Moduli spaces of arrangements of 11 projective lines with a quintuple point

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Abstract: In this paper, we try to classify moduli spaces of arrangements of 11 lines with quintuple points. We show that moduli spaces of arrangements of 11 lines with quintuple points can consist of more than 2 connected components. We also present defining equations of the arrangements whose moduli spaces are not irreducible after taking quotients by the complex conjugation by Maple and supply some “potential Zariski pairs”.

Key words: Line arrangements, moduli spaces, irreducibility

1. Introduction

Let \( A = \{H_1, H_2, \ldots, H_n\} \) be a line arrangement in the complex projective plane \( \mathbb{CP}^2 \), and denote by \( M(A) \) the corresponding complement of the arrangement.

An essential topic in hyperplane arrangement theory is to study the intersection between topology of complements and combinatorics of intersection lattices. It is important to study how closely topology and combinatorics of a given arrangement are related. For line arrangements, Jiang and Yau \cite{JY} showed that homeomorphism of the complement always implies lattice isomorphism. However, the converse is not true in general for line arrangements. In \cite{JL} and \cite{LW}, the authors found a large class of line arrangements whose intersection lattices determine topology of the complements, called nice arrangements and simple arrangements respectively. The notion of nice line arrangements has been generalized to arrangements of hyperplanes in higher dimensional projective spaces (see \cite{LY, NS, OS}).

We call a pair of line arrangements a Zariski pair if they are lattice isomorphic, but the fundamental groups of their complements are different. The first Zariski pair of line arrangements was constructed by Rybnikov \cite{Ry}. On the other hand, combining the results of Fan \cite{Fa}, Garber et al. \cite{Ga} proved that there is no Zariski pair of arrangements of up to 8 real lines. This result was recently generalized to arrangements of 8 complex lines by Nazir and Yoshinaga \cite{NY}. In the same paper, Nazir and Yoshinaga also claimed that there is no Zariski pair of arrangements of 9 complex lines. A complete proof of their claim was presented in \cite{NY}. Recently, Amram et al. classified arrangements of 10 complex lines in \cite{AC, AM} and found some “potential Zariski pairs”.

Let \( A \) be a complex line arrangement. We define the moduli space of line arrangements with the fixed lattice \( L(A) \) (or simply, the moduli space of \( A \)) as

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\[ M_A = \{ \mathcal{B} \in ((\mathbb{CP}^2)^{n}|\mathcal{B} \sim \mathcal{A})]/PGL_c(2) \],

where \( \mathcal{B} \sim \mathcal{A} \) means \( \mathcal{B} \) and \( \mathcal{A} \) are lattice isomorphic. We denote by \( M_{\overline{A}} \) the quotient of \( M_A \) under the complex conjugation. By Randell’s lattice-isotopy theorem in [10] and Cohen and Suciu’s theorem [3, Theorem 3.9], we know that arrangements in the same connected component of the moduli space, or in two complex conjugate components, cannot form Zariski pairs. Therefore, to investigate the existence of Zariski pairs of arrangements of 11 lines, it is very important to know the geometry of moduli spaces of arrangements. In this paper, we try to classify the moduli spaces of arrangements of 11 lines with quintuple points, and in particular we completely classify the arrangements of 11 lines with a quintuple point and at least one quadruple point. On this basis, we give forty new “potential Zariski pairs” of arrangements of 11 lines.

The classification of moduli spaces consists of three steps. First, we will roughly classify intersection lattices according to the number of multiple intersection points. Second, we divide our classification into some different cases according to different positions between quintuple point and the other multiple intersection points. Third, we will write down defining equations involving parameters for a given intersection lattice.

This paper is structured as follows. Section 2 provides preliminaries and ideas on classifying moduli spaces of arrangements of 11 lines. Section 3 shows that moduli spaces of arrangements with multiple points of high multiplicity are most likely irreducible. In Section 4 and Section 5, we completely classify the arrangements of 11 lines with a quintuple point and at least one quadruple point. In Section 6 we deal with the arrangements of 11 lines with a quintuple point and no quadruple point. Sections 4, 5, and 6 are the main parts of this work and in total forty “potential Zariski pairs” can be found there. In the Appendix (on the journal’s website), we give an example to show how to compute the defining equations of the arrangements by Maple.

### 2. Preliminaries

Let \( \mathcal{A} = \{ L_1, L_2, \cdots, L_n \} \) be a line arrangement in \( \mathbb{CP}^2 \). We say a singularity of \( L_1 \cup L_2 \cup \cdots \cup L_n \) is a multiple point of \( \mathcal{A} \) if it has multiplicity of at least 3. We call the set \( L(\mathcal{A}) = \{ \bigcap_{i \in S} L_i | S \subseteq \{1, 2, \ldots, n\} \} \) partially ordered by reverse inclusion in the intersection lattice of \( \mathcal{A} \).

**Definition 2.1** Two line arrangements \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are lattice isomorphic, denoted as \( \mathcal{A}_1 \sim \mathcal{A}_2 \), if their intersection lattices \( L(\mathcal{A}_1) \) and \( L(\mathcal{A}_2) \) are isomorphic, i.e. there is a permutation \( \phi \) of \( \{1, 2, \ldots, n\} \) such that

\[
\dim \left( \bigcap_{i \in S} L_i \right) = \dim \left( \bigcap_{j \in \phi(S)} H_j \right)
\]

for any nonempty subset \( S \subseteq \{1, 2, \ldots, n\} \).

**Definition 2.2** ([9, Definition 3.10]) Let \( k \in \mathbb{N} \). We say that a line arrangement \( \mathcal{A} \) is of type \( C_k \) if \( k \) is the minimum number of lines in \( \mathcal{A} \) containing all points of multiplicity of at least three.

**Definition 2.3** ([9, Definition 3.13]) Let \( \mathcal{A} \) be an line arrangement of type \( C_3 \). Then \( \mathcal{A} \) is a simple \( C_3 \) arrangement if there are three lines \( L_1, L_2, L_3 \in \mathcal{A} \) such that all points of multiplicity of at least three are contained in \( L_1 \cup L_2 \cup L_3 \) and one of the following holds:

1. \( L_1 \cap L_2 \cap L_3 \neq \emptyset \), or
2. \( L_1 \cap L_2 \cap L_3 = \emptyset \) and one of \( L_1, L_2, \) and \( L_3 \) contains only one multiple point apart from the other two lines.

**Theorem 2.4** ([9, Theorem 3.15]) Let \( \mathcal{A} \) be an arrangement of \( C_3 \) of simple type. Then the moduli space \( \mathcal{M}_A \) is irreducible.

**Theorem 2.5** ([9, Lemma 3.2]) Let \( \mathcal{A} = \{L_1, L_2, \ldots, L_n\} \) be a line arrangement. Assume that \( L_n \) passes through at most 2 multiple points. Set \( \mathcal{A}' = \{L_1, L_2, \ldots, L_{n-1}\} \), and then \( \mathcal{M}_A \) is irreducible if \( \mathcal{M}_{A'} \) is irreducible.

We say that a line arrangement is **nonreductive** if each line of the arrangement passes through at least 3 multiple points. Otherwise, we say the arrangement is **reductive**.

Denote by \( n_r \) the number of intersection points of multiplicity \( r \). We recall the following useful results.

**Lemma 2.6** (See for instance [6].) Let \( \mathcal{A} \) be an arrangement of \( k \) lines in \( \mathbb{CP}^2 \). Then
\[
\frac{k(k-1)}{2} = \sum_{r \geq 2} \frac{r(r-1)n_r}{2}.
\]

**Theorem 2.7** (See [6].) Let \( \mathcal{A} \) be an arrangement of \( k \) lines in \( \mathbb{CP}^2 \). Assume that \( n_k = n_{k-1} = n_{k-2} = 0 \). Then
\[
n_2 + \frac{3}{4}n_3 \geq k + \sum_{r \geq 5} (2r-9)n_r.
\]

The following lemma is well known and is used to facilitate the calculation in our paper.

**Lemma 2.8** Let \( \{L_1, L_2, L_3, L_4\} \) and \( \{L_5, L_6, L_7\} \) be two pencils of lines who intersect at one point and intersect transversally in 12 points. Then there is an automorphism of the dual projective plane such that the 7 lines under the automorphism are defined by \( Y = Z, Y = t_3Z, Y = t_2Z, Y = 0, X = 0, X = t_1Z, X = Z \).

**Remark 2.9** All the computations in Sections 4 and 5 are based on Lemma 2.8 above. First, we let \( L_1, \ldots, L_7 \) be as in Lemma 2.8 and let \( L_{11} \) be the line at infinity. Second, by the intersection points we obtain the defining equations of \( L_8, L_9, L_{10} \), and by the conditions of slope, parallel, and intersection points, we get the equations on the coefficients \( t_1, t_2, t_3 \). Third, using Maple, it is easy to get the solutions of \( t_1, t_2, t_3 \), and the defining equations of the arrangements or the arrangements cannot be realized. In Section 6, similarly as in Lemma 2.8, we can establish similar vertical nets and the methods of computing the defining equations of the arrangements is the same as in the above three steps.

### 3. Arrangements of 11 lines with multiple points of multiplicity at least 6

**Theorem 3.1** Let \( \mathcal{A} \) be an arrangement of \( n \) \((n \geq 9)\) lines. If there is a multiple point of multiplicity \( \geq n-4 \), then the moduli space \( \mathcal{M}_A \) is irreducible.

**Proof** For \( n = 9, 10 \), it was proved in [16, Prop 3.3] and [2, Theorem 3.1]. Now we consider \( n \geq 11 \). Assume that \( L_1 \cap L_2 \cap \cdots \cap L_{n-4} \neq \emptyset \). It is easy to see that at least one of the \( n-4 \) lines contains at most 2 multiple points. By Theorem 2.5 and [2, Theorem 3.1], we see that \( \mathcal{M}_A \) is irreducible.

In particular, if \( n = 11 \) and \( n_7 \geq 1 \), then \( \mathcal{M}_A \) is irreducible.
We may assume \( L \) has 1 point and 2 quadruple points. There must be another triple point on \( L \). Therefore, \( n \) points are not collinear. Let \( L \) be a reductive arrangement. If \( r \) lines pass through 4 points of \( L \) and \( n \) is a line with at most 2 multiple points. If \( \exists L \) lines with a multiple point of multiplicity 6 and no multiple points of higher multiplicities; then \( L \) is reductive.

**Proof** Assume that \( L \) is nonreductive, and then by Lemma 2.6 and Theorem 2.7 we have

\[
44 - 18n_6 \geq \frac{9}{4}(n_3 + n_4 + n_5).
\]

On the other hand, it is easy to see that there must be at least \( 13 - n_6 \) multiple points of multiplicity \( \leq 5 \). Thus, \( 13 - n_6 \leq n_3 + n_4 + n_5 \). Together with (1), we get \( n_6 \leq \frac{32}{15} < 1 \), a contradiction.

4. Arrangements of 11 lines with a quintuple point and 2 quadruple points

In this section, we investigate arrangements of 11 lines with a quintuple point and no multiple points of higher multiplicities.

