Abstract: In this article we introduce the concept of $r$-ideals in commutative rings (note: an ideal $I$ of a ring $R$ is called $r$-ideal, if $ab \in I$ and $\text{Ann}(a) = (0)$ imply that $b \in I$ for each $a, b \in R$). We study and investigate the behavior of $r$-ideals and compare them with other classical ideals, such as prime and maximal ideals. We also show that some known ideals such as $z^0$-ideals are $r$-ideals. It is observed that if $I$ is an $r$-ideal, then so too is a minimal prime ideal of $I$. We naturally extend the celebrated results such as Cohen’s theorem for prime ideals and the Prime Avoidance Lemma to $r$-ideals. Consequently, we obtain interesting new facts related to the Prime Avoidance Lemma. It is also shown that $R$ satisfies property $A$ (note: a ring $R$ satisfies property $A$ if each finitely generated ideal consisting entirely of zerodivisors has a nonzero annihilator) if and only if for every $r$-ideal $I$ of $R$, $I[x]$ is an $r$-ideal in $R[x]$. Using this concept in the context of $C(X)$, we show that every $r$-ideal is a $z^0$-ideal if and only if $X$ is a $\partial$-space (a space in which the boundary of any zeroset is contained in a zeroset with empty interior). Finally, we observe that, although the socle of $C(X)$ is never a prime ideal in $C(X)$, the socle of any reduced ring is always an $r$-ideal.

Key words: $r$-ideal, pr-ideal, annihilator, property $A$, zerodivisor, uz-ring, $z^0$-ideal, $r$-multiplicatively closed, almost $P$-space, $\partial$-space, socle

1. Introduction

Throughout this paper all rings are commutative with $1 \neq 0$. Let $R$ be a ring. For $a \in R$ we define $\text{Ann}_R(a) = \{r \in R : ra = 0\}$ (briefly, $\text{Ann}(a)$) and $a$ is said to be a regular (resp., zerodivisor) element if $\text{Ann}(a) = (0)$ (resp., $\text{Ann}(a) \neq (0)$). $aR$ denotes the principal ideal generated by $a \in R$. If $S$ is a subset of $R$ and $I$ is an ideal of $R$, then we define $(I : S) = \{a \in R : aS \subseteq I\}$, clearly $(0 : S) = \text{Ann}(S)$. By $r(R)$, $zd(R)$, and $u(R)$ we mean the set of all regular elements, zerodivisor elements, and unit elements of $R$, respectively. An ideal $I$ of $R$ is called a regular ideal if it contains at least a regular element, i.e. $I \cap r(R) \neq \emptyset$. If $I$ is an ideal of $R$, then $\text{Min}(I)$ denotes the set of all minimal prime ideals of $I$ and we use $\text{Min}(R)$ instead of $\text{Min}((0))$. Similarly, $\text{Max}(R)$ (resp., $\text{Spec}(R)$) denotes the set of all maximal (resp., prime) ideals of $R$. For each $a \in R$, $P_a$ (resp., $M_a$) is the intersection of all minimal prime (resp., maximal) ideals containing $a$. We use $\text{rad}(R)$ (resp., $\text{Jac}(R)$) instead of $P_0$ (resp., $M_0$). A proper ideal $I$ of $R$ is called a $z^0$-ideal (resp., $z$-ideal) if for each $a \in I$ we have $P_a \subseteq I$ (resp., $M_a \subseteq I$). Equivalently, $I$ is a $z^0$-ideal if $a \in I$, $b \in R$, and $\text{Ann}(a) = \text{Ann}(b)$ imply that $b \in I$. For more information about the aforementioned ideals in general commutative rings we refer the reader to [2], [8], [26]. If $S$ is a subset of $R$, then an element $a \in S$ is called a

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von Neumann regular element if there exists $b \in S$ such that $a = a^2b$. Whenever we say a ring $R$ or a subset of $R$ is von Neumann regular, it means that all of their elements are von Neumann regular. An ideal $I$ in a ring $R$ is called a pure ideal if for each $a \in I$ there exists $b \in I$ such that $a = ab$. Let us also recall the following properties: A ring $R$ satisfies a) property A: if each finitely generated (briefly, f.g.) ideal $I \subseteq \text{zd}(R)$ has nonzero annihilator; b) annihilator condition (briefly, a.c.): if for each f.g. ideal $I$ of $R$ there exists an element $b \in R$ with $\text{Ann}(I) = \text{Ann}(b)$; c) strong annihilator condition (briefly, s.a.c.): if for each f.g. ideal $I$ of $R$ there exists an element $b \in I$ with $\text{Ann}(I) = \text{Ann}(b)$. We refer the reader to [1, 2, 18, 25] for the necessary background about the above concepts.

Let $C(X)$ (resp., $C^*(X)$) be the ring of (resp., bounded) real valued continuous functions on a Tychonoff space $X$. If $f \in C(X)$, then $Z(f) = \{x \in X : f(x) = 0\}$ is the zero set of $f$ and by int$Z(f)$ we mean the interior of $Z(f)$. Recall that an ideal $I$ of $C(X)$ is a $z$-ideal if $f \in I$, $g \in C(X)$, and $Z(f) = Z(g)$ imply that $g \in I$. It is known that if $f, g \in C(X)$, then int$Z(f) = \text{int}Z(g)$ if and only if $\text{Ann}(f) = \text{Ann}(g)$; see [5]. Hence, an ideal $I$ in $C(X)$ is a $z^o$-ideal if $f \in I$, $g \in C(X)$ and int$Z(f) = \text{int}Z(g)$ imply that $g \in I$; see [7, 9]. For more information about the ideals in $C(X)$, see [7, 10, 12, 16], and for details about topological spaces, see [14, 16].

In Section 2, we introduce $r$-ideals and $pr$-ideals in general commutative rings. It is shown that every $z^o$-ideal is an $r$-ideal, and if $I$ is an $r$-ideal of $R$ and $P \in \text{Min}(I)$, then $P$ is an $r$-ideal, too. We also show in this section that the socle of every reduced ring is an $r$-ideal. In Section 3, we investigate the relations between $r$-ideals and prime ideals. We observe that every maximal $r$-ideal in a ring is a prime ideal. We show that in order for every prime $r$-ideal of a ring $R$ to be minimal prime, it is necessary and sufficient that the classical ring of quotients of $R$ be a von Neumann regular ring. Finally, we naturally extend the celebrated results such as Cohen’s theorem for prime ideals and the Prime Avoidance Lemma to $r$-ideals. In Section 4, we observe that whenever $I$ is an ideal of a ring $R$ and $I[x]$ is an $r$-ideal, then trivially $I$ is also an $r$-ideal, but the converse may not be true. In this section, we prove a ring $R$ satisfies property A if and only if for every $r$-ideal $I$ of $R$, $I[x]$ is an $r$-ideal in $R[x]$. Section 5 is devoted to the investigation of $r$-ideals in $C(X)$. We show that every $r$-ideal is a $z^o$-ideal if and only if $X$ is a $\partial$-space. It is observed that every ideal in $C(X)$ is an $r$-ideal if and only if $X$ is an almost $P$-space. Using some appropriate facts in $C(X)$, we answer some natural questions in general. By giving several examples, we compare and contrast $r$-ideals with some well-known ideals, such as $z$-ideals and $z^o$-ideals.

2. $r$-ideals

Our aim in this section is to study the $r$-ideals in commutative rings. We begin with the following definition.

**Definition 2.1** A proper ideal $I$ in a ring $R$ is called an $r$-ideal (resp., $pr$-ideal), if $ab \in I$ with $\text{Ann}(a) = (0)$ implies that $b \in I$ (resp., $b^n \in I$, for some $n \in \mathbb{N}$), for each $a, b \in R$.

Let $I$ be an ideal of $R$ and $S$ be a multiplicatively closed (briefly, m.c.) subset in $R$. Clearly, $I_S = \{x \in R : sx \in I \text{ for some } s \in S\}$ is an ideal of $R$ containing $I$. Now we call an ideal $I$ an $s$-ideal if $I = I_S$, for some m.c. subset $S$ of $R$. In case $S = r(R)$, each $s$-ideal is an $r$-ideal. Recall that if $S = r(R)$, then the ring $S^{-1}R$ is called the classical ring of quotients of $R$, which is denoted by $Q(R)$. Let $\varphi : R \to Q(R)$ be the natural homomorphism. For each ideal $\mathcal{J}$ in $Q(R)$, we put $\varphi^{-1}(\mathcal{J}) = \mathcal{J}^e$. Clearly, $\mathcal{J}^e$ is an ideal of $R$ and it is called the contraction of $\mathcal{J}$ in $R$. For the details of the concept of contraction, see [3].
Proposition 2.2 Let $R$ be a ring and $I$ be an ideal of $R$. Then the following statements are equivalent:

a) $I$ is an $r$-ideal.

b) $rR \cap I = rI$, for each $r \in r(R)$.

c) $I = (I : r)$, for each $r \in r(R) \setminus I$.

d) $I = J^c$, where $J$ is an ideal in $Q(R)$.

