On the second homology of the Schützenberger product of monoids

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Abstract: For two finite monoids $S$ and $T$, we prove that the second integral homology of the Schützenberger product $S \triangle T$ is equal to

$$H_2(S \triangle T) = H_2(S) \times H_2(T) \times (H_1(S) \otimes \mathbb{Z} H_1(T))$$

as the second integral homology of the direct product of two monoids. Moreover, we show that $S \triangle T$ is inefficient if there is no left or right invertible element in both $S$ and $T$.

Key words: Monoid, Schützenberger product, second integral homology, efficiency

1. Introduction

It was shown by SJ Pride (unpublished) that, for a finitely presented monoid $M$, $\text{def}_M(M) \geq \text{rank}(H_2(M))$ where $H_2(M)$ is the second integral homology of the monoid and

$$\text{def}_M(M) = \min\{|R| - |A| : \langle A \mid R \rangle \text{ is a finite monoid presentation for } M\}.$$

In [1] this result was extended to a finitely presented semigroup $S$, that is $\text{def}_S(S) \geq \text{rank}(H_2(S))$ where $H_2(S)$ is the second integral homology of $S^1$, the monoid obtained from $S$ by adjoining an identity if necessary, and

$$\text{def}_S(S) = \min\{|R| - |A| : \langle A \mid R \rangle \text{ is a finite semigroup presentation for } S\}.$$

Moreover, it was shown that the $n$th integral homology of a semigroup with a left or a right zero is trivial for $n \geq 1$ (see also [8, Lemma 1]), and the second integral homology of a finite rectangular band $R_{m,n}$ of order $mn$ is $\mathbb{Z}^{(m-1)(n-1)}$. A finite semigroup $S$ is called efficient as a semigroup if $\text{def}_S(S) = \text{rank}(H_2(S))$, and inefficient otherwise. The efficiency and inefficiency of a finite monoid are defined similarly. The first examples of efficient and inefficient semigroups were given in [1], which showed that finite zero semigroups and finite free semilattices are inefficient, and finite rectangular bands are efficient. More examples of efficient semigroups can be found in [2, 3, 4, 5, 6].

It was shown in [2] that the second integral homology of a finite Rees matrix semigroup $M[G; I, \Lambda; P]$ (finite simple semigroup) is $H_2(G) \times \mathbb{Z}^{(|I|-1)(|\Lambda|-1)}$ by using the Squier resolution (see [12]). In this paper, we also use this resolution to compute the second integral homology of the Schützenberger product of two finite monoids. We show that, for two finite monoids $S$ and $T$,
\[ H_2(S \triangleleft T) = H_2(S) \times H_2(T) \times (H_1(S) \otimes H_1(T)), \]

and it follows from [3, Equation (1)] that \( H_2(S \triangleleft T) = H_2(S \times T) \). Moreover, we consider the efficiency of \( S \triangleleft T \) and conclude that, if there is no left or right invertible element in both \( S \) and \( T \), then \( S \triangleleft T \) is inefficient.

2. Preliminaries

Since the Squier resolution given in [12] is defined by using a presentation in which the set of relations is a uniquely terminating rewriting system, we give some elementary concepts about rewriting systems.

Let \( A \) be an alphabet. We denote the free semigroup on \( A \) consisting of all nonempty words over \( A \) by \( A^+ \), and the free monoid \( A^+ \cup \{ \varepsilon \} \) where \( \varepsilon \) denotes the empty word by \( A^* \). A rewriting system \( R \) on \( A \) is a subset of \( A^* \times A^* \). For \( w_1, w_2 \in A^* \), if they are identical words then we write \( w_1 \equiv w_2 \), and if there exist \( u, v \in A^* \) and \( (r, s) \in R \) such that \( w_1 = uv \) and \( w_2 = usv \) then we write \( w_1 \rightarrow w_2 \) and we say that \( w_1 \) rewrites to \( w_2 \). We denote by \( \rightarrow^* \) the reflexive and transitive closure of \( \rightarrow \), and by \( \sim \) the equivalence relation generated by \( \rightarrow^* \). For a word \( w \in A^* \) we say that \( w \) is reducible (\( R \)-reducible) if there is a word \( z \in A^* \) such that \( w \rightarrow z \); otherwise we say that \( w \) is irreducible (\( R \)-irreducible). If \( w \rightarrow y \) and \( y \in A^* \) is irreducible, then we say that \( y \) is an irreducible form of \( w \). A rewriting system \( R \) is called terminating if there is no infinite sequence \((w_n)\) such that \( w_n \rightarrow w_{n+1} \) for all \( n \geq 1 \). Let \( |w| \) be the length of the word \( w \in A^* \). If \( |r| > |s| \) for all \( (r, s) \in R \) then the system \( R \) is called length-reducing.