First, we show the possible values of the numerical invariants \( n_4 \), \( n_5 \) such that the arrangement is nonreductive.

**Lemma 4.1** Let \( L \) be a nonreductive arrangement of 11 lines in \( \mathbb{CP}^2 \) with a quintuple point and \( n_r = 0 \) for \( r \geq 6 \). Then \( n_5 = 1 \) and \( n_4 \leq 2 \).

**Proof** By Lemma 2.6 and Theorem 2.7, we have \( n_4 + n_3 \geq \frac{4}{3}(44 - 11n_5) \). On the other hand, it is easy to see that there must be at least \( 11 - n_5 \) multiple points of multiplicity \( \leq 5 \). Thus, \( 11 - n_5 \leq \frac{4}{3}(44 - 11n_5) \). It follows that \( n_5 \leq 2 \). If \( n_5 = 2 \) and these 2 quintuple points are not collinear, then it is easy to see that there is a line with at most 2 multiple points. If \( n_5 = 2 \) and these 2 quintuple points are collinear, let \( L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_{11} \) and \( L_5 \cap L_6 \cap L_7 \cap L_8 \cap L_{11} \) be 2 quintuple points, and let \( L_{11} \) be the line at infinity. Each of \( L_9 \) and \( L_{10} \) passes through 4 points of \( L_i \cap L_j \), \( i = 1, 2, 3, 4; j = 5, 6, 7, 8 \). Assume that \( L_9 \) passes through \( L_1 \cap L_8, L_2 \cap L_7, L_3 \cap L_6, L_4 \cap L_5 \), and then to make the arrangement nonreductive, \( L_{10} \) should pass through \( L_1 \cap L_6, L_2 \cap L_5, L_3 \cap L_8, L_4 \cap L_7 \) and \( L_9 \cap L_{10} \) is on \( L_{11} \). After an easy computation, such an arrangement can not be realized. Therefore, \( n_5 = 1 \).

Also by Lemma 2.6 and Theorem 2.7, we obtain \( \frac{9}{4}n_3 + 6n_4 \leq 33 \). Since each line contains at least 3 multiple points, then there must be at least \( 11 - n_5 = 10 \) multiple points. It follows that \( n_4 \leq \frac{42}{15} \), and thus \( n_4 \leq 2 \). □

**Theorem 4.2** Let \( L \) be a nonreductive arrangement of 11 lines in \( \mathbb{CP}^2 \) with a quintuple point such that \( n_4 = 2 \) and \( n_r = 0 \) for \( r \geq 6 \). Then the moduli space \( \mathcal{M}_L \) is irreducible.

**Proof** First, we assume that the quintuple point and a quadruple point are not collinear in \( L \). We show that there is a line containing only 2 multiple points. Let \( L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5 \) be the quintuple point and let \( L_6 \cap L_7 \cap L_8 \cap L_9 \) be the quadruple point. Then the other quadruple point must be \( L_i \cap L_j \cap L_{10} \cap L_{11} \) for some \( i \in 1, 2, 3, 4, 5; j \in 6, 7, 8, 9 \). Then \( L_i \) passes through at most 2 multiple points. If there are two noncollinear quadruple points, each one being collinear with the quintuple point, it is easy to see that the arrangement is reductive.

Second, we consider that any 2 of the quintuple points and 2 quadruple points are collinear, but all of them are not collinear. Let \( L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_{11} \), \( L_5 \cap L_6 \cap L_7 \cap L_{11} \) and \( L_2 \cap L_4 \cap L_9 \cap L_{10} \) be the quintuple point and 2 quadruple points. There must be another triple point on \( L_{11} \) so that it contains 3 multiple points. We may assume \( L_8 \cap L_{10} \cap L_{11} \) is the triple point.
Case 1. $L_8 \cap L_9$ is not a triple point. It is easy to see that $L_9 \cup L_{10}$ pass through at most 4 points of $\Delta := \{L_i \cap L_j, \ i = 1, 2, 3, 4; \ j = 5, 6, 7\}$ except $L_2 \cap L_6$. Thus, $L_8$ passes through 3 points of $\Delta$ and $L_9 \cup L_{10}$ pass through 5 points of $\Delta$ to make the arrangement nonreductive. Up to a lattice isomorphism, we may assume that $L_8$ passes through $\{L_2 \cap L_5, L_3 \cap L_6, L_4 \cap L_7\}$. Then $L_1 \cap L_5$ and $L_3 \cap L_7$ are on $L_9$ or $L_{10}$. Up to a permutation, we can assume they are on $L_9$, and then $L_1 \cap L_7$ and $L_4 \cap L_5$ are on $L_{10}$ (see Figure 1).

After an easy computation, we see that Figure 1 cannot be realized.

Case 2. $L_8 \cap L_9$ is a triple point. We assume that $L_8 \cap L_9$ is on $L_1$, and up to a lattice isomorphism, we assume $L_1 \cap L_7$ is on $L_{10}$ so that $L_1$ contains 3 multiple points. Note that $L_2 \cap L_5$ or $L_2 \cap L_7$ is on $L_8$ so that $L_2$ contains 3 multiple points and $L_3 \cap L_6$ or $L_4 \cap L_6$ is on $L_8$ so that $L_6$ contains 3 multiple points.

Subcase 1. $L_2 \cap L_5$ and $L_3 \cap L_6$ are on $L_8$.

(I). $L_4 \cap L_7$ is on $L_8$, so then $L_4 \cap L_5$ is on $L_9$ or $L_{10}$ so that $L_4$ passes through 3 multiple points.

Ω. $L_4 \cap L_5$ is on $L_9$, so then $L_3 \cap L_5$ is on $L_{10}$ (Figure 2) or $L_3 \cap L_7$ is on $L_9$ (Figure 3).

ΩΩ. $L_4 \cap L_5$ is on $L_{10}$, so then $L_3 \cap L_5$ (Figure 4) or $L_3 \cap L_7$ is on $L_9$ (Figure 5).

(II). $L_4 \cap L_7$ is on $L_9$, so then $L_4 \cap L_5$ is on $L_{10}$ and $L_3 \cap L_5$ is on $L_9$ so that $L_3, L_4$ pass through 3 multiple points (Figure 6).

After an easy computation, we see that Figures 2, 3, 4, 5, and 6 cannot be realized.

Subcase 2. $L_2 \cap L_7$ and $L_3 \cap L_6$ are on $L_8$. If $L_4 \cap L_5$ is on $L_9$, then $L_4$ contains only 2 multiple points, and thus $L_4 \cap L_5$ is on $L_8$ or $L_{10}$.

(I). $L_4 \cap L_5$ is on $L_8$, so then to make the arrangement nonreductive, $L_4 \cap L_7$ is on $L_9$ and $L_3 \cap L_5$ is on $L_9$ or $L_{10}$ (Figure 7).

(II). $L_4 \cap L_5$ is on $L_{10}$, so then to make the arrangement nonreductive, $L_4 \cap L_7$ and $L_3 \cap L_5$ are on $L_9$ (Figure 8).
After an easy computation, we see that Figures 7 and 8 cannot be realized.

Assume that the quintuple point and 2 quadruple points are collinear in $\mathcal{A}$. We assume that $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_{11}$, $L_5 \cap L_6 \cap L_7 \cap L_{11}$, and $L_8 \cap L_9 \cap L_{10} \cap L_{11}$ are the quintuple point and 2 quadruple points. It is easy to see that $L_8$, $L_9$, $L_{10}$ pass through 8 or 9 points of $\Delta$ so that $L_1$, $L_2$, $L_3$, $L_4$ contain at least 3 multiple points each.

If $L_8$, $L_9$, $L_{10}$ pass through 9 points of $\Delta$, then each of $L_5$, $L_6$, $L_7$, $L_8$, $L_9$, $L_{10}$ contains 4 multiple points. Then $\mathcal{A} = \mathcal{A}\setminus \{L_i\}$, $i \in \{1, 2, 3, 4\}$ is a line arrangement of 10 lines with 3 quadruple points, which are collinear in $\mathcal{A}$. By the last paragraph in the proof [2, Theorem 4.2], $\mathcal{M}_{\mathcal{A}}$ is irreducible. Hence, $\mathcal{M}_{\Delta}$ is irreducible.

If $L_8$, $L_9$, $L_{10}$ pass through 8 points of $\Delta$, then each of $L_1$, $L_2$, $L_3$, $L_4$ passes through 2 triple points. We assume that 8 points of $\Delta$ are $L_2 \cap L_5$, $L_3 \cap L_5$, $L_4 \cap L_5$, $L_1 \cap L_6$, $L_4 \cap L_6$, $L_1 \cap L_7$, $L_2 \cap L_7$, $L_3 \cap L_7$. Furthermore, we assume that $L_1 \cap L_6$ is on $L_8$ and $L_4 \cap L_6$ is on $L_9$.

(I). $L_2 \cap L_5$ is on $L_8$, so then we see that $L_3 \cap L_7$ is on $L_8$, $L_3 \cap L_5$ is on $L_9$, and $L_4 \cap L_5$ is on $L_{10}$.

① $L_1 \cap L_7$ is on $L_9$ and $L_2 \cap L_7$ is on $L_{10}$ (Figure 9).

② $L_1 \cap L_7$ is on $L_{10}$ and $L_2 \cap L_7$ is on $L_9$ (Figure 10).

After an easy computation, we conclude that Figures 9 and 10 cannot be realized.

(II). $L_3 \cap L_5$ is on $L_8$, so then we see that $L_2 \cap L_7$ is on $L_8$, $L_2 \cap L_5$ is on $L_9$, and $L_4 \cap L_5$ is on $L_{10}$. Exchanging $L_2$, $L_3$, we see it is lattice isomorphic to (I).

(III). $L_4 \cap L_5$ is on $L_8$, and up to a lattice isomorphism, we assume that $L_3 \cap L_5$ is on $L_9$, so then $L_2 \cap L_5$ is on $L_{10}$ (Figures 11 and 12).

① $L_1 \cap L_7$ is on $L_9$, so then $L_2 \cap L_7$ is on $L_8$, $L_3 \cap L_7$ is on $L_{10}$.

② $L_2 \cap L_7$ is on $L_9$, so then $L_3 \cap L_7$ is on $L_8$, $L_1 \cap L_7$ is on $L_{10}$.
After an easy computation, we conclude that Figures 11 and 12 cannot be realized.

Thus, $M_A$ is irreducible.

5. Arrangements of 11 lines with a quintuple point and exactly 1 quadruple point

In this section, we investigate an arrangement of 11 lines with a quintuple point and exactly 1 quadruple point.

**Lemma 5.1** Let $\mathcal{A}$ be a nonreductive arrangement of 11 lines in $\mathbb{CP}^2$ with $n_5 = n_4 = 1$ and $n_r = 0$ for $r \geq 6$. If the quintuple point and the quadruple point are not collinear, then $M_A$ is empty.

**Proof** Assume that $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5$ is the quintuple point and $L_6 \cap L_7 \cap L_8 \cap L_9$ is the quadruple point. Since $L_{10}$ and $L_{11}$ pass through at most 8 triple points of $\{L_6 \cup L_7 \cup L_8 \cup L_9\}$, then one of $\{L_1, L_2, L_3, L_4, L_5\}$ contains at most 2 multiple points, and then $\mathcal{A}$ is reductive, contradiction. Then $M_A$ is empty.