Proof It is evident. □

Recall that part (c) of the previous proposition is similar to this statement about prime ideals, which says that a proper ideal $P$ of a ring $R$ is prime if and only if $P = (P : a)$, for each $a \in R \setminus P$. We should remind the reader that part (b) of the previous proposition may not be true if $I$ is a prime ideal. The reason that part (b) is valid for an $r$-ideal $I$ is the fact $I \cap r(R) = \emptyset$; this immediately implies that part (b) is trivially true for prime ideal $P$ with $P \cap r(R) = \emptyset$.

We observe several elementary properties concerning $r$-ideals in any ring $R$ as follows:

Remark 2.3 a) If $f : R \to S$ is an isomorphism, then $f[I]$ is an $r$-ideal in $S$ whenever $I$ is an $r$-ideal in $R$, and $f^{-1}[J]$ is an $r$-ideal in $R$ whenever $J$ is an $r$-ideal in $S$.

b) The zero ideal is an $r$-ideal.

c) The intersection of any family of $r$-ideals is an $r$-ideal.

d) If $I$ is an $r$-ideal, then $I \subseteq \zd(R)$.

e) Every $r$-ideal is a $pr$-ideal.

f) A prime ideal is an $r$-ideal if and only if it consists entirely of zerodivisors. Consequently, every minimal prime ideal is an $r$-ideal.

g) If $I$ is an $r$-ideal, $S \subseteq R$ and $S \nsubseteq I$, then $(I : S)$ is an $r$-ideal. In particular, $\Ann(S)$ is always an $r$-ideal.

h) It is well known that if $I$ is a minimal ideal of a reduced ring $R$, and then $I = eR = \Ann(1 - e)$, where $e \in R$ is an idempotent element, i.e. $e^2 = e$. Hence, by part (g), every minimal ideal in a reduced ring is an $r$-ideal.

i) Every pure ideal and also every von Neumann regular ideal is an $r$-ideal.

j) If $R$ satisfies the s.a.c., and $I$ is an ideal of $R$, then $I$ is an $r$-ideal if and only if for every ideal of $J$ and $K$ of $R$, whenever $JK \subseteq I$ and $\Ann(J) = (0)$, then $K \subseteq I$.

k) The product of two $r$-ideals is not necessarily an $r$-ideal; see Example [?].

l) The sum of two $r$-ideal is not necessarily an $r$-ideal; see Example [?].

Remark 2.4 It is well known that $I^cJ^c \subseteq (IJ)^c$ and $I^c + J^c \subseteq (I + J)^c$, where $I$ and $J$ are ideals of $Q(R)$. Now suppose that $I$ and $J$ are $r$-ideals of $R$; hence, by part (d) of Proposition [?], $I = I^c$ and $J = J^c$, for some ideals $I$ and $J$ in $Q(R)$. One can easily show that:

a) $IJ$ is an $r$-ideal in $R$ if and only if $(IJ)^c \subseteq I^c J^c$ (in fact, $(IJ)^c = I^c J^c$).

b) $I + J$ is an $r$-ideal in $R$ if and only if $(I + J)^c \subseteq I^c + J^c$ (in fact, $(I + J)^c = I^c + J^c$).

We need the following lemma in the sequel.

Lemma 2.5 Let $R$ be a ring and $I$ be an ideal of $R$. Then:
a) $I$ is an $r$-ideal if and only if whenever $J$ and $K$ are ideals of $R$ with $J \cap r(R) \neq \emptyset$ and $JK \subseteq I$, then $K \subseteq I$.

b) If $I \subseteq zd(R)$ is not an $r$-ideal, then there exist ideals $J$ and $K$ such that $J \cap r(R) \neq \emptyset$, $I \nsubseteq J,K$, and $JK \subseteq I$.

**Proof** a) It is evident.

b) Suppose that $I$ is not an $r$-ideal. Then there exist $r \in r(R)$, $x \in R$ with $rx \in I$ but $x \notin I$. Now put $J = (I : x)$ and $K = (I : J)$. Clearly, $r \in J \setminus I$, $J \cap r(R) \neq \emptyset$, $x \in K \setminus I$, and $JK \subseteq I$.

The proof of the following result is evident by the above lemma.

**Proposition 2.6** a) Let $R$ be a ring and $I$ be an ideal of $R$ with $I \cap r(R) \neq \emptyset$. If $J$ and $K$ are $r$-ideals of $R$ such that $IJ = IK$ or $I \cap J = I \cap K$, then $J = K$.

b) Let $R$ be a ring and $I$ and $J$ be ideals of $R$ with $J \cap r(R) \neq \emptyset$. If $IJ$ is an $r$-ideal of $R$, then $I = JJ$. In particular, $I$ is an $r$-ideal.

In Remark [?], we observe that an intersection of $r$-ideals is an $r$-ideal. In the following proposition we show that the converse is also true for prime ideals in the finite case. The result may not be true for an infinite number of primes; take the intersection of nonzero prime ideals in $\mathbb{Z}$.

**Proposition 2.7** Suppose that $P_1, \ldots, P_n$ are prime ideals in a ring $R$, which are not comparable. If $\bigcap_{i=1}^n P_i$ is an $r$-ideal, then $P_i$ is an $r$-ideal, for $i = 1, \ldots, n$.

**Proof** Let $rx \in P_i$ with $\text{Ann}(r) = (0)$ and take $y \in (\prod_{j \neq i} P_j) \setminus P_i$. Hence, $rxy \in \bigcap_{i=1}^n P_i$. Since $\bigcap_{i=1}^n P_i$ is an $r$-ideal, we infer that $xy \in \bigcap_{i=1}^n P_i$, and therefore $xy \in P_i$. This implies that $x \in P_i$, i.e., $P_i$ is an $r$-ideal. □

It is well known that a ring $R$ is a field if and only if $I = (0)$ is the only maximal ideal of $R$. However, we cannot extend this to domains by claiming that $R$ is a domain if and only if $I = (0)$ is its only prime ideal. By trading off the prime ideals with the $r$-ideals, we get the next interesting fact.

**Proposition 2.8** Let $R$ be a ring. Then the following statements are equivalent:

a) $R$ is a domain.

b) The zero ideal is the only $r$-ideal of $R$.

c) $\text{Ann}(ab) = \text{Ann}(a) \cup \text{Ann}(b)$, for every $a, b \in R$.

**Proof** (a $\Rightarrow$ b) Let $R$ be a domain and $(0) \neq I$ be a proper ideal of $R$. Hence, there exists $0 \neq a \in I$. By our hypothesis, we have $\text{Ann}(a) = (0)$, so $I$ is not an $r$-ideal (note: otherwise $1 \in I$, which is absurd).

(b $\Rightarrow$ c) We know that $\text{Ann}(x)$ is an $r$-ideal, for each $0 \neq x \in R$. Hence, by our hypothesis, we have $\text{Ann}(x) = (0)$, for each $0 \neq x \in R$. This immediately implies that $\text{Ann}(ab) = \text{Ann}(a) \cup \text{Ann}(b)$, for each $a, b \in R$.

(c $\Rightarrow$ a) Let $ab = 0$, where $a, b \in R$. Then $R = \text{Ann}(ab) = \text{Ann}(a) \cup \text{Ann}(b)$ implies that $1 \in \text{Ann}(a) \cup \text{Ann}(b)$. This means that $a = 0$ or $b = 0$, i.e. $R$ is a domain. □

**Remark 2.9** We should remind the reader that part (d) of Proposition [?] is quite natural with regard to some known facts. For example, if $Q$ is the quotient field of a domain $R$, the zero ideal of $R$, which is the only $r$-ideal of $R$, is the contraction of the only proper ideal of $Q$ (i.e. $(0)$). We also note that whenever $P$ is a

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prime ideal in a ring $R$ and $S = R \setminus P$, then each prime ideal of $S^{-1}R$ is contracted to a prime ideal of $R$. Finally, if in a ring $R$, we take $S = r(R)$, then the contractions of all proper ideals of $Q(R)$ are naturally $r$-ideals in $R$ (note: proper ideals of $Q(R)$ are all $r$-ideals).