It is well known that if there exists an ordering \( < \) on a set \( S \) such that, for each distinct pair \( s, s' \in S \), either \( s < s' \) or \( s' < s \), then the ordering \( < \) is called linear (or total) ordering and the set \( S \) is called linearly (or totally) ordered. For \( u, v \in A^* \), if \( |u| > |v| \) or if \( |u| = |v| \) and \( v \) precedes \( u \) in the lexicographic ordering induced by a linear ordering on \( A \) then we write \( v < u \) and \( < \) is called length-lexicographic ordering. A rewriting system \( R \) is called a length-lexicographic rewriting system if \( s < r \) for all \( (r, s) \in R \). It is clear that length-reducing systems and length-lexicographic rewriting systems are terminating.

A semigroup (monoid) presentation is an ordered pair \((A \mid R)\), where \( R \subseteq A^+ \times A^+ \) \((R \subseteq A^* \times A^*)\). Let \( S \) be a semigroup (monoid). \( S \) is called a semigroup (monoid) defined by the semigroup (monoid) presentation \((A \mid R)\) if \( S \) is isomorphic to \( A^+ / \rho \) \((A^* / \rho)\), where \( \rho \) is the congruence on \( A^+ \) \((A^*)\) generated by \( R \). For \( w_1, w_2 \in A^* \), we also write \( w_1 = w_2 \) if \((w_1, w_2) \in \rho\); that is, \( w_2 \) is obtained from \( w_1 \) by applying relations from \( R \), or, equivalently, there is a finite sequence \( w_1 \equiv \alpha_1, \alpha_2, ..., \alpha_n \equiv w_2 \) of words from \( A^* \) in which every \( \alpha_i \) is obtained from \( \alpha_{i-1} \) by applying a relation from \( R \) (see [9, Proposition 1.5.9]).

A rewriting system \( R \) is called confluent if, for any \( x, y, z \in A^* \) such that \( x \rightarrow y \), \( x \rightarrow z \), there exists \( w \in A^* \) such that \( y \rightarrow w \), \( z \rightarrow w \). Also, a rewriting system \( R \) is called complete if it is both terminating and confluent. For a given rewriting system \( R \), let the subset \( R_1 \subseteq A^* \) be the set of all \( r \in A^* \) such that there exists \((r, s) \in R\) for some \( s \in A^* \). The system \( R \) is called reduced if for each \((r, s) \in R\), \( R_1 \cap A^* r A^* = \{r\} \) and \( s \) is \( R \)-irreducible. Finally, a reduced complete rewriting system \( R \) is called a uniquely terminating rewriting system.

Lemma 2.1 ([7, Theorem 1.1] or [12, Theorem 2.1]) Let \( R \) be a terminating rewriting system on \( A \). Then the following are equivalent:

[Proof or statement of the lemma here]
(i) $R$ is confluent (and hence complete);
(ii) for any pair $(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3}) \in R$, where $r_2$ is nonempty, there exists a word $w \in A^*$ such that $s_{1,2}r_2 \rightarrow w$ and $r_1s_{2,3} \rightarrow w$; for any pair $(r_1r_2r_3, s_{1,2,3}), (r_2, s_2) \in R$, where $r_2$ is nonempty, there exists a word $w \in A^*$ such that $s_{1,2,3} \rightarrow w$ and $r_1s_{2,3} \rightarrow w$;
(iii) any word $w \in A^*$ has exactly one irreducible form. Moreover, $w \sim w'$ if and only if $w$ and $w'$ have the same irreducible form.

If there exists a pair $(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3}) \in R$ or $(r_1r_2r_3, s_{1,2,3}), (r_2, s_2) \in R$ such that $r_2$ is a nonempty word, then we define the overlaps to be the ordered pairs $[(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3})]$ and $[(r_1r_2r_3, s_{1,2,3}), (r_2, s_2)]$, respectively. Note that the overlaps of the form $[(r_1r_2r_3, s_{1,2,3}), (r_2, s_2)]$ do not exist in a reduced rewriting system.