In the following theorem, we assume that the quintuple point and the quadruple point are collinear.

**Theorem 5.2** Let $\mathcal{A}$ be a nonreductive arrangement of 11 lines in $\mathbb{CP}^2$ with $n_5 = n_4 = 1$ and $n_r = 0$ for $r \geq 6$. If the quintuple point and the quadruple point are collinear, then $M_A$ or $M_A'$ is irreducible except in the cases of Figures 14, 15, 17, 26, 27, 30, 31, 32, 33, 35, 37, 46, 55, 56, 57, 58, and 60 and the corresponding arrangements of these figures are “potential Zariski pairs”.

**Proof** Assume that $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_{11}$ is the quintuple point and $L_5 \cap L_6 \cap L_7 \cap L_{11}$ is the quadruple point. Then one of $\{L_8 \cap L_9, L_8 \cap L_{10}, L_9 \cap L_{10}\}$ is on $L_{11}$ so that it contains at least 3 multiple points. We may assume $L_8 \cap L_9$ is on $L_{11}$.

**Case 1.** Neither of $L_8 \cap L_{10}$ and $L_9 \cap L_{10}$ is a triple point. Then $L_{10}$ must pass through at 3 points of $\Delta$. Then $L_8, L_9$ must pass through at least 5 points of $\Delta$ so that $L_1, L_2, L_3, L_4$ contains at least 3 multiple points.

**Subcase 1.** Both $L_8$ and $L_9$ pass through 3 points of $\Delta$. Let $L_4$ be the line such that $L_4 \cap L_{10}$ is not a triple point and let $\mathcal{A}' = \mathcal{A} \setminus \{L_4\}$. Then $\mathcal{A}'$ is an arrangement of 10 lines with 2 quadruple points on the same line and none of $L_{10} \cap (L_8 \cup L_9)$ is a triple point, it is just [2, Theorem 4.4, Case 1]. Then $M_{A'}$ is either empty or irreducible, and then $M_A$ is either empty or irreducible.

**Subcase 2.** One of $L_8, L_9$ passes through 2 points of $\Delta$. We assume that $L_8$ passes through 2 points of $\Delta$. Up to a lattice isomorphism, we assume that $\{L_1 \cap L_7, L_2 \cap L_6, L_3 \cap L_5\}$ are on $L_{10}$. To make $L_4$ pass
through at least 3 multiple points, $L_8, L_9$ must pass through 2 points of $\{L_4 \cap L_5, L_4 \cap L_6, L_4 \cap L_7\}$. Up to a permutation, let $L_8$ contain $L_4 \cap L_6$ and let $L_9$ contain $L_4 \cap L_5$.

(I). $L_1 \cap L_5$ is on $L_8$. It is easy to see that $L_2 \cap L_7$ and $L_3 \cap L_6$ are on $L_9$ so that $L_2, L_3$ pass through 3 multiple points (Figure 13).

(II). $L_2 \cap L_5$ is on $L_8$. To make $L_1, L_3$ pass through 3 multiple points, $L_1 \cap L_6$ and $L_3 \cap L_7$ must be on $L_9$ (Figure 14).

(III). $L_2 \cap L_7$ is on $L_8$. Obviously, $L_9$ must pass through $L_1 \cap L_6$ and $L_3 \cap L_7$ so that $L_1, L_3$ pass through 3 multiple points (see Figure 15).

(IV). $L_3 \cap L_7$ is on $L_8$. Note that $L_1 \cap L_6$ and $L_2 \cap L_7$ should be on $L_9$, and then $L_1, L_2$ contains 3 multiple points (see Figure 16).

An easy computation shows that Figures 13, 14, 15, and 16 cannot be realized.

**Case 2.** One of $(L_8 \cup L_9) \cap L_{10}$ is a triple point in $A$. We assume that $L_8 \cap L_{10}$ is a triple point, and then $L_{10}$ passes through 2 or 3 points of $\Delta$.

**Subcase 1.** $L_{10}$ passes through 3 points of $\Delta$. We assume that $L_1 \cap L_7, L_2 \cap L_6, L_3 \cap L_5$ are on $L_{10}$ and $L_8 \cap L_{10}$ is on $L_4$. Note that $(L_8 \cup L_9)$ contain at least 4 points of $\Delta$ so that $L_1, L_2, L_3, L_4$ pass through at least 3 multiple points.

(I). Both $L_8$ and $L_9$ contain 2 points of $\Delta$. Note that $L_9$ must pass through one of $\{L_4 \cap L_5, L_4 \cap L_6, L_4 \cap L_7\}$. Up to a lattice isomorphism, we assume $L_4 \cap L_5$ is on $L_9$.

① $L_1 \cap L_6$ is on $L_9$, and then $L_2 \cap L_5, L_3 \cap L_7$ or $L_2 \cap L_7, L_3 \cap L_6$ is on $L_8$ so that $L_1, L_2, L_3, L_4$ pass through at least 3 multiple points (Figures 17 and 18).

Figure 17 can be defined by the following equation:
If \( L \) is on \( 5 \), then \( 8 \) is on \( 6 \); it is lattice isomorphic to \( \{1, 3, 7\} \).

(II). \( L_8 \) contains 2 points of \( \Delta \) and \( L_9 \) contains 3 points of \( \Delta \). Then \( L_9 \) passes through one of \( \{(L_1 \cap L_6, L_2 \cap L_7), (L_1 \cap L_6, L_4 \cap L_7)\}, (L_3 \cap L_6, L_2 \cap L_7) \).

\( L_1 \cap L_6, L_2 \cap L_7 \) is on \( L_9 \), and then \( L_3 \cap L_6 \) or \( L_3 \cap L_7 \) is on \( L_8 \); up to a permutation, we assume that \( L_3 \cap L_6 \) is on \( L_8 \). Then \( L_1 \cap L_5 \) or \( L_2 \cap L_5 \) is on \( L_8 \) (Figures 21 and 22).

Figure 21 cannot be realized.

Figure 22 can be defined by the following equation:

\( L_1 \cap L_6, L_3 \cap L_7 \) is on \( L_9 \). Then \( L_2 \cap L_5 \) or \( L_3 \cap L_7 \) is on \( L_8 \) so that \( L_2 \) contains 3 multiple points. If \( L_2 \cap L_5 \) is on \( L_8 \), then \( L_3 \cap L_6 \) is on \( L_8 \) (Figure 23). If \( L_2 \cap L_7 \) is on \( L_8 \), then \( L_1 \cap L_5 \) or \( L_3 \cap L_6 \) is on \( L_8 \) (Figures 24 and 25).
Figures 23, 24, and 25 cannot be realized.

(III). \( L_8 \) contains 3 points of \( \Delta \) and \( L_9 \) contains 2 points of \( \Delta \). Since \( L_4 \cap L_5 \) is on \( L_9 \), then one of \( \{L_1 \cap L_6, L_3 \cap L_6, L_2 \cap L_7, L_3 \cap L_7\} \) is on \( L_9 \).

(IV). \( L_8 \) contains 3 points of \( \Delta \) and \( L_9 \) contains 3 points of \( \Delta \). Since (III) cannot be realized, case (IV) cannot be realized.

(V). \( L_8 \) contains 1 point of \( \Delta \) and \( L_9 \) contains 3 points of \( \Delta \). From (III), we need to remove one \( L_8 \) intersecting with \( \Delta \), and it is easy to see that there are 3 cases (see Figures 28, 29, and 30).
Figure 28 cannot be realized.

Figure 29 can be defined by the following equation:
$$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(t_1 Y - X - t_1t_3Z)(Y + (t_2 - 1)X - t_2Z) = 0,$$
where $t_1 = 2t^2 - 2t + 2, t_2 = t, t_3 = 2t^2 - t + 1$, and $t$ satisfies $2t^3 - 2t^2 + 2t - 1 = 0$.

Figure 30 can be defined by the following equation:
$$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(t_1 Y - X - (t_1t_3-1)Z)(Y + (t_2 - 1)X - t_2Z) = 0,$$
where $t_1 = t^2 - t + 2, t_2 = t, t_3 = t^2 + 1$, and $t$ satisfies $t^3 - t^2 + 2t - 1 = 0$.

**Subcase 2.** $L_{10}$ passes through 2 points of $\Delta$. We assume that $L_1 \cap L_7, L_2 \cap L_6$ are on $L_{10}$ and $L_8 \cap L_{10}$ is on $L_4$. Note that $(L_8 \cup L_9)$ contain at least 5 points of $\Delta$ so that $L_1, L_2, L_3, L_4$ pass through at least 3 multiple points. To make $L_3, L_4$ contain at least 3 multiple points, $L_9 \cap (L_3 \cup L_4)$ is a triple point.

(I). $L_8$ contains 3 points of $\Delta$ and $L_9$ contains 2 points of $\Delta$.

1. $L_4 \cap L_5$ is on $L_9$.
   (a) $L_3 \cap L_6$ is on $L_9$. Then $L_1 \cap L_6$ is on $L_8$, and $(L_2 \cap L_5, L_3 \cap L_7)$ or $(L_2 \cap L_7, L_3 \cap L_5)$ is on $L_8$ (see Figures 28 and 29).
   (b) $L_3 \cap L_7$ is on $L_9$. By a permutation, it is lattice isomorphic to the previous case.

Figure 31 can be defined by the following equation:

Figure 32 can be defined by the following equation:
$$XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y+(t_3-t_2)X-t_3Z)(t_1 Y - t_2Z)(Y-\frac{t_3-t_1}{t_1-t_2}X - \frac{t_3-t_1}{t_1-t_2}Z) = 0,$$
where $t_1 = \frac{1}{3} + \frac{1}{3}t, t_2 = 1 - t, t_3 = t$, and $t$ satisfies $t^2 - t + 1 = 0$.

2. $L_4 \cap L_6$ is on $L_9$. Then $L_3 \cap L_5$ is on $L_9$ and $L_1 \cap L_5$ or $L_2 \cap L_5$ is on $L_8$ so that $L_5$ passes through 3 multiple points.
   (a) $L_1 \cap L_5$ is on $L_8$. Then $L_2 \cap L_7$ or $L_3 \cap L_6$ is on $L_8$ so that $\mathcal{A}$ is nonreductive (Figure 33).
(b) \( L_2 \cap L_5 \) is on \( L_8 \) Then \( L_1 \cap L_6 \) or \( L_3 \cap L_7 \) is on \( L_8 \) so that \( \mathcal{A} \) is nonreductive (Figure 34).