In Example [?], we will observe that the sum of two $r$-ideals need not be an $r$-ideal. In the following result we show that the sum of two special annihilator ideals of a ring and also the sum of a minimal prime ideal and an annihilator ideal in a reduced ring are $r$-ideal.

**Proposition 2.10**  

a) Let $R$ be a ring and $a, b \in R$ with $a + b = 1$. Then $I = \text{Ann}(a) + \text{Ann}(b)$ is an $r$-ideal.

b) Let $R$ be a reduced ring, $P \in \text{Min}(R)$ and $e \in R$ be an idempotent element. Then $I = P + \text{Ann}(e)$ is an $r$-ideal.

**Proof**  

a) Suppose that $xy \in I$ and $\text{Ann}(x) = (0)$. Hence, there exist $r \in \text{Ann}(a)$ and $s \in \text{Ann}(b)$ such that $xy = r + s$. Clearly, $xyab = 0$, and since $\text{Ann}(x) = (0)$, we infer that $yab = 0$. Consequently, $ya \in \text{Ann}(b)$ and $yb \in \text{Ann}(a)$. Therefore, $y = y(a + b) = ya + yb$, i.e., $y \in I$.

b) Let $rx \in I$ with $\text{Ann}(r) = (0)$ and $x \in R$. Hence, $rx = a + b$, where $a \in P$ and $be = 0$. Clearly, there exists $y \notin P$ such that $ay = 0$. Therefore, $e y r x = 0$, we have $e y x = 0$, and hence $e x \in P$. Now $x = e x + (1 - e)x \in P + \text{Ann}(e) = I$, and therefore $I$ is an $r$-ideal. 

If in the equality $a + b = 1$ of part (a) of the previous proposition, we replace 1 by $R$ and $a, b$ by two subsets $A, B$ in $R$, then $\text{Ann}(A) + \text{Ann}(B)$ will be also an $r$-ideal.

In general, if $R$ is a ring such that every ideal of $R$ is an annihilator ideal (i.e. for every ideal $I$ there exists $S \subseteq R$ such that $I = \text{Ann}(S)$), then every ideal of $R$ is an $r$-ideal. Also, if for any two ideals $I$ and $J$ in the ring $R$, there exists an ideal $K$ such that $\text{Ann}(I) + \text{Ann}(J) = \text{Ann}(K)$, then $\text{Ann}(I) + \text{Ann}(J)$ is an $r$-ideal. We should remind the reader that the latter case may happen in certain rings. In what follows we mention some examples. We recall that if $X$ is an extremally disconnected space (i.e. every open subset of $X$ has an open closure), then $C(X)$ has the above property; see [6]. In [11], the concepts of $SA$-ring and $IN$-ring are introduced and it is shown that these rings also satisfy the above property. We should also emphasize that in contrast with the latter fact the sum of two $r$-ideals is not necessarily an $r$-ideal in general; we refer the reader to Example 5.14 in this regard. However, it is worthwhile to remind the reader that any direct summand of an $r$-ideal is always an $r$-ideal (i.e. if $I = J \oplus K$, and $I$ is an $r$-ideal, then so too are $J$ and $K$).

**Remark 2.11** In contrast to the latter fact the summand of prime ideals may not be prime. To see this, take a von Neumann regular ring that is not a finite direct product of fields, and then take a prime ideal $P$ that is not $f.g.$ (note: von Neumann regular rings that are not a finite direct product of fields cannot be Noetherian; hence, by Cohen’s theorem, it contains a prime ideal that is not $f.g.$), and notice that all of its $f.g.$ subideals are direct summands, which are not prime ideal.

Recall that the socle of a ring $R$, which is denoted by $\text{soc}(R)$, is the sum of all minimal ideals of $R$. We also recall that the socle of a reduced ring $R$ is of the form $\text{soc}(R) = \bigoplus_{i \in A} e_i R$, where $\{e_i : i \in A\}$ is the set of idempotents of $R$; see [23]. By the following proposition we observe that the sum of principal ideals generated by idempotents is an $r$-ideal, from which the socle of a reduced ring is an $r$-ideal. We know that the socle plays an important role in the structure theory of rings, especially in the context of noncommutative rings and
$C(X)$. For details about the socle in general rings, see [[23]], and for a topological characterization of the socle of $C(X)$, see [[22]].

**Proposition 2.12** Let $R$ be a ring, and \( \{e_i : i \in A\} \) is a set of idempotents of $R$. Then $I = \sum_{i \in A} e_i R$ is an $r$-ideal.

**Proof** Let $rx \in I$, where $x \in R$ and $\text{Ann}(r) = (0)$. We are to show that $x \in I$. Since $I = \sum_{i \in A} e_i R$, we infer that $rx = \sum_{k=1}^{n} e_{ik} r_{ik}$ for some $i_1, \ldots, i_n \in A$ and $r_{i_1}, \ldots, r_{i_k} \in R$. Let us put $y = \prod_{k=1}^{n} (1 - e_{ik})$. It is manifest that $rxy = 0$, and hence $xy = 0$. On the other hand, there exists $s \in I$ such that $y = 1 - s$. Therefore, $x(1-s) = 0$, so $x = xs \in I$.

**Corollary 2.13** Let $R$ be a reduced ring. Then $\text{soc}(R)$ is an $r$-ideal. In particular, there exists an ideal $J$ of $Q(R)$ such that $\text{soc}(R) = J^c$.

It is interesting that in $C(X)$, where $X$ is an infinite topological space, the socle of $C(X)$ is an $r$-ideal that is not prime; see [[4], [15]].

**Remark 2.14** Let $M$ be a projective $R$-module, where $R$ is a von Neumann regular ring. Then $M$ is isomorphic to a direct sum of countably generated $r$-ideals. To see this, we note that by a celebrated theorem of Kaplansky $M = \bigoplus_{i \in A} M_i$, where each $M_i$ is a countably generated submodule of $M$. Since $M$ is a regular module (i.e. each cyclic submodule of $M$ is a direct summand), we infer that each $M_i = \bigoplus_{n=1}^{\infty} x_n R$ is regular too. Hence, by [[[20]], Lemma 2], we conclude that $M_i \cong \bigoplus_{n=1}^{\infty} e_n R$, where each $e_n$ is idempotent. Now by Proposition 2.12, each $M_i$ is isomorphic to an $r$-ideal, and we are done.

We recall that in the ring $C(X)$, the sum of two minimal prime ideals is either a prime ideal or all of $C(X)$; see [[16]]. In contrast to this fact, the sum of two minimal prime ideals in general is not necessarily an $r$-ideal; see also the next example.

**Example 2.15** Let $R = \frac{F[x,y]}{xy F[x,y]}$, where $F$ is a field. Then $P = \frac{x F[x,y]}{xy F[x,y]}$ and $Q = \frac{y F[x,y]}{xy F[x,y]}$ are minimal prime ideals of $R$. Clearly, $P + Q \neq R$ and $(x + y) + xy F[x,y] \in P + Q$ is a regular element. Hence, $P + Q$ is not an $r$-ideal.

The following is a counterpart of the well-known fact that $Q$ is a primary ideal of a ring $R$ if and only if $\sqrt{Q}$ is a prime ideal.

**Proposition 2.16** Let $R$ be a ring and $I$ be an ideal of $R$. Then $I$ is a $pr$-ideal if and only if $\sqrt{I}$ is an $r$-ideal.

**Proof** Suppose that $I$ is a $pr$-ideal and $ab \in \sqrt{I}$ with $\text{Ann}(a) = (0)$. Then there exists $n \in \mathbb{N}$ such that $a^n b^n \in I$. Clearly, $\text{Ann}(a^n) = (0)$, so there exists $m \in \mathbb{N}$ such that $b^{nm} \in I$ and therefore $b \in \sqrt{I}$. Conversely, we assume that $ab \in I$ with $\text{Ann}(a) = (0)$. Since $ab \in \sqrt{I}$ we infer that $b \in \sqrt{I}$ and so there exists $n \in \mathbb{N}$ such that $b^n \in I$.