Let $\langle A \mid R \rangle$ be a presentation for a monoid $S$ in which $R$ is a uniquely terminating rewriting system on $A$. Also, let $ZS$ denote the monoid ring of $S$ with coefficients in $Z$. In [12] Squier defined the free resolution of $Z$ as follows:

$$P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} Z \rightarrow 0,$$

where $P_0$ is the free $ZS$-module on a single formal symbol $[\ ]$ and the augmentation map $\varepsilon : P_0 \rightarrow Z$ is defined by $\varepsilon([\ ] \varepsilon a - 1[[a]]$.

$P_1$ is the free $ZS$-module on the set of formal symbols $[a]$ for each $a \in A$ and the map $\partial_1 : P_1 \rightarrow P_0$ is defined by

$$\partial_1([a]) = (a - 1)[[a]] .$$

$P_2$ is the free $ZS$-module on the set of formal symbols $[r, s]$, for each $(r, s) \in R$. For each $a \in A$, a function $\partial/\partial_a : A^* \rightarrow ZA^*$, which is called a derivation, is defined by induction as follows:

$$\partial/\partial_a(1) = 0,$$

and if $w \in A^*$ and $b \in A$, then

$$\partial/\partial_a(wb) = \begin{cases} \partial/\partial_a(w) & \text{if } b \neq a, \\ \partial/\partial_a(w) + w & \text{if } b = a. \end{cases}$$

Then the map $\partial_2 : P_2 \rightarrow P_1$ is defined by

$$\partial_2([r, s]) = \sum_{a \in A} \phi(\partial/\partial_a(r) - \partial/\partial_a(s))[a],$$

where $\phi : ZA^* \rightarrow ZS$ is the map induced by the natural homomorphism from $A^*$ to $S$. Finally, $P_3$ is the free $ZS$-module on the set of formal symbols $[(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3})]$, for each pair $(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3}) \in R$ where $r_2$ is not an empty word. Let $w$ be in $A^*$ and let $u$ be the irreducible form of $w$. Then we have a sequence

$$w = u_1r_1v_1 \rightarrow u_1s_1v_1 = u_2r_2v_2 \rightarrow \cdots \rightarrow u_qv_qv_q = u$$

where $u_i, v_i \in A^*$ and $(r_i, s_i) \in R$ for each $i = 1, \ldots, q$. Then the map $\Phi : A^* \rightarrow P_2$ is defined by

$$\Phi(w) = \sum_{i=1}^{q} \phi(u_i)[r_i, s_i].$$
and the map $\partial_3 : P_3 \rightarrow P_2$ is defined by

$$\partial_3([(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3})]) = r_1[r_2r_3, s_{2,3}] - [r_1r_2, s_{1,2}] + \Phi(r_1s_{2,3}) - \Phi(s_{1,2}r_3).$$

Squier showed that $P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0$ is an exact sequence if $R$ is a uniquely terminating rewriting system and we assume that for each word $w \in A^*$, the chosen relation chain from $w$ to the irreducible form of $w$ consists of reductions only; that is, if $(r, s) \in R$, then $(s, r) \notin R$.

If we apply the tensor product $\mathbb{Z} \otimes_{\mathbb{Z}S} -$ to the resolution of $\mathbb{Z}$ given above, we obtain the chain complex of abelian groups

$$\mathbb{Z} \otimes P_3 \xrightarrow{1 \otimes \partial_3} \mathbb{Z} \otimes P_2 \xrightarrow{1 \otimes \partial_2} \mathbb{Z} \otimes P_1 \xrightarrow{1 \otimes \partial_1} \mathbb{Z} \otimes P_0 \xrightarrow{1 \otimes \varepsilon} \mathbb{Z} \otimes \mathbb{Z} \rightarrow 0,$$

or simply,

$$\bar{P}_3 \xrightarrow{\bar{\partial}_3} \bar{P}_2 \xrightarrow{\bar{\partial}_2} \bar{P}_1 \xrightarrow{\bar{\partial}_1} \mathbb{Z} \rightarrow 0 \tag{1}$$

where $\bar{P}_1$, $\bar{P}_2$, and $\bar{P}_3$ are the free abelian groups on the set of formal symbols $[a]$, $[r, s]$, and $[(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3})]$ where $a \in A$; $(r, s), (r_1r_2, s_{1,2}), (r_2r_3, s_{2,3}) \in R$ with $r_2$ not an empty word, respectively. Clearly the map $\bar{\partial}_1 : \bar{P}_1 \rightarrow \mathbb{Z}$ is the zero map.