Figure 33 can be defined by the following equation:

\[
XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_3-1)X-Z)(Y+\frac {t_2}{t_1}X-t_2Z)(Y-\frac {1}{1-t_1}X-\frac {t_2}{1-t_1}Z) = 0,
\]
where \( t_1 = \pm t, t_2 = \frac {1}{2}, t_3 = \pm t - 1, \) and \( t \) satisfies \( 2t^2 - 4t + 1 = 0. \)

Figure 34 can be defined by the following equation:

\[
XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_2-2t)X-t_3Z)(Y+\frac {t_2}{t_1}X-t_2Z)(Y-\frac {1}{1-t_1}X-\frac {t_2}{1-t_1}Z) = 0,
\]
where \( t_1 = \pm t - 1, t_2 = \pm t - 1, t_3 = \pm t, \) and \( t \) satisfies \( t^2 - t - 1 = 0. \)

Figures 35 and 36 can be defined by the following equation:

\[
XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_3-1)X-Z)(Y-\frac {t_2}{t_1}X-t_2Z)(Y-\frac {1}{1-t_1}X-\frac {t_2}{1-t_1}Z) = 0,
\]
where \( t_1 = \frac {1}{4} t + \frac {1}{2} t_2, t_2 = \frac {1}{2} t_3, t = t, \) and \( t \) satisfies \( t^2 - 2t - 1 = 0. \)

Figures 35 and 36 can be defined by the following equation:

\[
XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_2-2t)X-t_3Z)(Y-\frac {t_2}{t_1}X-t_2Z)(Y-\frac {1}{1-t_1}X-\frac {t_2}{1-t_1}Z) = 0,
\]
where \( t_1 = \frac {1}{4} t + \frac {1}{2} t_2, t_2 = t, t_3 = t - 1, \) and \( t \) satisfies \( 2t^2 + t - 2 = 0. \)

If \( L_4 \) \( \cap L_5 \) is on \( L_9 \), then \( L_3 \cap L_5 \) or \( L_3 \cap L_7 \) is on \( L_9 \).

If \( L_4 \cap L_5 \) is on \( L_9 \), then \( L_1 \cap L_5, L_3 \cap L_6 \) or \( L_1 \cap L_5, L_3 \cap L_7 \) are on \( L_8 \) so that \( \mathcal{A} \) is nonreductive (Figures 37 and 38).

Figure 37 can be defined by the following equation:
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Figure 37.

Figure 38.

Figure 39.

Figure 40.

Figure 39 can be defined by the following equation:


Figure 40 can be defined by the following equation:

\[ XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_3-t_2)X-t_2Z)(Y-(t_2-1)X-Z)(Y- \frac{t_2}{t_1}X- \frac{t_3-t_2}{1-t_1}Z) = 0, \]
where \( t_1 = t, t_2 = \frac{1}{4}t + \frac{1}{4}, t_3 = -\frac{1}{2} + \frac{1}{2}t, \) and \( t \) satisfies \( t^2 - 3t + 4 = 0. \)

(2) \( L_2 \cap L_5 \) is on \( L_9, \) so then \( L_1 \cap L_5, L_3 \cap L_6 \) or \( L_2 \cap L_7, L_3 \cap L_5 \) is on \( L_8 \) so that \( A \) is nonreductive (Figures 41 and 42).

Figure 41 cannot be realized.

Figure 42 can be defined by the following equation:

\[ XYZ(X-Z)(X-t_1Z)(Y-Z)(Y-t_2Z)(Y-t_3Z)(Y-(t_2-t_3)X-t_3Z)(Y- \frac{1-t_2}{t_1}X-t_2Z)(Y- \frac{1-t_3}{1-t_1}X- \frac{t_2-t_3}{1-t_1}Z) = 0, \]
where \( t_1 = -t, t_2 = 1 + t, t_3 = t, \) and \( t \) satisfies \( t^2 + 2t - 1 = 0. \)

(3) \( L_4 \cap L_7 \) is on \( L_9. \) Up to a permutation \((6, 7)(1, 2),\) it is lattice isomorphic to (2).

(III). Both \( L_8 \) and \( L_9 \) contain 3 points of \( \Delta. \) We only need to add 1 point of \( \Delta \) to \( L_8 \) for (I) or to \( L_9 \) for (II), so we obtain 5 cases (Figures 43, 44, 45, 46, and 47).
Figures 43, 44, 45, 46, and 47 cannot be realized.

**Case 3.** \((L_9 \cup L_9) \cap L_{10}\) are triple points in \(A\). Then \(L_{10}\) passes through at least 1 point of \(\Delta\) so that it contains at least 3 multiple points.

**Subcase 1.** \(L_{10}\) passes through 2 points of \(\Delta\). We assume that \(L_{10}\) passes through \((L_1 \cap L_7, L_2 \cap L_6)\), \(L_8 \cap L_{10}\) is on \(L_4\), and \(L_9 \cap L_{10}\) is on \(L_3\). Note that \(L_8 \cup L_9\) pass through at least 4 points of \(\Delta\) so that \(A\) is nonreductive.

(I). \(L_8\) contains 3 points of \(\Delta\) and \(L_9\) contains 1 point of \(\Delta\). To make \(L_4, L_5\) contain at least 3 multiple points, \(L_4 \cap L_5\) is on \(L_9\).

1. \(L_3 \cap L_5\) is on \(L_8\). Then \(L_1 \cap L_6, L_2 \cap L_7\) are on \(L_8\) (Figure 48).
2. \(L_3 \cap L_6\) is on \(L_8\). Then \(L_1 \cap L_5, L_2 \cap L_7\) are on \(L_8\) (Figure 49).
3. \(L_3 \cap L_7\) is on \(L_8\). Then \(L_1 \cap L_6, L_2 \cap L_5\) are on \(L_8\). After a permutation \((6, 7)(1, 2)\), it is lattice isomorphic to \(\Box\).

Figure 48 cannot be realized.

Figure 49 can be defined by the following equation:

\[
XY Z(X-Z)(X-t_1 Z)(Y-Z)(Y-t_2 Z)(Y-(t_3-1)X)(Y-(t_3-1)X-Z)(Y-\frac{1+t_1+t_2}{1+t_1+t_2} X - \frac{t_1+t_2}{1+t_1+t_2} Z) = 0,
\]
where \(t_1 = t, t_2 = 1 + t^2 - 2t, t_3 = t - 1,\) and \(t\) satisfies \(t^3 - 4t^2 + 5t - 3 = 0\).

(II). \(L_8\) contains 3 points of \(\Delta\) and \(L_9\) contains 2 points of \(\Delta\).

1. \(L_3 \cap L_5\) is on \(L_8\). Then \(L_1 \cap L_6, L_2 \cap L_7\) are on \(L_8\). To make \(L_4, L_5\) contain at least 3 multiple points, up to a permutation \((6, 7)(1, 2)\), \(L_1 \cap L_5, L_4 \cap L_6\), or \(L_2 \cap L_5, L_4 \cap L_6\) must be on \(L_9\) (Figures 50 and 51).

2. \(L_3 \cap L_6\) is on \(L_8\). Then \(L_1 \cap L_5, L_2 \cap L_7\) are on \(L_8\). To make \(L_4, L_5\) contain at least 3 multiple points, \(L_9\) passes through one of \(\{(L_4 \cap L_5, L_1 \cap L_6), (L_4 \cap L_6, L_2 \cap L_5)\}\) (Figures 52, 53, and 54).

Figures 50, 51, 52, 53, and 54 cannot be realized.

3. \(L_3 \cap L_7\) is on \(L_8\). Up to a permutation \((6, 7)(1, 2)\), it is lattice isomorphic to \(\Box\).
(III). Both $L_8$ and $L_9$ contain 3 points of $\Delta$.

1. $L_3 \cap L_5$ is on $L_8$, and it is easy to see $L_9$ passes through at most 2 points of $\Delta$.
2. $L_3 \cap L_6$ is on $L_8$, so then $L_1 \cap L_5, L_2 \cap L_7$ are on $L_8$ and $L_9$ passes through $L_1 \cap L_6, L_2 \cap L_5, L_4 \cap L_7$ (Figure 55).
3. $L_3 \cap L_7$ is on $L_8$. Up to a permutation (6,7)(1,2), it is lattice isomorphic to 2.

Figure 55 cannot be realized.

(IV). $L_8$ contains 1 or 2 points of $\Delta$ and $L_9$ contains 3 points of $\Delta$. Up to a permutation (8,9)(3,4), it is lattice isomorphic to (I) or (II).

(V). Both $L_8$ and $L_9$ contain 2 points of $\Delta$.

If $L_4 \cap L_5$ is on $L_9$, then $L_1 \cap L_6$ or $L_2 \cap L_7$ is on $L_9$.

1. $L_1 \cap L_6$ is on $L_9$. Then $(L_2 \cap L_5, L_3 \cap L_7)$ or $(L_2 \cap L_7, L_3 \cap L_5)$ are on $L_8$ so that $\mathcal{A}$ is nonreductive (Figures 56 and 57).
2. $L_2 \cap L_7$ is on $L_9$. Up to a permutation (6,7)(1,2), it is lattice isomorphic to 1.

Figures 56 and 57 cannot be realized.

If $L_4 \cap L_6$ is on $L_9$, then $L_1 \cap L_5$ or $L_2 \cap L_5$ is on $L_9$.

(III). Both $L_8$ and $L_9$ contain 3 points of $\Delta$.

1. $L_3 \cap L_5$ is on $L_8$, and it is easy to see $L_9$ passes through at most 2 points of $\Delta$.
2. $L_3 \cap L_6$ is on $L_8$, so then $L_1 \cap L_5, L_2 \cap L_7$ are on $L_8$ and $L_9$ passes through $L_1 \cap L_6, L_2 \cap L_5, L_4 \cap L_7$ (Figure 55).
3. $L_3 \cap L_7$ is on $L_8$. Up to a permutation (6,7)(1,2), it is lattice isomorphic to 2.

Figure 55 cannot be realized.

(IV). $L_8$ contains 1 or 2 points of $\Delta$ and $L_9$ contains 3 points of $\Delta$. Up to a permutation (8,9)(3,4), it is lattice isomorphic to (I) or (II).

(V). Both $L_8$ and $L_9$ contain 2 points of $\Delta$.

If $L_4 \cap L_5$ is on $L_9$, then $L_1 \cap L_6$ or $L_2 \cap L_7$ is on $L_9$.

1. $L_1 \cap L_6$ is on $L_9$. Then $(L_2 \cap L_5, L_3 \cap L_7)$ or $(L_2 \cap L_7, L_3 \cap L_5)$ are on $L_8$ so that $\mathcal{A}$ is nonreductive (Figures 56 and 57).
2. $L_2 \cap L_7$ is on $L_9$. Up to a permutation (6,7)(1,2), it is lattice isomorphic to 1.

Figures 56 and 57 cannot be realized.

If $L_4 \cap L_6$ is on $L_9$, then $L_1 \cap L_5$ or $L_2 \cap L_5$ is on $L_9$.
3 points of $\Delta$. Assume that $L$ (Figures 1).

Subcase 2.

Figure 52.

Figure 53.

Figure 54.

Figure 55.

Figure 56.

Figure 57.

1. $L_1 \cap L_5$ is on $L_9$. Then $(L_2 \cap L_5, L_3 \cap L_7)$ or $(L_2 \cap L_7, L_3 \cap L_5)$ are on $L_8$ so that $A$ is nonreductive (Figures 58 and 59).