As we observed in the previous proposition, whenever $\sqrt{I}$ is an $r$-ideal, then $I$ is an $pr$-ideal. In the following example, we show that $\sqrt{I}$ may be an $r$-ideal where $I$ may not be an $r$-ideal. This example also shows that a $pr$-ideal is not necessarily an $r$-ideal.
Example 2.17 Let $S$ be a reduced ring with subring $\mathbb{Z}$ and $P \neq (0)$ be a minimal prime ideal in $S$ with $P \cap \mathbb{Z} = (0)$. By [[[10]], Lemma 3.6], $Q = xP[x] \subseteq S[x]$ is a minimal prime ideal in $R = \mathbb{Z} + xS[x]$, and hence it is also an $r$-ideal. Now we consider $Q_n = x^nP[x]$ with $1 \neq n \in \mathbb{N}$. Clearly, $\sqrt{Q_n} = Q$ is an $r$-ideal and by Proposition [?] we conclude that $Q_n$ is a $pr$-ideal. We claim that $Q_n$ is not an $r$-ideal. To see this, put $f(x) = x^{n-1}a$, where $0 \neq a \in P$ and $g(x) = x$. Thus, $f(x)g(x) = x^na \in Q_n$. Now it is clear that $\text{Ann}(g) = (0)$ and $f \notin Q_n$. Consequently, $Q_n$ is not an $r$-ideal.

Clearly, if $I$ and $J$ are $r$-ideals in a ring $R$, then $IJ$ is a $pr$-ideal of $R$, but it may not be an $r$-ideal; for instance, in the previous example, the ideal $Q$ is an $r$-ideal, while $Q^2$ is not an $r$-ideal (note: for a prime ideal $P$, $P^2$ is prime if and only if $P^2 = P$).

Using the previous proposition and Proposition [?], we have the next corollary.

Corollary 2.18 Let $R$ be a ring and $I$ be an ideal of $R$. Then the following statements are equivalent:

a) $I$ is a $pr$-ideal.

b) $rR \cap \sqrt{I} = r\sqrt{I}$, for any $r \in r(R)$.

c) $\sqrt{I} = \sqrt{(I : r)}$, for any $r \in r(R) \setminus I$.

d) $I = J^c$, where $J$ is a primary ideal in $Q(R)$.

In the next section we will show that an $r$-ideal is not necessarily a $z^0$-ideal; see part (d) of Remark [?]. In the following theorem, however, we observe that the converse holds.

Theorem 2.19 a) Every $z^0$-ideal in a ring $R$ is an $r$-ideal.

b) Every ideal consisting entirely of zerodivisors in a ring is contained in a prime $r$-ideal.

Proof a) Let $I$ be a $z^0$-ideal, $ab \in I$ and $\text{Ann}(a) = (0)$. Clearly, $\text{Ann}(b) = \text{Ann}(ab)$. Since $I$ is a $z^0$-ideal, we conclude that $b \in I$.

b) It is evident. \( \Box \)

Let $S$ be a m.c. subset of a reduced ring $R$. Clearly, $I = \sum_{a \in S} \text{Ann}(a)$ is a $z^0$-ideal, so by part (a) of the previous theorem, $I$ is also an $r$-ideal.

We remind the reader that if $I$ is a $z^0$-ideal (resp., $z$-ideal) and $P \in \text{Min}(I)$, then $P$ is a $z^0$-ideal (resp., $z$-ideal); see [[[8]], Theorem 1.16] (resp., see [[[10], [26]])]. The following is a similar result.

Theorem 2.20 Let $R$ be a ring and $P \in \text{Min}(I)$, where $I$ is an $r$-ideal of $R$. Then $P$ is an $r$-ideal.

Proof Suppose that $ab \in P$ and $\text{Ann}(a) = (0)$. By [[[18]], Theorem 1.2], there exist $x \notin P$ and $n \in \mathbb{N}$ such that $x(ab)^n = xa^nb^n \in I$. Since $\text{Ann}(a^n) = (0)$ and $I$ is an $r$-ideal, we infer that $xb^n \in I \subseteq P$. Since $x \notin P$, we infer that $b^n \in P$ and therefore $b \in P$. \( \Box \)

We conclude this section with the following example and the proposition that follows it.

Example 2.21 For two $r$-ideals $I$ and $J$ of $R$, with $J \supseteq I$, the ideal $\frac{I}{J}$ of $\frac{R}{J}$ may not be an $r$-ideal in $\frac{R}{J}$.

To see this, suppose that $P \in \text{Min}(R)$ and $M \in \text{Max}(R)$ such that $P \subsetneq M \subseteq zd(R)$; for maximal ideals of this kind, see [[[8]]]. Clearly, $P$ and $M$ are $r$-ideals of $R$. However, $(0) \neq \frac{M}{P}$ and $\frac{R}{P}$ is a domain, so $\frac{M}{P}$ is not an $r$-ideal of $\frac{R}{P}$.
Proposition 2.22 Let $I$ be an $r$-ideal in $R$ contained in ideal $J$. If $\frac{J}{I}$ is an $r$-ideal in $\frac{R}{I}$, then $J$ is also an $r$-ideal in $R$.

Proof It is evident. \hfill $\Box$

3. $r$-ideals vs. prime ideals

This section is devoted to the relations between $r$-ideals and prime ideals and natural extensions of Cohen’s theorem and the Prime Avoidance Lemma for $r$-ideals. We start with the following proposition.

Proposition 3.1 Let $R$ be a ring. Then every maximal $r$-ideal of $R$ is a prime ideal.

Proof Suppose that $P$ is a maximal $r$-ideal of $R$, $xy \in P$ and $x \notin P$, and we are to show that $y \in P$. Clearly, $(P : x)$ is an $r$-ideal, $P \subseteq (P : x)$ and $y \in (P : x)$. Now by the maximality of $P$ we have $P = (P : x)$. This implies that $y \in P$. \hfill $\Box$

Using [[8]], Corollary 1.22, every maximal ideal consisting entirely of zerodivisors in a reduced ring with property $A$ is a $z^o$-ideal. In the following proposition we show that maximal $r$-ideals in reduced rings with property $A$ are also $z^o$-ideals.

Proposition 3.2 Let $R$ be a reduced ring with property $A$. Then every maximal $r$-ideal of $R$ is a $z^o$-ideal.

Proof Suppose that $P$ is a maximal $r$-ideal of $R$. Therefore, $P \subseteq \text{zd}(R)$, and so by [[8]], Proposition 1.21, there is a $z^o$-ideal $J$ such that $P \subseteq J$. By part (a) of Theorem [8], $J$ is an $r$-ideal. Now the maximality of $P$ implies that $P = J$. Hence, $P$ is a $z^o$-ideal. \hfill $\Box$

Recall that a nonzero ideal $I$ in a ring $R$ is called essential if for every nonzero ideal $J$ of $R$ we have $I \cap J \neq (0)$.

Proposition 3.3 Let $I$ be a nonzero $r$-ideal of a reduced ring $R$, which is not essential. Then there is a minimal prime ideal $P$ containing $I$, which is a maximal $r$-ideal.

Proof Since $I$ is not an essential ideal, there is a nonzero ideal $J$ of $R$ such that $I \cap J = (0)$. Since $R$ is reduced and $(0) \neq J$, we infer that there exists $P \in \text{Min}(R)$ such that $J \nsubseteq P$ and hence there exists $x \in J \setminus P$. On the other hand, by Zorn’s Lemma, there exists a maximal $r$-ideal $N$ containing $I$ such that $N \cap J = (0)$. Hence, $JN = (0)$; that is to say, $xN = (0) \subseteq P$. Now we conclude that $N \subseteq P$ and so $I \subseteq N = P$. (Note that $N$ is a prime ideal by Proposition [8].) \hfill $\Box$

It is well known that every element of $Q(R)$ is either a unit or a zerodivisor. Motivated by this fact, we call a ring $R$ a $uz$-ring if every element of $R$ is either a unit or a zerodivisor. In this case, clearly $R = Q(R)$. For example, every von Neumann regular ring and any Artinian ring is a $uz$-ring. If $R$ is a domain, then obviously $R$ is a field if and only if $R$ is a $uz$-ring. Clearly, a ring $R$ is a field if and only if every ideal in $R$ is prime. Similarly, $R$ is a $uz$-ring if and only if every ideal in $R$ is an $r$-ideal. More generally, we have the following result.

Proposition 3.4 For any ring $R$ the following statements are equivalent:

a) $R$ is a $uz$-ring.

b) Every essential ideal of $R$ is an $r$-ideal.

c) Every principal ideal of $R$ is an $r$-ideal.
d) Every prime ideal of $R$ is an $r$-ideal.

e) Every maximal ideal of $R$ is an $r$-ideal.

**Proof** It is evident. □

The proof of the next result is similar to the proof of [[8], Proposition 1.26].

**Proposition 3.5** Let $R$ be a reduced ring. Then $Q(R)$ is a von Neumann regular ring if and only if every prime $r$-ideal of $R$ is a minimal prime ideal.