For $a \in A$ and $w \in A^*$, the number of occurrences of the letter $a$ in the word $w$ is called $a$-length of $w$ and denoted by $\|w\|_a$. Moreover, if $w = a_1a_2 \cdots a_m$, then we denote the list $[a_1, a_2, \ldots, a_m]$ by $C[w]$. (Note that in any list some of the elements can be the same; for example, $C[ab^2a^2] = [a, b, b, a]$.)

The maps $\bar{\partial}_2 : \bar{P}_2 \rightarrow \bar{P}_1$ and $\bar{\partial}_3 : \bar{P}_3 \rightarrow \bar{P}_2$ are defined by

$$\bar{\partial}_2([r, s]) = \sum_{a \in A} ([\|r\|_a - \|s\|_a])[a]$$

and

$$\bar{\partial}_3([(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3})]) = [r_2r_3, s_{2,3}] - [r_1r_2, s_{1,2}] + \bar{\Phi}(r_1s_{2,3}) - \bar{\Phi}(s_{1,2}r_3),$$

respectively, where $\bar{\Phi} : A^* \rightarrow \bar{P}_2$ is the map defined by

$$\bar{\Phi}(w) = \sum_{i=1}^{q} [r_i, s_i] \text{ if } \Phi(w) = \sum_{i=1}^{q} \phi(u_i)[r_i, s_i].$$

With this notation we have the following immediate result:

**Lemma 2.2** ([8, Lemma 3.1]) If a monoid $S$ has a presentation $(A \mid R)$ such that $R$ is a uniquely terminating rewriting system on $A$, then

$$H_1(S) = H_1(G) = G/G' = (A \mid \sum_{a \in A} ([\|r\|_a - \|s\|_a])[a] = 0 \quad ((r, s) \in R)),$$

where $G$ is the group defined by $(A \mid R)$ as a group presentation and $G'$ is the derived subgroup of $G$.

**Lemma 2.3** ([11, Chapter 6]) Let $(A \mid R)$ and $(B \mid Q)$ ($A$ and $B$ are distinct) be presentations for the monoids $S$ and $T$, respectively. Then the tensor product of their first homologies, namely $H_1(S) \otimes_{\mathbb{Z}} H_1(T)$,
can be given by the abelian group presentation

\[
\langle [A, B] | \sum_{a \in A} ((a - b)ab, ba) = 0 \quad (b \in B, (r, s) \in R) \\
\sum_{b \in B} ((u - v)ab, ba) = 0 \quad (a \in A, (u, v) \in Q) \rangle,
\]

where \([A, B] = \{[ab, ba] | a \in A, b \in B\}\).

### 3. The second integral homology of the Schützenberger product of monoids

Let \(S\) and \(T\) be two finite monoids, and let \(\mathcal{P}(S \times T)\) denote the set of all subsets of \(S \times T\). Now we define the sets

\[
sX = \{(sx, y) : (x, y) \in X\} \quad \text{and} \quad Xt = \{(x, yt) : (x, y) \in X\},
\]

where \(X \in \mathcal{P}(S \times T)\), \(s \in S\), and \(t \in T\). Then the set \(S \times \mathcal{P}(S \times T) \times T\) is a monoid, denoted by \(S \diamond T\) and called the Schützenberger product of \(S\) and \(T\), with identity \((1_S, 0, 1_T)\) by the multiplication

\[
(s_1, X_1, t_1)(s_2, X_2, t_2) = (s_1s_2, X_1t_2 \cup s_1X_2, t_1t_2).
\]

If \(S\) is a finitely presented monoid then it is clear that \(S\) is linearly ordered by considering the length-lexigraphic ordering. In this section we consider that the monoids \(S\) and \(T\) are well ordered. Moreover, the direct product \(S \times T\) is also linearly ordered, with the ordering \((s, t) \prec (s', t')\) if \(s < s'\) or if \(s = s'\) and \(t < t'\).

If the monoid presentations \(\langle A | R\rangle\) and \(\langle B | Q\rangle\) (\(A\) and \(B\) are distinct) define the monoids \(S\) and \(T\), respectively, then the presentation \(\langle A \cup B \cup C | R \cup Q \cup Z \rangle\) where \(C = \{c_{s,t} : s \in S, \ t \in T\}\) and

\[
Z = \{ c^2_{s,t} = c_{s,t} \ \ (s \in S, \ t \in T), \ c_{s,t}c_{s',t'} = c_{s',t'}c_{s,t} \ \ ((s', t') \prec (s, t) \in S \times T), \ ac_{s,t} = ac_{s,t}a \ \ (a \in A, \ s \in S, \ t \in T), \ c_{s,t}b = bc_{s,t}b \ \ (b \in B, \ s \in S, \ t \in T), \ ab = ba \ \ (a \in A, \ b \in B) \}
\]

defines \(S \diamond T\) in terms of the generating set

\[
\{(a, 0, 1_T), (1_S, 0, b), (1_S, (s, t), 1_T) : a \in A, \ b \in B, \ (s, t) \in S \times T\}.
\]

(For a proof, see [10, Theorem 3.2].)

