \(u_i\) and \(v_i\), respectively. Then we get \(us \leq vs\) in \(S^I\). Since \(S^I\) satisfies Condition \((PWP)_w\), we have that \(u \leq wp\), \(wq \leq v\) and \(ps \leq qs\) for some \(p, q \in S\) and \(w \in S^I\). Then \((p, q) \in L(s, s)\), and for each \(i \in I\) we have \(u_i \leq w_i p\), \(w_i q \leq v_i\) where \(w_i\) is the \(i\)th component of \(w\). Thus, we have \(L(s, s) = \overline{S(p, q)}\), as desired.

(3) \(\Rightarrow\) (1). Let \(A = \prod_{j \in J} A_j\) be a direct product of right \(S\)-posets satisfying Condition \((PWP)_w\). Suppose that \(s \in S\), and \(a = (a_j), b = (b_j) \in A\) are such that \(as \leq bs\). Then we have \(a_j s \leq b_j s\) for each \(j \in J\). Since \(A_j\) satisfies Condition \((PWP)_w\), there are elements \(u_j, v_j \in S\) and \(c_j \in A_j\) with \(a_j \leq c_j u_j\), \(c_j v_j \leq b_j\), and \(u_j s \leq v_j s\). Therefore, \((u_j, v_j) \in L(s, s) \neq \emptyset\) and by assumption there exists \((p, q) \in L(s, s)\) such that \(L(s, s) = \overline{S(p, q)}\). Then for each \((u_j, v_j) \in L(s, s)\) there exists \(w_j \in S\) with \(u_j \leq w_j p\) and \(w_j q \leq v_j\). Thus, \(ps \leq qs\), and for each \(j \in J\) we can calculate that \(a_j \leq c_j w_j p\) and \(c_j w_j q \leq b_j\). Taking \(a' = (c_j w_j)_{j \in J} \in A\), we have \(a \leq a'p\) and \(a'q \leq b\), as required.

Note that the fact that not every pomonoid \(S\) has a diagonal \(S\)-poset \(D(S)\) satisfying Condition \((PWP)_w\) is shown by the following example.

**Example 4.4** Let \(S = \{0, x, 1 \mid x^2 = 0\}\) be a monoid with the nontrivial order relations \(0 < x < 1\). Then \(S\) is a pomonoid, and the diagonal \(S\)-poset \(D(S)\) does not satisfy Condition \((PWP)_w\).

**Proof** It is clear that \(S\) is a pomonoid. We use Theorem 4.1 to check that \(D(S)\) fails to satisfy Condition \((PWP)_w\). Note that \((1, 1), (x, 0) \in L(x, x)\) for \(x \in S\). However, there is no element \((p, q) \in L(x, x)\) such that \((1, 1), (x, 0) \in \overline{S(p, q)}\).

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