**Proof** Let $Q(R)$ be a von Neumann regular ring and $P$ be a prime $r$-ideal of $R$ that is not minimal prime, and seek a contradiction. Therefore, there exists $a \in P$ such that $\text{Ann}_R(a) \subseteq P$. Hence, $\frac{a}{1} \in S^{-1}P$ and $\text{Ann}_R(\frac{a}{1}) \subseteq S^{-1}P$. We conclude that $S^{-1}P \notin \text{Min}(Q(R))$, which is a contradiction. Conversely, since $R$ is reduced, by a well-known theorem of Kaplansky on characterization of von Neumann regular rings, it suffices to show that each prime ideal is a minimal prime ideal. To see this, we prove in fact that each maximal ideal is a minimal prime ideal. Let $\mathcal{M} \in \text{Max}(Q(R))$; since $Q(R)$ is a $uz$-ring, we have $\mathcal{M} \subseteq zd(Q(R))$, so $\mathcal{M}$ is a $z^a$-ideal of $Q(R)$. Hence, $\mathcal{M}^c = \mathcal{M} \cap R$ is a prime $z^a$-ideal of $R$ and so it is a prime $r$-ideal of $R$, too. Now by our hypothesis we conclude that $\mathcal{M}^c \in \text{Min}(R)$. Therefore, $\mathcal{M} \in \text{Min}(Q(R))$. This implies that $Q(R)$ is a von Neumann regular ring. □

In the following result we characterize the regularity of $Q(R)$ in terms of $r$-ideals of $R$. Recall that an ideal $I$ is semiprime if $\sqrt{I} = I$.

**Proposition 3.6** Let $R$ be a ring. Then:

a) $Q(R)$ is a von Neumann regular ring if and only if every $r$-ideal of $R$ is a semiprime ideal.

b) If $IJ = I \cap J$, where $I$ and $J$ are $r$-ideals of $R$, then $Q(R)$ is a von Neumann regular ring.

c) If every $r$-ideal of $R$ is idempotent, then $Q(R)$ is a von Neumann regular ring.

**Proof** It is evident. □

The following proposition is a counterpart of the celebrated Prime Avoidance Lemma for $r$-ideals; see [[21]] for recent work on this lemma. First we need the next definition.

**Definition 3.7** Let $B \subseteq \bigcup_{i=1}^n A_i$, where $B$, $A_i$s are subsets of a ring $R$. This inclusion is called irreducible if no $A_i$ can be removed from the union.

**Theorem 3.8** Let $I \subseteq \bigcup_{i=1}^n J_i$, where $I$ and $J_i$s are ideals of a ring $R$, be an irreducible inclusion. If $J_1$ is an $r$-ideal and the others have regular elements, then $I \subseteq J_1$.

**Proof** Since $I \notin \bigcup_{i=2}^n J_i$, there exists $a \in I \setminus \bigcup_{i=2}^n J_i$. This implies that $a \in J_1$. Let $x \in I \cap (\bigcap_{i=2}^n J_i)$; clearly $x + a \notin \bigcup_{i=2}^n J_i$. Since $x + a \in I \subseteq \bigcup_{i=1}^n J_i$, we infer that $x \in J_1$. This implies that $I \cap (\bigcap_{i=2}^n J_i) \subseteq J_1$ and hence $I(\bigcap_{i=2}^n J_i) \subseteq J_1$. Since $(\bigcap_{i=2}^n J_i) \cap r(R) \neq \emptyset$, by part (a) of Lemma [?], we conclude that $I \subseteq J_1$. □

The following fact is an interesting variant of the Prime Avoidance Lemma.

**Corollary 3.9** Let $Q \subseteq \bigcup_{i=1}^n P_i$, where $Q$ and $P_i$s are ideals of a ring $R$, be an irreducible inclusion. If $P_1 \in \text{Min}(R)$ and $P_i \cap r(R) \neq \emptyset$, for all $i \geq 2$, then $Q \subseteq P_1$. Moreover, if $Q$ is a prime ideal, then $Q = P_1$, i.e. $Q \in \text{Min}(R)$.  

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Proposition 3.10 Let $R$ be a reduced ring with $|\text{Min}(R)| < \infty$ and $Q \subseteq \bigcup_{i=1}^{n} P_i$, where $Q$ and $P_i$s are ideals of the ring $R$, be an irreducible inclusion. If $P_i \in \text{Min}(R)$ and $P_i$ is an essential ideal for all $i \geq 2$, then $Q \subseteq P_1$. Moreover, if $Q$ is a prime ideal, then $Q = P_1$, i.e. $Q \in \text{Min}(R)$.

Proof Since $R$ is a Goldie ring (see [[[23]], Theorem 11.43]), we infer that each $P_i$ contains a regular element for all $i \geq 2$; see [[[23]], Theorem 11.46]. Consequently, by the above corollary we are done. \hfill\Box

Definition 3.11 Let $R$ be a ring and $S$ be a subset of $R$. We say that $S$ is an r-multiplicatively closed (briefly, r-m.c.) set if $0 \notin S$, $1 \in S$, $S$ contains at least a regular element $t \neq 1$, and $rt \in S$ for all regular elements $r \in S$ and all $x \in S$ (e.g., $S = R \setminus I$, where $I$ is an r-ideal).

We remind the reader that if $S$ is a m.c. subset, then $S' = S \cup u(R) \cup \{ax : a \in A, x \in S\}$ is a m.c. subset containing all units. Clearly, if $I$ is an ideal, then $I \cap S = \emptyset$ if and only if $I \cap S' = \emptyset$. Hence, for all practical purposes we may assume that whenever $S$ is a m.c. subset, then $u(R) \subseteq S$. Note that $P$ is a prime ideal if and only if $S = R \setminus P$ is a m.c. set.

Similarly, let $S$ be an r-m.c. subset and $A$ be a m.c. subset containing a regular element (e.g., $A = \{r^n : n = 0, 1, 2, \ldots\}$, where $r \in r(R)$); then $S' = S \cup A \cup \{ax : a \in A, x \in S\}$ is an r-m.c. subset. In particular, we may take $A$ to be $r(R)$. Hence, from now on we may assume that whenever $S$ is an r-m.c. subset, then $r(R) \subseteq S$ (note: if $I$ is an r-ideal, then $S = R \setminus I$ naturally contains $r(R)$). Therefore, $I$ is an r-ideal of $R$ if and only if $S = R \setminus I$ is an r-m.c. subset.

The following theorem is the counterpart of the celebrated theorem of IS Cohen for r-ideals.

Theorem 3.12 Let $I$ be an ideal of a ring $R$ and $S$ be an r-m.c. subset in $R$ with $I \cap S = \emptyset$. Then there exists an r-ideal $J$ such that $I \subseteq J$ and $J \cap S = \emptyset$.

Proof Put $A = \{K : K$ is an ideal of $R$ such that $I \subseteq K$ and $K \cap S = \emptyset\}$. Clearly, $A \neq \emptyset$, and by Zorn’s Lemma, $A$ has a maximal element, namely $J$, with $I \subseteq J$ and $J \cap S = \emptyset$. We now claim that $J$ is an r-ideal.

Let $rx \in J$, $\text{Ann}_r(rx) = (0)$, and $x \notin J$. We are to seek a contradiction. Clearly, $x \in (J : r)$ and so $J \subseteq (J : r)$. Now it is sufficient to show that $(J : r) \cap S = \emptyset$. To see this, let $t \in (J : r) \cap S$, and then $t \in S$ and $rt \in J$. Since $r \in r(R) \subseteq S$, we infer that $rt \in S$, i.e. $rt \in J \cap S$, which is a contradiction. \hfill\Box

Definition 3.13 Let $S$ be a subset of a ring $R$. We say that $S$ is an r-saturated m.c. subset if $S$ is an r-m.c. Subset, and moreover, when $xy \in S$, then $x, y \in S$ for every $x, y \in R$.

We should bring to the attention of the reader that whenever $A$ is a set of r-ideals, then clearly $S = R \setminus \bigcup_{I \in A} I$ is an r-saturated m.c. subset of $R$. In the following result we aim to show that every r-saturated m.c. subset of $R$ is of the latter form, which is the counterpart of its corresponding fact for saturated m.c. sets.

Proposition 3.14 Let $S$ be an r-saturated m.c. subset of a ring $R$ and

$$A = \{I : I \text{ is an r-ideal of } R \text{ with } I \cap S = \emptyset\}.$$ 

Then $S = R \setminus \bigcup_{I \in A} I$. 

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Proof Since \((0) \cap S = \emptyset\), we infer that \((0) \in \mathcal{A}\). This implies that \(\mathcal{A} \neq \emptyset\) and it is manifest that \(S \subseteq R \setminus \bigcup_{I \in \mathcal{A}} I\).