Note that, for ease of notation, we write \(c_{a, s, t}\) and \(c_{s, t, b}\) instead of \(c_{\pi_S(a), s', t}\) and \(c_{s, t, \pi_T(b)}\) where \(\pi_S : A^* \to S\) and \(\pi_T : B^* \to T\) are the natural homomorphisms, respectively. Thus, for \(r, p \in A^*\) and \(u, v \in TB^*\), the words \(c_{r, u}\) and \(c_{p, v}\) are identical if the relations \(r = p\) and \(u = v\) hold in \(S\) and \(T\), respectively.

**Lemma 3.1** Let \(S\) and \(T\) be two finite monoids, and let \(\langle A | R\rangle\) and \(\langle B | Q\rangle\) be their finite monoid presentations such that \(R\) and \(Q\) are uniquely terminating rewriting systems on \(A\) and \(B\), respectively. With the above notations, the rewriting system \(R \cup Q \cup Z\) is uniquely terminating on \(A \cup B \cup C\).

**Proof** For an arbitrary word \(w\) in \((A \cup B \cup C)^*\), it is clear that the reduced form of \(w\) has the form \(w_1w_2w_3\) where \(w_1, w_2,\) and \(w_3\) are reduced words in \(B, C,\) and \(A\), respectively. It is also clear that \(R \cup Q \cup Z\) is
terminating and reduced. The overlaps are:

\[
\begin{align*}
V_1 &= [(r_1r_2, p_{1,2}), (r_2r_3, p_{2,3})], \\
V_2 &= [(ra, p), (ac_s, t, c_{as,t}a)], \\
V_3 &= [(ra, p), (ab, ba)], \\
V_4 &= [(u_1u_2, v_{1,2}), (u_2u_3, v_{2,3})], \\
V_5 &= [(c_s, t, c_{as,t}, c_{as,t}), (c_{as,t}, c_{as,t}, c_{as,t})], \\
V_6 &= [(c_s, t, c_{as,t}, c_{as,t}), (c_{as,t}, c_{as,t}, c_{as,t})][((s', t') \prec (s, t)), \\
V_7 &= [(c_s, t, c_{as,t}, c_{as,t}), (c_{as,t}, b, bc_{as,t}b)], \\
V_8 &= [(c_s, t, c_{as,t}, c_{as,t}, c_{as,t}), (c_{as,t}, c_{as,t}, c_{as,t})][((s', t') \prec (s, t)), \\
V_9 &= [(c_s, t, c_{as,t}, c_{as,t}, c_{as,t}), (c_{as,t}, c_{as,t}, c_{as,t})][((s'', t'') \prec (s', t') \prec (s, t)), \\
V_{10} &= [(c_s, t, c_{as,t}, c_{as,t}, c_{as,t}), (c_{as,t}, b, bc_{as,t}v_b)][((s', t') \prec (s, t)), \\
V_{11} &= [(ac_s, t, c_{as,t}a), (c_{as,t}, c_{as,t}, c_{as,t})], \\
V_{12} &= [(ac_s, t, c_{as,t}a), (c_{as,t}, c_{as,t}, c_{as,t})][((s', t') \prec (s, t)), \\
V_{13} &= [(ac_s, t, c_{as,t}a), (c_{as,t}, b, bc_{as,t}b)], \\
V_{14} &= [(c_s, t, b, bc_{as,t}b), (bu, v)], \\
V_{15} &= [(ab, ba), (bu, v)],
\end{align*}
\]

where \( a \in A; b \in B; (ra = p), (r_1r_2 = p_{1,2}), (r_2r_3 = p_{2,3}) \in R; (bu = v), (u_1u_2 = v_{1,2}), (u_2u_3 = v_{2,3}) \in Q; (s, t), (s', t'), (s'', t'') \in S \times T \). Now it follows from Lemma 2.1 that \( R \cup Q \cup Z \) is confluent and so a uniquely terminating rewriting system.

\[ \square \]

**Theorem 3.2** If \( S \) and \( T \) are two finite monoids, then

\[
H_2(S \ast T) = H_2(S) \times H_2(T) \times (H_1(S) \otimes \mathbb{Z} H_1(T)).
\]

**Proof** We consider the uniquely terminating rewriting system \( R \cup Q \cup Z \) on \( A \cup B \cup C \) given in Lemma 3.1 and the chain complex (1) arising from it.