2. $L_2 \cap L_5$ is on $L_9$. Then $(L_1 \cap L_5, L_3 \cap L_7)$ are on $L_8$ so that $A$ is nonreductive (Figure 60).

Figure 58 can be defined by the following equation:

$$XYZ(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)(Y - t_3Z)(Y - (t_2 - t_3)X)(Y - (t_2 - t_3)X - t_3Z)(Y - \frac{1-t_1}{1-t_2}X - \frac{t_1-t_2}{1-t_2}Z) = 0,$$

where $t_1 = -1, t_2 = 1 \pm i, t_3 = \pm i$.

Figure 59 can be defined by the following equation:

$$XYZ(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)(Y - t_3Z)(Y - (t_3 - t_2)X - Z)(Y - (t_3 - t_2)X - t_2Z)(Y - \frac{1-t_3}{1-t_2}X - \frac{t_3-t_2}{1-t_2}Z) = 0,$$

where $t_1 = 2t - t^2, t_2 = 1 + t^2, t_3 = t$, and $t$ satisfies $t^3 - 2t^2 + t - 1 = 0$.

Figure 60 can be defined by the following equation:

$$XYZ(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)(Y - t_3Z)(Y - (t_2 - 1)X - Z)(Y - (t_2 - 1)X - t_3Z)(Y - \frac{1-t_1}{1-t_2}X - \frac{t_1-t_2}{1-t_2}Z) = 0,$$

where $t_1 = -1, t_2 = \pm \sqrt{2}, t_3 = \sqrt{2} - 1$.

Subcase 2. $L_{10}$ passes through 1 point of $\Delta$. Then $L_8 \cup L_9$ passes through at least 5 points of $\Delta$. We assume that $L_1 \cap L_7$ is on $L_{10}, L_8 \cap L_{10}$ is on $L_4$, and $L_9 \cap L_{10}$ is on $L_3$.

(I). $L_8 \cup L_9$ passes through 5 points of $\Delta$. We assume that $L_8$ contains 2 points of $\Delta$ and $L_9$ contains 3 points of $\Delta$.
1. \( L_1 \cap L_5 \) is on \( L_9 \). Then \( L_1 \cap L_6, L_2 \cap L_7 \) are on \( L_9 \). To make \( L_2, L_3, L_5, L_6 \) pass through at least 3 multiple points, \( L_8 \) passes through \((L_2 \cap L_5, L_3 \cap L_6)\) or \((L_2 \cap L_6, L_3 \cap L_5)\) (Figures 61 and 62).

2. \( L_1 \cap L_6 \) is on \( L_9 \). Up to a permutation \((5,6)\), it is lattice isomorphic to 1.

3. \( L_1 \cap L_7 \) is on \( L_9 \). \( L_9 \) passes through \((L_1 \cap L_5, L_2 \cap L_6)\) or \((L_1 \cap L_6, L_2 \cap L_5)\). Up to a permutation \((5,6)\), we assume \((L_1 \cap L_5, L_2 \cap L_6)\) are on \( L_9 \) (Figure 63).

Figure 61 can be defined by the following equation:

\[ XYZ(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)(Y - t_3Z)(Y - t_3X)(Y - t_3X - t_3Z)(Y - \frac{1-t_2}{1-t_1} X - \frac{t_2-t_1}{1-t_1} Z) = 0, \]

where \( t_1 = \pm \sqrt{2}, t_2 = 1 \pm \frac{1}{\sqrt{2}}, t_3 = \pm \frac{1}{\sqrt{2}}. \)

Figure 62 can be defined by the following equation:


Figure 63 can be defined by the following equation:

\[ XYZ(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)(Y - t_3Z)(Y + X - Z)(Y + X - t_3Z)(Y - \frac{1-t_2}{1-t_1} X - \frac{t_2-t_1}{1-t_1} Z) = 0, \]

where \( t_1 = -\frac{1+\sqrt{3}}{2}, t_2 = 2 \pm \sqrt{3}, t_3 = \frac{3\pm\sqrt{3}}{2}. \)

(II). \( L_8 \cup L_9 \) passes through 6 points of \( \Delta \). It is easy to see from (I) that this case is impossible. \( \square \)

6. Arrangements of 11 lines with a quintuple point and no quadruple point

Let \( \mathcal{A} \) be a nonreductive arrangement of 11 with a quintuple point and no quadruple point. By Lemma 2.6 and Theorem 2.7, we know that there are at most 14 triple points.

We say that 2 multiple points of \( \mathcal{A} \) are disjoint if they are not on the same line of \( \mathcal{A} \). We say that 2 multiple points of \( \mathcal{A} \) are adjoint if they are on the same line of \( \mathcal{A} \).

We claim that there is at most 1 disjoint triple point apart from the quintuple point. Assume that \( L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5 \) is the quintuple point and \( L_6 \cap L_7 \cap L_8 \) is the triple point apart from the quintuple point. Suppose there is another triple point apart from the quintuple point. It is easy to see that \( L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5 \) pass through at most 9 triple points, but \( L_1 \cup L_2 \cup L_3 \cup L_4 \cup L_5 \) pass through at least 10 triple points, a contradiction.

6.1. One disjoint triple point apart from the pencil of the quintuple point

First, we show that there are at most 13 triple points in \( \mathcal{A} \).

Lemma 6.1 Let \( L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5 \) be the quintuple point and let \( L_6 \cap L_7 \cap L_8 \) be the triple point apart from the quintuple point. Then there are at most 13 triple points in \( \mathcal{A} \).
Then the triple points. Then by Bézout’s theorem, the intersection number of \((L_1 \cup L_2 \cdots \cup L_8)\) and \((L_9 \cup L_{10} \cup L_{11})\) is 24. Since the intersection multiplicity of a triple point is 2, there will be at most 12 triple points in \((L_1 \cup L_2 \cdots \cup L_8) \cap (L_9 \cup L_{10} \cup L_{11})\). In addition with \(L_6 \cap L_7 \cap L_8\), we will have at most 13 triple points.

**Theorem 6.2** Let \(\mathcal{A}\) be a nonreductive line arrangement of 11 lines in \(\mathbb{CP}^2\) with a quintuple point \(L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5\). Assume that \(L_6 \cap L_7 \cap L_8\) is the triple point apart from the quintuple point; then there are exactly 11 triple points in \(\mathcal{A}\). Then there are 28 cases that can be realized, 7 of whose moduli spaces are irreducible and 21 of them are “potential Zariski pairs”.

**Proof** Note that we have at least 11 triple points in \(\mathcal{A}\), because it is nonreductive.

**Case 1.** There are 13 triple points in \(\mathcal{A}\). Then \((L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11})\) are triple points, and each of \(L_9, L_{10}, L_{11}\) passes through 3 triple points on \((L_6 \cup L_7 \cap L_8)\). Up to a lattice isomorphism, we assume that \((L_1 \cap L_8, L_2 \cap L_7, L_3 \cap L_6)\) are on \(L_{11}\), and \(L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11}\) are on \(L_1, L_4, L_5\), respectively. Note that \(L_9\) must pass through one of \(L_5 \cap L_6, L_5 \cap L_7, L_5 \cap L_8\) so that \(L_5\) contains at least 3 multiple points. Up to lattice isomorphism, we assume that \(L_5 \cap L_6\) is on \(L_9\). Then \(L_3 \cap L_7, L_2 \cap L_8\) are on \(L_9\) and \((L_2 \cap L_6, L_3 \cap L_8, L_4 \cap L_7)\) are on \(L_{10}\) so that \(\mathcal{A}\) is nonreductive (Figure 64). Figure 64 cannot be realized.

**Case 2.** There are 12 triple points in \(\mathcal{A}\). Then at least 2 points of \((L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11})\) are triple points.

**Subcase 1.** All of \((L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11})\) are triple points. Note that \(L_9, L_{10}, L_{11}\) pass through 8 triple points on \((L_6 \cup L_7 \cap L_8)\), and we assume \(L_{10}\) passes through 2 triple points on \((L_6 \cup L_7 \cap L_8)\). Similarly as in **Case 1**, we assume that \((L_1 \cap L_8, L_2 \cap L_7, L_3 \cap L_6)\) are on \(L_{11}\), \((L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11})\) are on \(L_1, L_4, L_5\), respectively, and \(L_5 \cap L_6\) is on \(L_9\). Then \(L_3 \cap L_7, L_2 \cap L_8\) are on \(L_9\). Hence, \(L_{10}\) contains one of \((L_4 \cap L_6, L_3 \cap L_8), (L_4 \cap L_7, L_2 \cap L_6), (L_4 \cap L_7, L_3 \cap L_8), (L_4 \cap L_8, L_2 \cap L_6)\)\).

Subcase 1 cannot be realized.

**Subcase 2.** Two of \((L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11})\) are triple points. Let \(L_9 \cap L_{11}, L_{10} \cap L_{11}\) be the triple points. Then \(L_9, L_{10}, L_{11}\) pass through 3 triple points on \((L_6 \cup L_7 \cap L_8)\). We assume that \((L_1 \cap L_8, L_2 \cap L_7, L_3 \cap L_6)\) are on \(L_{11}\), \((L_9 \cap L_{10}, L_{10} \cap L_{11})\) are on \(L_4, L_5\) respectively, and \(L_5 \cap L_6\) is on \(L_9\). Then \(L_9\) must pass through one of \((L_1 \cap L_7, L_2 \cap L_8), (L_3 \cap L_7, L_2 \cap L_8), (L_1 \cap L_7, L_3 \cap L_8)\)\).

(1) \(L_9\) passes through \((L_1 \cap L_7, L_2 \cap L_8)\). Up to a permutation \((7, 8)(1, 2)\), \(L_{10}\) contains \((L_1 \cap L_6, L_3 \cap L_8, L_4 \cap L_7)\) or \((L_2 \cap L_6, L_3 \cap L_8, L_4 \cap L_7)\).
(2). \( L_9 \) passes through \((L_3 \cap L_7, L_2 \cap L_8)\). To make \( \mathcal{A} \) nonreductive, \( L_{10} \) contains one of \( \{(L_4 \cap L_6, L_3 \cap L_8, L_1 \cap L_7), (L_4 \cap L_7, L_3 \cap L_8, L_1 \cap L_6), (L_4 \cap L_8, L_2 \cap L_6, L_1 \cap L_7)\} \).

(3). \( L_9 \) passes through \((L_1 \cap L_7, L_3 \cap L_8)\). After a permutation \((7,8)(1,2)\), it is lattice isomorphic to \((2)\).

Subcase 2 cannot be realized.

Case 3. There are 11 triple points in \( \mathcal{A} \).

Subcase 1. One of \((L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11})\) is a triple point. Let \( L_{10} \cap L_{11} \) be the triple point. Then each of \( L_9, L_{10}, \) and \( L_{11} \) passes through 3 triple points in \((L_6 \cup L_7 \cap L_8)\). We assume that \((L_1 \cap L_8, L_2 \cap L_7, L_3 \cap L_6)\) are on \( L_{11} \) and \( L_{10} \cap L_{11} \) is on \( L_4 \). Note that \( L_9 \) must pass through one of \((L_5 \cap L_6, L_5 \cap L_7, L_5 \cap L_8)\). Up to a lattice isomorphism, let \( L_9 \) pass through \((L_5 \cap L_6, L_4 \cap L_7, L_2 \cap L_8)\) or \((L_5 \cap L_6, L_4 \cap L_7, L_3 \cap L_8)\).