Now suppose that \(x \in R \setminus \bigcup_{I \in \mathcal{A}} I\) but \(x \notin S\) and seek a contradiction. Since \(xR \cap S = \emptyset\), by the previous theorem there exists an \(r\)-ideal \(I\) containing \(x\) such that \(I \cap S = \emptyset\). Consequently, \(I \in \mathcal{A}\). By our assumption \(x\) does not belong to any member of \(\mathcal{A}\), whereas \(x \in I \in \mathcal{A}\), which is the desired contradiction. \(\square\)

Remark 3.15 Let \(R \subseteq T\) be rings. It is possible that \(J\) is an \(r\)-ideal of \(T\), but \(J \cap R = I\) is not an \(r\)-ideal of \(R\). To see this, let \(A = \mathbb{Z}\) and \(T = \mathbb{Z} \times \mathbb{Z}\). Clearly, \(\varphi: \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}\) defined by \(\varphi(x) = (x, 0)\) is a monomorphism. Then \(R = \varphi(\mathbb{Z})\) is a domain. Also, it is clear that \(J = \text{Ann}((0,1))\) is a nonzero \(r\)-ideal in \(T\). On the other hand, \(R \subseteq J\), and hence \(I = R = J \cap R\) is not an \(r\)-ideal in \(R\).

Definition 3.16 Let \(R\) and \(T\) be rings with \(R \subseteq T\). We say that \(R\) is essential in \(T\), if \(R \cap I \neq (0)\), for every nonzero ideal of \(T\).

For example, \(C^*(X)\) is essential in \(C(X)\). To see this, let \(I\) be an ideal in \(C(X)\) and \(0 \neq f \in I\), and clearly \(0 \neq g = \frac{f}{1 + x^2} \in I \cap C^*(X)\). More generally, \(R\) is essential in \(Q(R)\).

In contrast to the fact in Remark [?], we have the following result.

Proposition 3.17 Let \(R \subseteq T\) be rings such that \(R\) is essential in \(T\). If \(I\) is an \(r\)-ideal in \(T\), then \(I \cap R = J\) is an \(r\)-ideal in \(R\).

Proof Suppose that \(r, x \in R\) and \(rx \in J\) with \(\text{Ann}_R(r) = (0)\). We are to show that \(x \in J\). Clearly, \(rx \in I\). We claim that \(\text{Ann}_T(r) = (0)\). To see this, let \(\text{Ann}_T(r) \neq (0)\), and then by our hypothesis, we have \(\text{Ann}_T(r) \cap R \neq (0)\), so there exists \(0 \neq y \in R\) such that \(y \in \text{Ann}_T(r)\), i.e. \(yr = 0\). Consequently, we have \(y \in \text{Ann}_R(r)\), which is a contradiction. Thus, \(x \in I\) and hence \(x \in J\). \(\square\)

4. \(r\)-ideals in polynomial rings

Let \(R[x]\) denote the ring of polynomials with coefficients in \(R\). If \(f = \sum_{i=0}^{n} f_i x^i \in R[x]\), then the content of \(f\), by definition, is the ideal of \(R\) generated by the coefficients of \(f\) and is denoted by \(c(f)\), and the set of coefficients of \(f\) is denoted by \(C(f)\), i.e. \(C(f) = \{f_0, f_1, \ldots, f_n\}\). If \(I\) is an ideal of \(R\) then \(I[x]\) is denoted by the set \(\{f \in R[x] : C(f) \subseteq I\}\). Also let \(R[[x]]\) be the ring of formal power series with coefficients in \(R\). If \(f = \sum_{i=0}^{n} f_i x^i \in R[[x]]\), then \(C(f)\) is the sequence \(\{f_n\}_{n \in \mathbb{N}}\).

Remark 4.1 a) Let \(R\) be a reduced ring and \(f \in R[x]\); then by [[3]], Theorem 3.3, we have \(\text{Ann}(f) = \text{Ann}(C(f))[x]\). Also, if \(f \in R[[x]]\), then clearly \(\text{Ann}(f) = \text{Ann}(C(f))[x]\).

b) If \(I[x]\) is an \(r\)-ideal in \(R[x]\), then \(I\) is an \(r\)-ideal in \(R\). The converse is true if and only if \(R\) satisfies property \(A\); see Theorem [?]. (Note: \(R[x]\) and \(C(X)\) have property \(A\).) We should also remind the reader that if \(I = \text{Ann}(a)\) with \(0 \neq a \in R\), then \(I[x]\) is an \(r\)-ideal in \(R[x]\).

c) Let \(I[[x]]\) be an \(r\)-ideal in \(R[[x]]\), and then \(I\) is an \(r\)-ideal in \(R\). The converse is true if \(R\) satisfies the c.a.c.; see Proposition [?]. It is also clear that if \(I = \text{Ann}(a)\) where \(0 \neq a \in R\), then \(I[[x]]\) is an \(r\)-ideal in \(R[[x]]\).
d) Let $I$ be a semiprime ideal of a reduced ring $R$. Assume that $f, g \in R[[x]]$, where $f = \sum_{i=0}^{\infty} f_i x^i$ and $g = \sum_{i=0}^{\infty} g_i x^i$. Then one can easily show that $fg \in I[[x]]$ if and only if $f_i g_m \in I$, for $n, m = 0, 1, 2, \ldots$.

e) If $(I, x)$ is an $r$-ideal in $R[x]$, then $I$ is an $r$-ideal in $R$. The converse is not true in general. For example, the ideal $I = (0)$ in $R$ is an $r$-ideal, but $(I, x) = xR[x]$ is not an $r$-ideal in $R[x]$.

f) If $M \in \text{Max}(R[x])$, then by [[[19]], Theorem 150] there exists $f \in M$ such that $\text{Ann}_{R[x]}(f) = (0)$, so $M$ is not an $r$-ideal. This implies that $R[x]$ is never a uz-$r$-ring.

g) If $R$ satisfies property A, $f \in R[x]$ and $\text{Ann}_{R[x]}(f) = (0)$, then by [[[18]], Theorem 2.6], there exists $a \in c(f)$ such that $\text{Ann}_R(a) = (0)$, and hence $c(f)$ is not an $r$-ideal.

h) Let $R$ be a uz-$r$-ring and $M \in \text{Max}(R[x])$, and then there is $f \in M$ such that $\text{Ann}_{R[x]}(f) = (0)$, by part (f). Whenever $I = c(f) \neq R$, then $I$ is an $r$-ideal, whereas $I[x]$ is not an $r$-ideal.

In the following proposition we show that if $I$ is an $r$-ideal in a reduced ring $R$, then $I[x]$ is an $r$-ideal in $R[x]$ if and only if $R$ satisfies property $A$.

**Theorem 4.2** Let $R$ be a ring. Then the following statements are equivalent:

a) $R$ satisfies property $A$.

b) $I$ is an $r$-ideal in $R$ if and only if $I[x]$ is an $r$-ideal in $R[x]$, for every ideal $I$ of $R$.

**Proof** $(a \Rightarrow b)$ Let $I$ be an $r$-ideal of $R$, $f, g \in R[x]$ and $fg \in I[x]$ with $\text{Ann}_{R[x]}(g) = (0)$. Hence, by [[[2]], Proposition 3.5], we conclude that $c(g) \subseteq \text{zd}(R)$. Therefore, there exists $r \in c(g)$ such that $\text{Ann}_R(r) = (0)$. Clearly, $C(fg) \subseteq I$ and so $c(fg) \subseteq I$. Now by [[[17]], Theorem 28.1], we have $c(g)^{n+1}c(f) = c(g)^nc(fg)$, where $n$ is the degree of $f$. This implies that $c(g)^{n+1}c(f) \subseteq I$. Since $r^{n+1} \in c(g)^{n+1}$, we infer that $r^{n+1}c(f) \subseteq I$. On the other hand, we have $\text{Ann}_R(r^{n+1}) = (0)$. Now we conclude that $c(f) \subseteq I$. Thus, $f \in I[x]$. The converse is evident.

$(b \Rightarrow a)$ Suppose, on the contrary, that $R$ does not satisfy property $A$. We are to seek a contradiction. By [[[2]], Proposition 3.5], there exists $f \in R[x]$ such that $\text{Ann}_{R[x]}(f) = (0)$ and $I = c(f) \subseteq \text{zd}(R)$. Now by part (b) of Theorem [?], there exists a prime $r$-ideal $P$ such that $I \subseteq P$, i.e. $c(f) \subseteq P$. Hence, $f \in P[x]$, while $f$ is a regular element. Thus, $P[x]$ is not an $r$-ideal, which is the desired contradiction.