Before we compute the second integral homology of \( S \ast T \), that is \( H_2(S \ast T) = \ker \tilde{\partial}_2 / \text{im} \tilde{\partial}_3 \), we assume that \( H_2(S) = \ker \tilde{\partial}_2 / \text{im} \tilde{\partial}_3 \) and \( H_2(T) = \ker \tilde{\partial}_2 / \text{im} \tilde{\partial}_3 \) where \( \ker \tilde{\partial}_2, \text{im} \tilde{\partial}_3, \ker \tilde{\partial}_2, \text{im} \tilde{\partial}_3 \) are the free abelian groups on \( \{ X_i : i \in I \}, \{ Y_j : j \in J \}, \{ U_k : k \in K \}, \) and \( \{ W_l : l \in L \} \) (which are found by using the Squier resolution), respectively.

Now we find a generating set for the free abelian group \( \text{im} \tilde{\partial}_3 \) by using the overlaps in the proof of Lemma 3.1. We compute the following.

\[
\begin{align*}
\tilde{\partial}_3(V_1) &= \text{im} \tilde{\partial}_3, \\
\tilde{\partial}_3(V_2) &= [ac_s, t, c_{as,t}a] - [ra, p] + \Phi(r_{c_{as,t}a}) - \Phi(pc_{as,t}) \\
\tilde{\partial}_3(V_3) &= \sum_{a \in C[ra]} [ab, ba] - \sum_{a \in C[p]} [ab, ba]
\end{align*}
\]

768
\[ \tilde{\partial}_3(V_4) = \text{im } \tilde{\partial}_3; \]
\[ \tilde{\partial}_3(V_5) = 0 \]
\[ \tilde{\partial}_3(V_6) = [c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}] \]
\[ \tilde{\partial}_3(V_7) = [c_{s,t}b, bc_{s,t}] - [c_{s,t}c_{s,t} + [c_{s,t}b, c_{s,t}] \]
\[ \tilde{\partial}_3(V_8) = -[c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}] \]
\[ \tilde{\partial}_3(V_9) = 0 \]
\[ \tilde{\partial}_3(V_{10}) = -[c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}] + [c_{s,t}b c_{s',t'}, c_{s',t'}c_{s,t}] \]
\[ \tilde{\partial}_3(V_{11}) = [c_{s,t}^2, c_{s,t}] - [ac_{s,t}, c_{as,t}a] - [c_{as,t}^2, c_{as,t}] \]
\[ \tilde{\partial}_3(V_{12}) = [c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}] - [c_{as,t}c_{as',t'}, c_{as',t'}c_{as,t}] \]
\[ \tilde{\partial}_3(V_{13}) = [c_{s,t}b, bc_{s,t}] - [ac_{s,t}, c_{as,t}a] + [ac_{s,t}, b, c_{as,t}a] - [c_{as,t}b, bc_{s,t}] \]
\[ \tilde{\partial}_3(V_{14}) = -[c_{s,t}b, bc_{s,t}] + \Phi(c_{s,t}) - \Phi(c_{s,t}b) \]
\[ \tilde{\partial}_3(V_{15}) = \sum_{b \in C[v]} [ab, ba] - \sum_{b \in C[bu]} [ab, ba] \]

Now let
\[ W(ra, p) = \sum_{a \in C[ra]} [ab, ba] - \sum_{a \in C[p]} [ab, ba], \]
\[ W(bu, v) = \sum_{b \in C[v]} [ab, ba] - \sum_{b \in C[bu]} [ab, ba], \]
\[ W(a, s, t) = [c_{s,t}^2, c_{s,t}] - [ac_{s,t}, c_{as,t}a] - [c_{as,t}^2, c_{as,t}], \]
\[ W(b, s, t) = [c_{s,t}b, bc_{s,t}] - [c_{s,t}^2, c_{s,t}] + [c_{s,t}b, c_{s,t}], \]
\[ W(s', t', s, t) = [c_{s,t}c_{s',t'}, c_{s',t'}c_{s,t}] \quad ((s', t') \prec (s, t)) \]

where \( a \in A, b \in B, s, s' \in S, t, t' \in T, (ra, p) \in R, \) and \((bu, v) \in Q.\) Then we show that the set
\[ \{ Y_j, W_1, W(ra, p), W(bu, v), W(a, s, t), W(b, s, t), W(s', t', s, t) ((s', t') \prec (s, t)) : j \in J; l \in L; a \in A; b \in B; s, s' \in S; t, t' \in T; (ra, p) \in R; (bu, v) \in Q \} \]
is a generating set for the free abelian group \( \text{im } \tilde{\partial}_3 \) as follows.