(I). \( L_9 \) passes through \((L_5 \cap L_6, L_4 \cap L_7, L_2 \cap L_8)\). Then \( L_{10} \) passes through \((L_4 \cap L_6, L_1 \cap L_7, L_3 \cap L_8)\) or \((L_4 \cap L_8, L_3 \cap L_7, L_1 \cap L_6)\). The first equation can be defined by

\[
XY(X - Z)(X - t_1Z)(Y - Z)(Y - t_3Z)(Y - t_4Z)(Y - (t_3 - t_2)X - t_2Z)(Y - (1 - t_3)X - t_3Z) = 0, \text{ where } t_1 = t^3, t_2 = t^2, t_3 = t, t_4 = t^3 - 1, \text{ and } t \text{ satisfies } t^4 - t - 1 = 0.
\]

The second equation can be defined by

\[
XY(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)(Y - t_3Z)(Y - t_4X)(Y - (t_2 - 1)X - Z)(Y - (1 - t_3)X - t_3Z) = 0, \text{ where } t_1 = t, t_2 = t^3, t_3 = t, t_4 = t^2, \text{ and } t \text{ satisfies } t^4 + 1 = 0.
\]

(II). \( L_9 \) passes through \((L_5 \cap L_6, L_4 \cap L_7, L_3 \cap L_8)\). Then \( L_{10} \) passes through \((L_4 \cap L_6, L_1 \cap L_7, L_2 \cap L_8)\) or \((L_4 \cap L_8, L_2 \cap L_6, L_1 \cap L_7)\) (the first case cannot be realized).

The equation can be defined by

\[
XY(X - Z)(X - t_1Z)(Y - Z)(Y - t_3Z)(Y - t_4Z)(Y - (t_2 - t_4)X - t_4Z)(Y - (1 - t_3)X - t_3Z) = 0, \text{ where } t_1 = t^3 + t^2 - 2t - 2, t_2 = t^3 - t - 1, t_3 = t, t_4 = t^3 - 1, \text{ and } t \text{ satisfies } t^4 - 2t^2 - t + 1 = 0.
\]

Subcase 2. Two of \((L_9 \cap L_{10}, L_9 \cap L_{11}, L_{10} \cap L_{11})\) are triple points. Let \( L_9 \cap L_{11} \) and \( L_{10} \cap L_{11} \) be triple points on \( L_4 \) and \( L_5 \), respectively. Then \( L_9, L_{10}, \) and \( L_{11} \) pass 8 triple points in \((L_6 \cup L_7 \cap L_8)\).

(I). \( L_{11} \) passes through 2 triple points in \((L_6 \cup L_7 \cap L_8)\). Assume that \( L_1 \cap L_8 \) and \( L_2 \cap L_7 \) are on \( L_{11} \). Note that \( L_9 \) must pass through one of \((L_5 \cap L_6, L_5 \cap L_7, L_5 \cap L_8)\), so that \( L_5 \) contains 3 multiple points.

① \( L_5 \cap L_6 \) is on \( L_9 \), so then \((L_1 \cap L_7, L_3 \cap L_8)\) or \((L_3 \cap L_7, L_2 \cap L_8)\) is on \( L_9 \).

If \((L_1 \cap L_7, L_3 \cap L_8)\) are on \( L_9 \), then \( L_{10} \) must pass through one of \((L_4 \cap L_6, L_3 \cap L_7, L_2 \cap L_8)\), \((L_4 \cap L_7, L_3 \cap L_6, L_2 \cap L_8)\), \((L_4 \cap L_8, L_3 \cap L_7, L_2 \cap L_6)\) (only the first case can be realized).

The equation can be defined by

\[
XY(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)(Y - t_3Z)(Y - t_4X)(Y - (t_4 - t_2)X - t_2Z)(Y - \frac{1-t_4}{1-t_1}X - \frac{t_4-t_1}{1-t_1}Z) = 0, \text{ where } t_1 = t, t_2 = -2t + 6, t_3 = t + 3, t_4 = -2 + t, \text{ and } t \text{ satisfies } t^2 - 3t + 1 = 0.
\]

If \((L_3 \cap L_7, L_2 \cap L_8)\) are on \( L_9 \), then by a permutation \((7,8)(1,2)\), it is lattice isomorphic to the case that \((L_1 \cap L_7, L_3 \cap L_8)\) are on \( L_9 \).

② \( L_5 \cap L_7 \) is on \( L_9 \), so then \( L_9 \) must pass through one of \((L_1 \cap L_6, L_3 \cap L_8)\), \((L_2 \cap L_6, L_3 \cap L_8)\), \((L_3 \cap L_6, L_2 \cap L_8)\).

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If \( L_1 \cap L_6, L_3 \cap L_8 \) are on \( L_9 \), then \( L_{10} \) must pass through one of \( \{ (L_4 \cap L_6, L_3 \cap L_7, L_2 \cap L_8), (L_4 \cap L_7, L_3 \cap L_6, L_2 \cap L_8), (L_4 \cap L_8, L_3 \cap L_7, L_2 \cap L_6) \} \).

The first equation can be defined by
\[
XY(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)Y - t_3Z)(Y - (t_4 - t_3)X - t_3Z)(Y - (t_3 - t_2)X - t_2Z)(Y - \frac{1-t_4}{1-t_1}X - t_1Z - \frac{1-t_4}{1-t_1}Z = 0, \]
where \( t_1 = 1 \pm (t - t^2 + t^3), t_2 = 4 \mp (t^2 + 2t^3), t_3 = 2 \mp (t^2 + t^3), t_4 = \pm t, \) and \( t \) satisfies \( t^4 - t^3 + 2t - 1 = 0 \).

The second equation can be defined by
\[
XY(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)(Y - t_3Z)(Y - (t_4 - t_3)X - t_3Z)(Y - (t_3 - 1)X - Z)(Y - \frac{1-t_4}{1-t_1}X - t_1Z - \frac{1-t_4}{1-t_1}Z = 0, \]
where \( t_1 = 1 \mp t, t_2 = -\frac{1}{2} \pm t, t_3 = -1 \pm t, t_4 = \pm t, \) and \( t \) satisfies \( 2t^2 - 4t + 1 = 0 \).

The third equation can be defined by
\[
XY(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)(Y - t_3Z)(Y - t_4Z)(Y - (t_4 - t_3)X - t_3Z)(Y - (t_4 - 1)X - Z)(Y - \frac{1-t_4}{1-t_1}X - t_1Z - \frac{1-t_4}{1-t_1}Z = 0, \]
where \( t_1 = 1 \pm t, t_2 = -3 \pm (4t - t^2), t_3 = 2 \mp t, t_4 = -1 \mp (t^2 - 3t), \) and \( t \) satisfies \( t^3 - 3t^2 - 2t + 1 = 0 \).

If \( L_2 \cap L_6, L_3 \cap L_8 \) are on \( L_9 \), then \( L_{10} \) must pass through \( \{ (L_4 \cap L_6, L_3 \cap L_6, L_1 \cap L_7), (L_3 \cap L_7, L_1 \cap L_6) \} \) (the second case cannot be realized).

The first equation can be defined by
\[
XY(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)(Y - t_3Z)(Y - t_4Z)(Y - (t_4 - t_3)X - t_3Z)(Y - (t_3 - 1)X - Z)(Y - \frac{1-t_4}{1-t_1}X - t_1Z - \frac{1-t_4}{1-t_1}Z = 0, \]
where \( t_1 = 1 \pm t, t_2 = -3 \pm (4t - t^2), t_3 = 2 \mp t, t_4 = -1 \mp (t^2 - 3t), \) and \( t \) satisfies \( t^3 - 3t^2 - 2t + 1 = 0 \).

If \( L_3 \cap L_6, L_2 \cap L_8 \) are on \( L_9 \), then \( L_{10} \) must pass through one of \( \{ (L_4 \cap L_6, L_3 \cap L_8, L_1 \cap L_7), (L_4 \cap L_7, L_3 \cap L_8, L_1 \cap L_6), (L_4 \cap L_8, L_3 \cap L_7, L_1 \cap L_6) \} \).

The first equation can be defined by
\[
XY(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)(Y - t_3Z)(Y - t_4Z)(Y - (t_4 - t_3)X - t_3Z)(Y - (t_4 - 1)X - Z)(Y - \frac{1-t_4}{1-t_1}X - t_1Z - \frac{1-t_4}{1-t_1}Z = 0, \]
where \( t_1 = 1 \pm t, t_2 = -3 \pm (4t - t^2), t_3 = 2 \mp t, t_4 = -1 \mp (t^2 - 3t), \) and \( t \) satisfies \( t^3 - 3t^2 - 2t + 1 = 0 \).

The second equation can be defined by
\[
XY(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)(Y - t_3Z)(Y - t_4Z)(Y - (t_4 - t_3)X - t_3Z)(Y - (t_4 - 1)X - Z)(Y - \frac{1-t_4}{1-t_1}X - t_1Z - \frac{1-t_4}{1-t_1}Z = 0, \]
where \( t_1 = 1 \pm t, t_2 = -3 \pm (4t - t^2), t_3 = 2 \mp t, t_4 = -1 \mp (t^2 - 3t), \) and \( t \) satisfies \( t^3 - 3t^2 - 2t + 1 = 0 \).

The third equation can be defined by
\[
XY(X - Z)(X - t_1Z)(Y - Z)(Y - t_2Z)(Y - t_3Z)(Y - t_4Z)(Y - (t_4 - t_3)X - t_3Z)(Y - (t_4 - 1)X - Z)(Y - \frac{1-t_4}{1-t_1}X - t_1Z - \frac{1-t_4}{1-t_1}Z = 0, \]
where \( t_1 = 1 \pm t, t_2 = -3 \pm (4t - t^2), t_3 = 2 \mp t, t_4 = -1 \mp (t^2 - 3t), \) and \( t \) satisfies \( t^3 - 3t^2 - 2t + 1 = 0 \).

\( \oplus \) \( L_5 \cap L_8 \) is on \( L_9 \). After a permutation \( (7, 8)(1, 2) \), it is lattice isomorphic to \( \bowtie \).

\( \boxminus \) \( L_{10} \) passes through 2 triple points in \( (L_6 \cup L_7 \cap L_8) \). We assume that \( (L_1 \cap L_8, L_2 \cap L_7, L_3 \cap L_6) \) are on \( L_{11} \), \( (L_9 \cap L_{10}, L_{10} \cap L_{11}) \) are on \( L_4, L_5 \) respectively, and \( L_5 \cap L_6 \) is on \( L_9 \). Then \( L_9 \) must pass through one of \( \{ (L_1 \cap L_7), (L_2 \cap L_8), (L_3 \cap L_7, L_2 \cap L_8), (L_1 \cap L_7, L_3 \cap L_8) \} \).