**Corollary 4.3** Let $R$ be a uz-$r$-ring. Then $R$ satisfies property $A$ if and only if $I[x]$ is an $r$-ideal in $R[x]$, for every ideal $I$ of $R$.

A ring $R$ is said to have the finite (resp., countable) annihilator condition or briefly to have the f.a.c. (resp., the c.a.c.) if for every finite (resp., countable) subset $S$ of $R$ there exists an element $a \in S$ with $\text{Ann}(S) = \text{Ann}(a)$.

For example, the ring $\mathbb{Z}_{p^n}$, where $p$ is a prime number and $n \in \mathbb{N}$, satisfies the f.a.c. To see this, let $a \in \mathbb{Z}_{p^n}$, and hence there exists $0 \leq r \leq n$, such that $a = p^r a_1$, with $a_1$ and $p$ being relatively prime. One can easily show that $\text{Ann}_{\mathbb{Z}_{p^n}}(a) = p^{n-r}\mathbb{Z}_{p^n}$. Now if $b = p^s b_1$, with $r \leq s$, then $\text{Ann}(a, b) = \text{Ann}(a) \cap \text{Ann}(b) = p^{n-r}\mathbb{Z}_{p^n} \cap p^{n-s}\mathbb{Z}_{p^n} = p^{n-s}\mathbb{Z}_{p^n} = \text{Ann}(b)$. More generally, if in a ring $R$, the set of all $\text{Ann}(r)$, where $r \in R$, is a chain, then $R$ satisfies the f.a.c. Clearly, if $R$ is a finite ring, which satisfies the f.a.c., then $R$ satisfies the c.a.c. Also, if $F$ is a field, then $R = \frac{F[x]}{x^n F[x]}$ satisfies the c.a.c.
It is clear that if $R$ satisfies the f.a.c., then it satisfies the s.a.c., and so it satisfies the a.c. A ring $R$ may satisfy property $A$, but it may not satisfy a.c. and also f.a.c.; see [[2]], Example 4.1.

**Proposition 4.4** Let $R$ be a ring satisfying the f.a.c. (c.a.c.) and $I$ be a semiprime ideal of $R$. Then $I$ is an $r$-ideal in $R$ if and only if $I[x]$ ($I[[x]]$) is an $r$-ideal in $R[x]$ ($R[[x]]$).

**Proof** Let $f, g \in R[x]$ and $fg \in I[x]$ with $\text{Ann}_{R[x]}(f) = (0)$. Thus, $\text{Ann}_{R}(C(f)) = (0)$. By our hypothesis, there exists $a \in C(f)$ such that $\text{Ann}_{R}(C(f)) = \text{Ann}_{R}(a)$. Therefore, $\text{Ann}_{R}(a) = (0)$. It is easy to show that $aC(g) \subseteq I$. Since $I$ is an $r$-ideal in $R$, we infer that $C(g) \subseteq I$. This implies that $g \in I[x]$, i.e. $I[x]$ is an $r$-ideal in $R[x]$. The converse is evident. In case ($I[[x]]$), whenever $R$ satisfies the c.a.c., the proof is similar. 

\qed

5. $r$-ideals in $C(X)$

In this section we will investigate the relations between $r$-ideals, $z^\circ$-ideals, and $z$-ideals in $C(X)$. We characterize the topological spaces $X$ for which $r$-ideals coincide with others. In this section, for the sake of brevity, $r(C(X))$, $zd(C(X))$, and $u(C(X))$ are replaced by $r(X)$, $zd(X)$, and $u(X)$. It is easy to see that $f \in C(X)$ is a regular element if and only if $\text{int}Z(f) = \emptyset$; see also [[7]]. Let us recall the following definitions.

**Definitions 5.1** A topological space $X$ is said to be:

a) $P$-space if every prime ideal of $C(X)$ is a $z$-ideal.

b) $F$-space if finitely generated ideals of $C(X)$ are principal.

c) Almost $P$-space if every nonempty zeroset has a nonempty interior, or equivalently every $z$-ideal of $C(X)$ is a $z^\circ$-ideal.

d) Quasi $F$-space if finitely generated ideals containing a nondivisor of 0 in $C(X)$ are principal, or equivalently the sum of two $z^\circ$-ideals of $C(X)$ is a $z^\circ$-ideal.

e) $m$-space if every prime $z^\circ$-ideal of $C(X)$ is minimal prime ideal, or equivalently if for every zeroset $Z$ in $X$ there exists a zeroset $F$ in $X$ such that $Z \cup F = X$ with $\text{int}Z \cap \text{int}F = \emptyset$.

f) Quasi $m$-space if every prime $z^\circ$-ideal of $C(X)$ is either a minimal prime or a maximal ideal.

g) W, almost $P$-space if for every two zerosets $Z$ and $F$, with $\text{int}Z \subseteq \text{int}F$, there exists a zeroset $E$ in $X$ such that $Z \subseteq F \cup E$ and $\text{int}E = \emptyset$.

h) $\partial$-space if for every zeroset $Z$ in $X$ there exists a zeroset $F$ in $X$ such that $\partial(Z) \subseteq F$ and $\text{int}F = \emptyset$, where $\partial(Z) = Z \setminus \text{int}Z$ is the boundary of $Z$.

For more details about $P$-spaces and $F$-spaces, see [[16]]. For almost $P$-spaces, see [[5], [24]]; for quasi $F$-spaces, see [[13]]; and for other spaces, see [[9]].

We cite the following facts from [[9]].

**Proposition 5.2** a) Every $z$-ideal $I \subseteq zd(X)$ of $C(X)$ is a $z^\circ$-ideal if and only if $X$ is an almost $P$-space.

b) Every prime $z$-ideal $P \subseteq zd(X)$ of $C(X)$ is a $z^\circ$-ideal if and only if $X$ is a w, almost $P$-space.

c) Every prime ideal $P \subseteq zd(X)$ of $C(X)$ is a $z^\circ$-ideal if and only if $X$ is a $\partial$-space.

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Proposition 5.3 For a topological space $X$ the following statements are equivalent:

a) $X$ is an almost $P$-space.

b) Every ideal $I$ of $C(X)$ is an $r$-ideal.

c) Every ideal $I \subseteq zd(X)$ of $C(X)$ is an $r$-ideal.

Proof (a $\Leftrightarrow$ b) By [[5]], Theorem 2.2 we know that $X$ is an almost $P$-space if and only if $C(X)$ is a $uz$-ring. Therefore, every ideal in $C(X)$ is an $r$-ideal if and only if $X$ is an almost $P$-space.

(b $\Rightarrow$ c) It is clear.

c $\Rightarrow$ a) Suppose that $0 \neq f \in C(X)$ and $\text{int} Z(f) = \emptyset$, and we are to show that $Z(f) = \emptyset$. Assume that $x \notin Z(f)$; therefore, there exist $g, h \in C(X)$ such that $x \in \text{int} Z(g)$, $Z(f) \subseteq \text{int} Z(h)$, and $Z(g) \cap Z(h) = \emptyset$. Now we put $I = fgC(X)$. Clearly, $I$ is consisting entirely of zero-divisors, for $\text{int} Z(fg) = \text{int} Z(g) \neq \emptyset$. Thus, by our hypothesis, $I$ is an $r$-ideal. Since $fg \in I$ and $f$ is regular, we conclude that $g \in I$ and hence $g = fgk$ for some $k \in C(X)$. Now using $Z(f) \subseteq Z(g)$, we have $Z(f) = Z(f) \cap Z(g) \subseteq Z(h) \cap Z(g) = \emptyset$. This implies that $Z(f) = \emptyset$ and we are done.

Proposition 5.4 Every $r$-ideal of $C(X)$ is a $z^\circ$-ideal if and only if $X$ is a $\partial$-space.

Proof The necessary is clear by part (c) of Proposition [?]. For sufficiency, the proof is similar to that of [[9]], Theorem 4.4.

Let us remind the reader that in part (l) of Remark [?], we have noticed that the sum of two $r$-ideals is not necessarily an $r$-ideal. It is interesting to observe, in what follows, that in a $\partial$-space quasi $F$-space, the sum of $r$-ideals becomes an $r$-ideal.

Corollary 5.5 Let $X$ be a $\partial$-space. Then the following statements hold:

a) $I$ is an $r$-ideal in $C(X)$ if and only if it is a $z^\circ$-ideal.

b) $I$ is an $r$-ideal in $C(X)$ if and only if $\sqrt{I}$ is an $r$-ideal.

c) $I$ is an $r$-ideal in $C(X)$ if and only if every minimal prime ideal of $I$ is an $r$-ideal.

d) Every prime ideal in $C(X)$ is an $r$-ideal in $C(X)$ if and only if every prime ideal is a $z^\circ$-ideal.

e) The sum of two $r$-ideals of $C(X)$ is an $r$-ideal if and only if $X$ is a quasi $F$-space.