If \( r \equiv a_1 \cdots a_m \) and \( p \equiv a_1' \cdots a_n' \) \((a_1, \ldots, a_m, a_1', \ldots, a_n' \in A)\) then we define
\[ W_0 = W(a_m, as, t), \]
\[ W_i = W(a_{m-i}, a_{m+1-i} \cdots a_{m} as, t) \quad (1 \leq i \leq m - 1), \]
\[ W_0' = W(a_n', s, t), \]
\[ W_j' = W(a_n'-j, a_{n+1-j} \cdots a_n' s, t) \quad (1 \leq j \leq n - 1). \]
Thus, we have

\[
\tilde{\partial}(V_2) = [ac, t, c_{as, t} a] + \tilde{\Phi}(rc_{as, t}) - \tilde{\Phi}(pc_{s, t}) = [ac, t, c_{as, t} a] \\
+ [a_m c_{as, t}, c_{as, t} a_m] + \sum_{i=1}^{m-1} [a_{m-i} c_{as, t} a_{m-i}] \\
- [a'_n c_{as, t}, c_{as, t} a'_n] - \sum_{j=1}^{n-1} [a_{n-j} c_{as, t} a_{n-j}] \\
= - W(a, s, t) + \sum_{j=0}^{n-1} W_j - \sum_{i=0}^{n-1} W_i,
\]

and so \( \tilde{\partial}(V_2) \) is a linear combination of \( W(a, s, t) \)s. Similarly, it can be shown that \( \tilde{\partial}(V_{i+4}) \) is a linear combination of \( W(b, s, t) \)s. Moreover, it is clear that all of \( \tilde{\partial}(V_6), \tilde{\partial}(V_8), \tilde{\partial}(V_{10}), \) and \( \tilde{\partial}(V_{12}) \) are linear combinations of \( W(s', t', s, t) \)s, and that

\[
\tilde{\partial}(V_{13}) = W(b, s, t) + W(a, s, t) - W(a, s, tb) - W(b, as, t).
\]

Next we find a generating set for \( \ker \tilde{\partial}_2 \). Since any \( \alpha \in \tilde{P}_2 \) has the form

\[
\alpha = \sum_{(r=s) \in R} \alpha_{(r, s)}[r, s] + \sum_{(u=v) \in Q} \alpha_{(u, v)}[u, v] + \sum_{a \in A, b \in B} \alpha_{(a, b)}[ab, ba] \\
+ \sum_{s \in S, t \in T} \alpha_{(s, t)}[c_{s, t}^2, c_{s, t}] + \sum_{(s', t') \in (s, t) \in S \times T} \alpha_{(s', t', s, t)}[c_{s', t'}, c_{s, t}, c_{s, t}]
\]

where all the coefficients are integers, then \( \alpha \in \ker \tilde{\partial}_2 \) if and only if

\[
\tilde{\partial}_2\left( \sum_{(r=s) \in R} \alpha_{(r, s)}[r, s] \right) = 0, \quad \tilde{\partial}_2\left( \sum_{(u=v) \in Q} \alpha_{(u, v)}[u, v] \right) = 0 \quad \text{and}
\]

\[
\sum_{s \in S, t \in T} \alpha_{(s, t)}[c_{s, t}] + \sum_{a \in A} \alpha_{(a, s)}([c_{as, t}] - [c_{as, t}]) + \sum_{b \in B} \alpha_{(b, s, t)}([c_{s, t}] - [c_{as, t}]) = 0.
\]