\( \oplus \) \( (L_1 \cap L_7, L_2 \cap L_8) \) are on \( L_9 \), so then up to lattice isomorphism \( L_{10} \) must pass through \( \{ L_4 \cap L_6, L_3 \cap L_7 \} \) or \( \{ L_4 \cap L_7, L_3 \cap L_8 \} \).

The first equation can be defined by
\[
\]

The second equation can be defined by
\[
\]
(L₃ ∩ L₇, L₂ ∩ L₈) are on L₉, so then L₁₀ must pass through one of \{(L₄ ∩ L₆, L₁ ∩ L₇), (L₄ ∩ L₇, L₁ ∩ L₆), (L₄ ∩ L₈, L₁ ∩ L₆), (L₄ ∩ L₈, L₁ ∩ L₇)\}.

The first equation can be defined by
\[ XY(X - Z)(X - t₁Z)(Y - Z)(Y - t₂Z)(Y - t₃Z)(Y - t₄Z)(Y - (1 - t₃)X - t₃Z) = 0, \]
where \( t₁ = t₂ = t₃ = -1 + 2t, t₄ = 2t, \) and \( t \) satisfies \( 2t² - 2t + 1 = 0. \)

The second equation can be defined by
\[ XY(X - Z)(X - t₁Z)(Y - Z)(Y - t₂Z)(Y - t₃Z)(Y - t₄Z)(Y - t₅Z)(Y - t₄X)(Y - \frac{t₂ - 1}{t₁}X - Z)(Y - (1 - t₃)X - t₃Z) = 0, \]
where \( t₁ = t₂ = t₃ = t₄ = t₅ = t₆ = t₇ = 2t, \) and \( t \) satisfies \( t³ - 2t² + 3t - 1 = 0. \)

The third equation can be defined by
\[ XY(X - Z)(X - t₁Z)(Y - Z)(Y - t₂Z)(Y - t₃Z)(Y - t₄Z)(Y - (t₂ - 1)X - Z)(Y - (1 - t₃)X - t₃Z) = 0, \]
where \( t₁ = t² + t₂ = t₃ = t₄ = t₅ = t₆ = t₇ = 2t², \) and \( t \) satisfies \( t^4 + t³ - 2t² + 2t + 1 = 0. \)

The fourth equation can be defined by
where \( t₁ = \frac{1}{2} + \frac{3}{2}t², t₂ = \frac{1}{2}t² + t + \frac{1}{2}, t₃ = t₄ = t₅ = t₆ = t₇ = 1, \) and \( t \) satisfies \( t^4 + t³ - 2t² + 2t + 1 = 0. \)

(3) \( (L₁ ∩ L₇, L₃ ∩ L₈) \) are on \( L₉. \) After a permutation \((7, 8)(1, 2), \) it is lattice isomorphic to \( (2). \)

Subcase 3. All of \((L₉ ∩ L₁₀, L₉ ∩ L₁₁, L₁₀ ∩ L₁₁)\) are triple points. Then \( L₀, L₁₀, \) and \( L₁₁ \) pass through 8 triple points in \((L₆ ∩ L₇ ∩ L₈)\) and at least one of \( L₉, L₁₀, L₁₁ \) passes through 3 triple points in \((L₆ ∩ L₇ ∩ L₈)\). We always assume that \( L₉ \) can be such a line. Furthermore, up to a lattice isomorphism, we assume that \((L₁ ∩ L₈, L₂ ∩ L₇, L₃ ∩ L₆)\) are on \( L₁₁, \) and \( L₉ ∩ L₁₀, L₉ ∩ L₁₁, L₁₀ ∩ L₁₁ \) are on \( L₁, L₄, L₅ \) respectively.

(I). \( L₉ \) passes through 3 triple points in \((L₆ ∩ L₇ ∩ L₈)\) and \( L₁₀ \) must pass through 1 triple point in \((L₆ ∩ L₇ ∩ L₈)\). Note that \( L₉ \) must pass through one of \{L₅ ∩ L₆, L₅ ∩ L₇, L₅ ∩ L₈\} so that \( L₅ \) contains 3 multiple points. Up to a permutation \((6, 7)(2, 3), \) we assume that \{L₅ ∩ L₆, L₃ ∩ L₇, L₂ ∩ L₈\} or \{L₅ ∩ L₈, L₃ ∩ L₇, L₂ ∩ L₆\} are on \( L₉. \)

(1) \( L₈ ∩ L₆, L₃ ∩ L₇, L₂ ∩ L₈ \) are on \( L₀. \) Then \( L₁₀ \) must pass through one of \{L₄ ∩ L₆, L₄ ∩ L₇, L₄ ∩ L₈\}.

The first equation can be defined by
\[ XY(X - Z)(X - t₁Z)(Y - Z)(Y - t₂Z)(Y - t₃Z)(Y - t₄Z)(Y + t₄(t₂ - 1)X - t₂Z)(Y - (1 - t₃)X - t₃Z) = 0, \]
where \( t₁ = \frac{1}{2}, t₂ = \frac{1}{4}t + 1, t₃ = t₄ = 2, \) and \( t \) satisfies \( t^3 - 3t² + 4 = 0. \)

The second cannot be realized.

The third equation can be defined by
\[ XY(X - Z)(X - t₁Z)(Y - Z)(Y - t₂Z)(Y - t₃Z)(Y - t₄Z)(Y - t₅Z)(Y - t₆Z)(Y - t₇Z)(Y - (t₄(t₂ - 1)X + \frac{t₂ - 1}{t₄}Z)(Y - (1 - t₃)X - t₃Z) = 0, \]
where \( t₁ = t₂ = t₃ = -1, t₄ = 2t - 1, \) and \( t \) satisfies \( 2t² - t + 1 = 0. \)

(2) \( L₈ ∩ L₆, L₃ ∩ L₇, L₂ ∩ L₈ \) are on \( L₀. \) Then \( L₁₀ \) must pass through one of \{L₄ ∩ L₆, L₄ ∩ L₇, L₄ ∩ L₈\}.

The first equation can be defined by
\[ XY(X - Z)(X - t₁Z)(Y - Z)(Y - t₂Z)(Y - t₃Z)(Y - t₄Z)(Y + t₄X)(Y - \frac{t₃(t₄ - 1)}{t₃}X - t₃Z)(Y - (1 - t₃)X - t₃Z) = 0, \]
where \( t₁ = \frac{5}{2}, t₂ = \frac{3}{2}, t₃ = 3, t₄ = -2. \)

The second cannot be realized.

The third equation can be defined by
\[ XY(X - Z)(X - t₁Z)(Y - Z)(Y - t₂Z)(Y - t₃Z)(Y - t₄Z)(Y + t₂(t₃ - 1)X - t₂t₃Z)(Y - (1 - t₃)X - t₃Z) = 0, \]
where \( t₁ = -t₂ + t₃ = 2t, t₄ = -2t + 1, \) and \( t \) satisfies \( 2t² - t + 1 = 0. \)
(II). Both $L_9$ and $L_{10}$ pass through 2 triple points in $(L_6 \cup L_7 \cap L_8)$. Up to a lattice isomorphism, assume that $L_5 \cap L_6, L_3 \cap L_7$ or $L_5 \cap L_8, L_3 \cap L_7$ are on $L_9$.

1. $L_5 \cap L_6, L_3 \cap L_7$ are on $L_9$. Then $L_{10}$ must pass through one of $\{ (L_4 \cap L_6, L_2 \cap L_8), (L_4 \cap L_7, L_2 \cap L_6), (L_4 \cap L_7, L_2 \cap L_8), (L_4 \cap L_8, L_2 \cap L_6) \}$.

The first equation can be defined by

$$XY(X - Z)(X - t_1 Z)(Y - Z)(Y - t_2 Z)(Y - t_3 Z)(Y - t_4 Z)(t_1 Y - t_3 X)(Y - (t_4 - t_2) X - t_2 Z)(Y - (1 - t_3) X - t_3 Z) = 0,$$

where $t_1 = t, t_2 = -2t + 4, t_3 = 2t, t_4 = 2t - 2$, and $t$ satisfies $2t^2 - t - 2 = 0$.

The second case cannot be realized.

The third equation can be defined by

$$XY(X - Z)(X - t_1 Z)(Y - Z)(Y - t_2 Z)(Y - t_3 Z)(Y - t_4 Z)(t_1 Y - t_3 X)(Y - \frac{t_2 - t_4}{1 - t_4} X + \frac{t_2 - t_4}{1 - t_4} Y)(Y - (1 - t_3) X - t_3 Z) = 0,$$

where $t_1 = -\frac{2}{3}t^3 + \frac{1}{3}t^2 - t - \frac{2}{7}, t_2 = -4t^3 + t^2 + 7t - 12, t_3 = t, t_4 = -t^3 + 2t - 2$, and $t$ satisfies $t^4 + t^3 - 2t^2 + t + 4 = 0$.

The fourth equation can be defined by

$$XY(X - Z)(X - t_1 Z)(Y - Z)(Y - t_2 Z)(Y - t_3 Z)(Y - t_4 Z)(t_1 Y - t_3 X)(Y - (t_2 - t_4) X - t_4 Z)(Y - (1 - t_3) X - t_3 Z) = 0,$$

where $t_1 = t, t_2 = -2t + 1, t_3 = -1, t_4 = 2t - 1$, and $t$ satisfies $2t^2 - t - 1 = 0$.

2. $L_5 \cap L_8, L_3 \cap L_7$ are on $L_9$. Then $L_{10}$ must pass through one of $\{ (L_4 \cap L_6, L_2 \cap L_8), (L_4 \cap L_7, L_2 \cap L_6), (L_4 \cap L_7, L_2 \cap L_8), (L_4 \cap L_8, L_2 \cap L_6) \}$.

The first equation can be defined by

$$XY(X - Z)(X - t_1 Z)(Y - Z)(Y - t_2 Z)(Y - t_3 Z)(Y - t_4 Z)(Y - \frac{t_2 - t_4}{1 - t_4} X + \frac{t_2 - t_4}{1 - t_4} Z)(Y - (t_4 - t_2) X - t_2 Z)(Y - (1 - t_3) X - t_3 Z) = 0,$$

where $t_1 = -t^2 - t, t_2 = t^4, t_3 = t, t_4 = t^3$, and $t$ satisfies $t^5 + t^4 - t^2 - t - 1 = 0$.

The second equation can be defined by $XY(X - Z)(X - t_1 Z)(Y - Z)(Y - t_2 Z)(Y - t_3 Z)(Y - t_4 Z)(Y - \frac{t_2 - t_4}{1 - t_4} X + \frac{t_2 - t_4}{1 - t_4} Z)(Y - (t_4 - t_2) X - t_2 Z)(Y - (1 - t_3) X - t_3 Z) = 0$, where $t_1 = \frac{1}{2} - \frac{1}{2} t, t_2 = \frac{1}{2} t + \frac{1}{2}, t_3 = t, t_4 = 2$, and $t$ satisfies $t^5 - 2t^2 - 1 = 0$.