Since a $\partial$-space almost $P$-space is a $P$-space, the following corollary is immediate.

Corollary 5.6 Let $X$ be a $\partial$-space. Then the following statements are equivalent:

a) $X$ is a $P$-space.

b) Every ideal is an $r$-ideal in $C(X)$.

c) Every prime ideal is an $r$-ideal in $C(X)$.

Proposition 5.7 Every prime $r$-ideal of $C(X)$ is a $z^\circ$-ideal if and only if $X$ is an $m$-space.

Proof It is evident.

Lemma 5.8 Let $X$ be an $m$-space. Then every $r$-ideal of $C(X)$ is a $z$-ideal.
Conversely, it suffices to show that every prime ideal consisting entirely of zerodivisors is an \( r \)-ideal. To this end, we just notice that every prime ideal consisting entirely of zerodivisors is an \( f \)-space. Then the following statements are equivalent:

a) \( \text{int}\{Z(f) = \{0\} \)

b) \( I \) is an \( r \)-ideal.

c) \( I \) is a \( z \)-ideal.

d) \( I \) is a \( z^o \)-ideal.

Using Proposition \([6] \) and the fact that every almost \( P \)-space that is also a \( \partial \)-space is a \( P \)-space, the following corollary is now evident.

**Corollary 5.9** Let \( X \) be an \( m \)-space, \( f \in C(X) \) and \( I = fC(X) \). Then the following statements are equivalent:

a) \( \text{int}\{Z(f) = \{0\} \)

b) \( I \) is an \( r \)-ideal.

c) \( I \) is a \( z \)-ideal.

d) \( I \) is a \( z^o \)-ideal.

Let us recall that the socle of \( C(X) \), denoted by \( C_F(X) \), is of the form \( C_F(X) = \{ f \in C(X) : X \setminus Z(f) \text{ is a finite subset of } X \} \); see \([22] \), Proposition 3.3. It is also shown that \( C_F(X) \) is never a prime ideal in \( C(X) \); see \([4] \), Proposition 2.5 and \([15] \). One can easily show that \( C_F(X) \) is a \( z^o \)-ideal. Note that we have already shown (see Corollary \([?3] \)) that the socle of any reduced ring is an \( r \)-ideal.

**Theorem 5.11** Every \( r \)-ideal in the class of all \( z \)-ideals of \( C(X) \) is a \( z^o \)-ideal if and only if \( X \) is \( w \). almost \( P \)-space.

**Proof** Let \( I \) be an \( r \)-ideal that is also a \( z \)-ideal. Assume that \( \text{int}\{Z(f) \subseteq \text{int}\{Z(g) \) and \( f \in I \), and we must show that \( g \in I \). By definition of \( w \). almost \( P \)-spaces, there exists \( h \in C(X) \) such that \( \text{int}\{h) = \{0\} \) and \( Z(f) \subseteq Z(gh) \). Since \( I \) is a \( z \)-ideal, we infer that \( gh \in I \). Since \( I \) is an \( r \)-ideal we conclude that \( g \in I \). Conversely, it suffices to show that every prime \( z \)-ideal consisting entirely of zerodivisors is a \( z^o \)-ideal, by \([9] \), Theorem 4.2. To this end, we just notice that every prime ideal consisting entirely of zerodivisors is an \( r \)-ideal.

Let us recall that the socle of \( C(X) \), denoted by \( C_F(X) \), is of the form \( C_F(X) = \{ f \in C(X) : X \\setminus Z(f) \text{ is a finite subset of } X \} \); see \([22] \), Proposition 3.3. It is also shown that \( C_F(X) \) is never a prime ideal in \( C(X) \); see \([4] \), Proposition 2.5 and \([15] \). One can easily show that \( C_F(X) \) is a \( z^o \)-ideal. Note that we have already shown (see Corollary \([?4] \)) that the socle of any reduced ring is an \( r \)-ideal.

**Remark 5.12** We should emphasize that \( C_F(X) \) is an \( r \)-ideal, as we may present in a direct proof, in which we do not need to use Theorem \([?6] \) or Corollary \([?7] \). Let \( fg \in C_F(X) \), \( \text{int}\{Z(f) = \{0\} \), and \( g \in C(X) \). Clearly, \( \text{cl}(X \setminus Z(fg)) = X \), and hence

\[
X \setminus Z(g) \subseteq \text{cl}(X \setminus Z(fg)) = \text{cl}(X \setminus Z(fg)) = X \setminus Z(fg).
\]

Therefore, \( X \setminus Z(g) \) is a finite subset of \( X \), i.e. \( g \in C_F(X) \).
One can easily see that other ideals in $C(X)$ of this kind, such as $C_K(X) = \{ f \in C(X) : \text{cl}(X \setminus Z(f)) \text{ is a compact subset of } X \}$, are $r$-ideals, too.

**Remark 5.13** Suppose that $X$ is an almost $P$-space that is not $P$-space.

a) $C(X)$ is a uz-$r$-ring but it is not a von Neumann regular ring.

b) Any $r$-ideal is not necessarily a pure ideal. For example, by [[1]], Corollary 2.4 there exists $x \in X$ such that $M_x = \{ f \in C(X) : f(x) = 0 \}$ is not a pure ideal, while this ideal is an $r$-ideal. More generally, whenever $A$ is regular closed in $X$, i.e. cl(int($A$)) = $A$ ($X$ is not necessarily an almost $P$-space), then $M_A = \{ f \in C(X) : A \subseteq Z(f) \}$ is an $r$-ideal.

c) Any $r$-ideal is not necessarily a von Neumann regular ideal. Since $X$ is not a $P$-space, there exists $f \in C(X)$ such that $f$ is not a von Neumann regular element. Now ideal $I = fC(X)$ is not von Neumann regular ideal, while this ideal is an $r$-ideal.

d) Any $r$-ideal is not necessarily a $z$-ideal and so is not a $z^o$-ideal either. Since $X$ is not a $P$-space, there exists an ideal $I$ in $C(X)$ such that it is not a $z$-ideal, while this ideal is an $r$-ideal.

It is well known that the sum of two prime ideals ($z$-ideals) in $C(X)$ is either $C(X)$ or is a prime ideal ($z$-ideal); see [[16]]. The next example shows that $r$-ideals do not have this property.

**Example 5.14** The sum of two $r$-ideals may not be an $r$-ideal. For example, we consider two ideals in $C(\mathbb{R})$, namely $M_{[0,\infty)} = \{ f \in C(\mathbb{R}) : [0,\infty) \subseteq Z(f) \}$ and $M_{(-\infty,0]} = \{ f \in C(\mathbb{R}) : (-\infty,0] \subseteq Z(f) \}$. Clearly, these ideals are $z^o$-ideals and by part (a) of Theorem 2.4 are $r$-ideals. Now we put $f(x) = 0$ if $0 \leq x$, $f(x) = x$ if $x < 0$, and $g(x) = 0$ if $x \leq 0$, $g(x) = x$, if $0 < x$. Clearly, $f \in M_{[0,\infty)}$, $g \in M_{(-\infty,0]}$ and $f + g = i$, where $i \in C(\mathbb{R})$ is the identity function. Hence, $i \in M_{[0,\infty)} + M_{(-\infty,0]}$. On the other hand, $Z(i) = \{0\}$ implies int$Z(i) = \emptyset$, and so $i$ is a regular element. Therefore, $M_{[0,\infty)} + M_{(-\infty,0]}$ is not an $r$-ideal.

The next example shows that every ideal consisting of zerodivisors is not necessarily an $r$-ideal (even if it is a semiprime or even a $z$-ideal). Recall that every $z$-ideal in $C(X)$ is a semiprime ideal.

**Example 5.15** Any $z$-ideal consisting entirely of zerodivisors is not necessarily an $r$-ideal. For example, in $C(\mathbb{R})$ we consider $I = \{ f \in C(\mathbb{R}) : [0,1] \cup \{2\} \subseteq Z(f) \}$. Clearly, $I$ is a $z$-ideal consisting entirely of zerodivisors. Now suppose that $Z(g) = [0,1]$ and $Z(h) = \{2\}$, where $g,h \in C(\mathbb{R})$. It is obvious that $[0,1] \cup \{2\} = Z(g) \cup Z(h) = Z(gh)$, so $gh \in I$. Since int$Z(h) = \emptyset$ and $g \notin I$, we conclude that $I$ is not an $r$-ideal.

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