From the first two equations given above we obtain the generators \( \{X_i : i \in I\} \) and \( \{U_k : k \in K\} \) for \( \ker \tilde{\partial}_{2|S} \) and \( \ker \tilde{\partial}_{2|T} \), respectively. Now we concentrate on the last equation. By rearranging it, we have

\[
\alpha_{(s, t)} = - \sum_{a \in A} \alpha_{(a, s)} - \sum_{b \in B} \alpha_{(b, s, t)} + \sum_{a' \in A, s' \in S, a' = a} \alpha_{(a', s', t)} + \sum_{b' \in B, t' \in T, b' = t} \alpha_{(b', s', t')}
\]

for each \( (s, t) \in S \times T \). For fixed \( \alpha_{(a, s, t)} \), we assume that \( \alpha_{(a, s, t)} = 1 \) and all the other variables on the right-hand side of Equation (2) are zero, and so we obtain \( \alpha_{(s, t)} = -1 \) and \( \alpha_{(a, s, t)} = 1 \). Thus, we have the following generators:

\[
W_1(a, s, t) = [ac, t, c_{as, t} a] - [c_{as, t}^2, c_{as, t}] + [c_{as, t}^2, c_{as, t}].
\]

770
Similarly, we have
\[ W_2(b, s, t) = [c_{s,t}b, bc_{s,t}] - [c_{s,t}, c_{s,t}] + [c_{s,t}b, c_{s,t}]. \]
Therefore,
\[ \{X_i, U_k, [ba, ab], W_1(a, s, t), W_2(b, s, t), [c_{s,t}, c_{s,t}', c_{s,t}', c_{s,t}]: i \in I; k \in K; a \in A; \]
\[ b \in B; s, s' \in S; t, t' \in T ((s', t') \prec (s, t)) \}

is a generating set for \( \ker \tilde{\theta}_2 \).

Notice that \( W_1(a, s, t), W_2(b, s, t) \) and \([c_{s,t}, c_{s,t}', c_{s,t}', c_{s,t}] \) are also in the generating set for \( \im \tilde{\theta}_3 \) given above, and so
\[ H_2(S \square T) = \langle X_i, U_k, [ab, ba] (i \in I, k \in K, a \in A, b \in B) \mid \]
\[ Y_j = 0, W_1 = 0, W(ra, p) = 0, W(bu, v) = 0 \]
\[ (j \in J, l \in L, (ra, p) \in R, (bu, v) \in Q) \]
\[ = H_2(S) \times H_2(T) \times \langle [ab, ba] (a \in A, b \in B) \mid W(ra, p) = 0, \]
\[ W(bu, v) = 0 ((ra, p) \in R, (bu, v) \in Q) \].

Since \( \langle [ab, ba] (a \in A, b \in B) \mid W(ra, p) = 0, W(bu, v) = 0, ((ra, p) \in R, (bu, v) \in Q) \) is equal to \( H_1(S) \otimes_2 H_1(T) \), from Lemma 2.3, the proof is complete. \( \square \)

Notice that one may consider the Schützenberger product \( S \square T \) as “a kind of direct product” of the monoids \( S \times T \) and the free semilattice over \( S \times T \) (the monoid considered as the set of all subsets of \( S \times T \) with set-theoretical union as a multiplication). Therefore, from [1, Proposition 3.1] and [3, Equation (1), p. 282], the result in the last theorem is perhaps not surprising.

4. Remark
In [1, Theorem 3.3] it was shown that if \( A \) is a finite nonempty set of size \( n \), then
\[ \text{def}_{SL} = n(n - 1)/2, \tag{3} \]
and for \( n \geq 2 \) \( SL_A \) is inefficient, where \( SL_A \) is the set of all nonempty subsets of \( A \) with set-theoretic union as multiplication.

For convenience, first we state a probably well-known lemma that can be proved easily.

**Lemma 4.1** Let \( S \) be a monoid, \( P = \langle A \mid R \rangle \) be a presentation of \( S \), \( T \) be a subsemigroup of \( S \), and \( S \backslash T \) be an ideal of \( S \). Then \( T \) has a presentation \( \langle B \mid Q \rangle \) such that \( B \subset A \) and \( Q \subset R \).

**Corollary 4.2** If \( S \) and \( T \) are two finite monoids without any left or right invertible element, then \( S \square T \) is inefficient.

**Proof** Consider the sets
\[ U = \{(1_S, X, 1_T) \mid X \subset S \times T \} \]
\[ V = (S \square T) \setminus U = \{(s, X, t) \in S \square T \mid (s, t) \neq (1_S, 1_T) \}. \]
It is clear that \( U \) is a subsemigroup of \( S \square T \) and isomorphic to the free semilattice \( SL_{S \times T} \). Moreover, \( V \) is an ideal of \( S \square T \). It follows from Lemma 4.1, Equation (3), and Theorem 3.2 that \( S \square T \) is inefficient. \( \square \)
References