The third equation can be defined by $XY(X - Z)(X - t_1 Z)(Y - Z)(Y - t_2 Z)(Y - t_3 Z)(Y - t_4 Z)(Y - \frac{t_2 - t_4}{1 - t_4} X + \frac{t_2 - t_4}{1 - t_4} Z)(Y - (t_4 - t_2) X - t_2 Z)(Y - (1 - t_3) X - t_3 Z) = 0$, where $t_1 = -t, t_2 = t + 2, t_3 = t, t_4 = 2$, and $t$ satisfies $t^5 - 2 = 0$.

The fourth equation can be defined by $XY(X - Z)(X - t_1 Z)(Y - Z)(Y - t_2 Z)(Y - t_3 Z)(Y - t_4 Z)(Y - \frac{t_2 - t_4}{1 - t_4} X + \frac{t_2 - t_4}{1 - t_4} Z)(Y - (t_4 - t_2) X - t_2 Z)(Y - (1 - t_3) X - t_3 Z) = 0$, where $t_1 = 2 - t, t_2 = t, t_3 = t - 2, t_4 = 2$, and $t$ satisfies $t^2 - 4t + 2 = 0$. \qed

6.2. All triple points are in the pencil of the quintuple point

Assume that all the triple points are on the lines passing through the quintuple point. We first show that there are at most 13 triple points, and at least 10 triple points if the arrangement is nonreductive.

Lemma 6.3 Let $\mathcal{A}$ be a nonreductive arrangement of 11 lines with 1 quintuple point so that all triple points are on the lines passing through the quintuple point. Then there are at most 13 triple points and at least 10 triple points.

**Proof** From Lemma 2.6 and Theorem 2.7, we have the following equations:
\[
\begin{align*}
  n_2 + 3n_3 + 6n_4 + 10n_5 &= 55, \\
  n_2 + \frac{3}{2}n_3 &\geq 11 + n_5.
\end{align*}
\]

From the above equations, we compute \( n_3 \leq 14 \).

Let \( L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5 \) be the quintuple point. Since there are 14 triple points on those 5 lines and we know that each of the 5 lines passes through at least 2 and at most 3 triple points, then we may assume that each of \( L_1, L_2, L_3, \) and \( L_4 \) passes through 3 triple points. On the other hand, each of the other six lines passes through at least 3 and at most 5 triple points. Let \( a, b, \) and \( c \) be the numbers of lines in \( \{L_6, L_7, L_8, L_9, L_{10}, L_{11}\} \) that pass through 3, 4, and 5 triple points, respectively. Then \( a \) and \( b \) should satisfy the following system of equations:

\[
\begin{align*}
  a + b + c &= 6, \\
  3a + 4b + 5c &= 28.
\end{align*}
\]

From the above equations, we have two solutions:

One is

\[ a = 1, b = 0, c = 5. \]

The other is

\[ a = 0, b = 2, c = 4. \]

Because there are 14 triple points, one of \( \{L_1, L_2, L_3, L_4, L_5\} \) has only two triple points. This fact tells us that \( c \leq 4 \). The first case does not exist. For the second case, we consider \( \mathcal{A}' = \mathcal{A} \setminus L_5 \). Now \( \mathcal{A}' \) is an arrangement of 10 lines with 1 quadruple point. All triple points are in the pencil of the quadruple point, and \( \mathcal{A}' \) has 12 triple points. By [2, Lemma 5.3], \( \mathcal{A}' \) cannot be realized, so \( \mathcal{A} \) does not exist.

Because each line of \( \{L_1, L_2, L_3, L_4, L_5\} \) has at least two triple points, then \( n_3 \geq 10 \).

The classification will run on the numbers of triple points.

**Theorem 6.4** Let \( \mathcal{A} \) be a nonreductive arrangement of 11 lines with a quintuple point and 13 triple points such that all triple points are on the 5 lines passing through the quintuple point. Then the moduli space \( \mathcal{M}_\mathcal{A} \) is irreducible, and in fact is one point.

**Proof** Let \( L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5 \) be the quintuple point. Since there are 14 triple points on those 5 lines and we know that each of 5 lines passes through at least 2 and at most 3 triple points, then we may assume that each of \( L_1, L_2, \) and \( L_3 \) passes through 3 triple points. On the other hand, each of the other six lines passes through at least 3 and at most 5 triple points. Let \( a, b, \) and \( c \) be the numbers of lines in \( \{L_6, L_7, L_8, L_9, L_{10}, L_{11}\} \) that pass through 3, 4, and 5 triple points, respectively. Then \( a \) and \( b \) should satisfy the following system of equations:

\[
\begin{align*}
  a + b + c &= 6, \\
  3a + 4b + 5c &= 26.
\end{align*}
\]

From the above equations, we have three solutions:

\[ a = 2, b = 0, c = 4; \quad a = 1, b = 2, c = 3; \quad a = 0, b = 4, c = 2. \]
For the first case, we consider $\mathcal{A}' = \mathcal{A} \setminus L_4$. Now $\mathcal{A}'$ is an arrangement of 10 lines with 1 quadruple point. All triple points are in the pencil of the quadruple point, and $\mathcal{A}'$ has 11 triple points. From [2, Theorem 5.4], $\mathcal{M}_{\mathcal{A}'}$ is irreducible. In fact, it is one point, because every case has only one solution from the proof of [2, Theorem 5.4]. The moduli space $\mathcal{M}_{\mathcal{A}'}$ is irreducible.

For the second case, because $b = 2$, one of $\{L_6, L_7, L_8, L_9, L_{10}, L_{11}\}$ must intersect 4 lines, and one of the intersect lines must be $L_4$ or $L_5$. We assume it is $L_4$. Now we consider $\mathcal{A}' = \mathcal{A} \setminus L_4$. The rest of this proof is similar to the first case.

For the third case, we consider $\mathcal{A}' = \mathcal{A} \setminus L_4$. The rest of this proof is similar to the first case.

\begin{remark}
The example of Theorem 6.4 is easy to construct (see Figure 65).
\end{remark}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure65.png}
\caption{Figure 65.}
\end{figure}

The equation is defined as follows:


\begin{theorem}
Let $\mathcal{A}$ be a nonreductive arrangement of 11 lines with a quintuple point and 12 triple points such that all triple points are on the 5 lines passing through the quintuple point. Then the quotient moduli space $\mathcal{M}_{\mathcal{A}}^c$ is irreducible, and in fact is one or two points.
\end{theorem}

\begin{proof}
Let $L_1 \cap L_2 \cap L_3 \cap L_4 \cap L_5$ be the quintuple point. On the one hand, since there are 12 triple points on those 5 lines and we know that each of 5 lines passes through at least 2 and at most 3 triple points, then we may assume that each of $L_1$ and $L_2$ passes through 3 triple points. On the other hand, each of the other six lines passes through at least 3 and at most 5 triple points. Let $a$, $b$, and $c$ be the numbers of lines in $\{L_6, L_7, L_8, L_9, L_{10}, L_{11}\}$ that pass through 3, 4, and 5 triple points, respectively. Then $a$ and $b$ should satisfy the following system of equations:

$$a + b + c = 6,$$

$$3a + 4b + 5c = 24.$$ 

From the above equations, we have these solutions:

$$a = 0, b = 6, c = 0; \quad a = 1, b = 4, c = 1; \quad a = 2, b = 2, c = 2; \quad a = 3, b = 0, c = 3.$$
For the first case, we consider $A' = A \setminus L_4$. Now $A'$ is an arrangement of 10 lines with 1 quadruple point. All triple points are in the pencil of the quadruple point, and $A'$ has 10 triple points. From [2, Theorem 5.5], $\mathcal{M}_{A'}$ is irreducible. In fact, it is one point. Because every case has only conjugation solutions from the proof of [2, Theorem 5.5], the moduli space $\mathcal{M}_{A'}$ is irreducible.

For the second case, because $a = 1$, we assume that the line is $L_6$. Two of $\{L_1, L_2, L_3, L_4, L_5\}$ cannot intersect with $L_6$. We assume that these lines are $L_4$ and $L_5$, and $L_4$ must have two triple points. Now we consider that $A' = A \setminus L_4$, and $A'$ is an arrangement of 10 lines with 1 quadruple point. All triple points are in the pencil of the quadruple point, and $A'$ has 10 triple points. From [2, Theorem 5.5], $\mathcal{M}_{A'}$ is irreducible, and in fact is one point, so the moduli space $\mathcal{M}_{A'}$ is irreducible.

For the third case, because $a = 2$, we assume the lines are $L_6$ and $L_7$. One of $\{L_1, L_2, L_3, L_4, L_5\}$ must not intersect $L_6$ and $L_7$. We assume that this line is $L_4$, and $L_4$ must have two triple points. Now we consider that $A' = A \setminus L_4$, and $A'$ is an arrangement of 10 lines with 1 quadruple point. The rest of this proof is similar to the second case.

For the fourth case, up to lattice isomorphism, there is only one case (see Figure 66).

It is easy to see that line 11 and line 10 must have a double point that is not on the 5 lines passing through the quintuple point, so Figure 66 and Figure 67 are equivalent.

It is easy to compute that $A$ does not exist.

From the above discussions, we have the following corollary:

**Corollary 6.7** Let $A$ be a nonreductive line arrangement of 11 lines with $n_5 = 1$, $n_4 = 0$, and $n_r = 0, r \geq 6$. Moreover, all triple points are on the 5 lines passing through the quintuple point. If it contains more than 11 triple points, then there is no Zariski pair.

Let $A$ be a nonreductive arrangement of 11 lines with a quintuple point and all triple points are on the 5 lines passing through the quintuple point. If the number of the triple points is less than 12, then there are many cases in which $\mathcal{M}_A$ is more than 2 points or even one dimension. Now we give two examples.
Example 6.1 The line arrangements are with 11 triple points, and all triple points are on the 5 lines passing through the quintuple point (see Figure 68).

After some easy computation, we get the equation as follows: $X(X - Z)(Y - Z)(Y - t_3Z)(Y - t_2Z)(Y - t_1Z)Y((t_2 - t_3)X - Y + t_3Z)(Y - Z + X)(Y - (t_1 - \frac{t_1 - t_2}{t_2})Z - (\frac{t_1 - t_2}{t_2})X)(Y - t_2Z - \frac{t_1 - t_2}{1-t_1}X) = 0$, where $t_1 = \frac{1}{5}t^2 - \frac{2}{5}t + \frac{4}{5}$, $t_2 = t$, $t_3 = 1 - t$, and $t$ satisfies $t^2 - 2t - t^3 + 1 = 0$, so that the moduli space $M_A$ is three points.

Example 6.2 The line arrangements are with 10 triple points, and all triple points are on the 5 lines passing through the quintuple point (see Figure 69).

After some easy computation, we get the equation as follows: $X(X - Z)(Y - Z)(Y - t_3Z)(Y - t_2Z)(Y - t_1Z)(Y(1 - t_1))X - Y + t_1Z)((t_3 - t_2)X - Y + t_2Z)(t_2X - Y)(t_2(t_1 - 1)X - (t_2 - 1)Y + (t_2 - t_1)Z)$, where $t_1 = \frac{1 - 3t - t^2}{t}$, $t_2 = t$, and $t_3 = \frac{t^3 - 3t^2 + t}{1 - 2t}$, so the moduli space $M_A$ is of one dimension.

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References


