Approximate duals and nearly Parseval frames

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Abstract: In this paper we introduce approximate duality of g-frames in Hilbert $C^*$-modules and we show that approximate duals of g-frames in Hilbert $C^*$-modules share many useful properties with those in Hilbert spaces. Moreover, we obtain some new results for approximate duality of frames and g-frames in Hilbert spaces; in particular, we consider approximate duals of $\varepsilon$-nearly Parseval and $\varepsilon$-close frames.

Key words: Hilbert $C^*$-module, g-frame, frame, approximate duality, $\varepsilon$-nearly Parseval frame

1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [10] in 1952 to study some problems in nonharmonic Fourier series, and they were reintroduced in 1986 by Daubechies et al. [9]. Frames have important applications in signal and image processing, wireless communications, and many other fields. There exist various generalizations of frames. A recent and general one is called g-frame [26].

As we know, duals play an important role in frame theory, especially they are used in the reconstruction of signals. It is well known that every frame in a Hilbert space has at least one dual (see [7]), and if a dual of a frame is found, then each signal can be reconstructed easily. However, it is usually difficult to calculate a dual. Here, approximate duals can be useful. Approximate duals in frame theory have important applications (see [4, 12, 27]). Approximate duality of frames in Hilbert spaces was recently investigated in [8]. Khosravi and Mirzaee Azandaryani (the present author) also introduced approximate duality of g-frames in Hilbert spaces and obtained some properties and applications of approximate duals (see [20]). In particular, it was shown that approximate duals are stable under small perturbations and they are useful for erasures (see [20, Section 3]).

Hilbert $C^*$-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a $C^*$-algebra rather than in the field of complex numbers.

Frank and Larson presented a general approach to the frame theory in Hilbert $C^*$-modules (see [11]). They showed that every countably generated Hilbert $C^*$-module over a unital $C^*$-algebra admits a frame. It was also shown in [25] that every Hilbert $C^*$-module that is countably generated in the set of adjointable operators admits a frame of multipliers. Furthermore, g-frames in Hilbert $C^*$-modules were introduced in [16].

Frames in Hilbert $C^*$-modules are not trivial generalizations of Hilbert space frames due to the complex structure of $C^*$-algebras. Since there are important differences between the theory of Hilbert $C^*$-modules
and Hilbert spaces (see Chapter 1 in [21]), it is expected that problems about frames in Hilbert $C^*$-modules are more complicated than those in Hilbert spaces.

In this paper we generalize the concept of approximate duality of g-frames to Hilbert $C^*$-modules and we get some results for approximate duals of frames and g-frames in Hilbert spaces. In particular, approximate duals of $\varepsilon$-nearly Parseval and $\varepsilon$-close frames are studied.

First, in the following section, we have a brief review of the definitions and basic properties of frames and g-frames in Hilbert $C^*$-modules.

In this note, all index sets are finite or countable subsets of $\mathbb{Z}$.

### 2. Frames and g-frames in Hilbert $C^*$-modules

Suppose that $\mathfrak{A}$ is a unital $C^*$-algebra and $E$ is a left $\mathfrak{A}$-module such that the linear structures of $\mathfrak{A}$ and $E$ are compatible. $E$ is a pre-Hilbert $\mathfrak{A}$-module if $E$ is equipped with an $\mathfrak{A}$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathfrak{A}$, such that

(i) $\langle ax + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for each $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in E$;

(ii) $\langle ax, y \rangle = a \langle x, y \rangle$, for each $a \in \mathfrak{A}$ and $x, y \in E$;

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$, for each $x, y \in E$;

(iv) $\langle x, x \rangle \geq 0$, for each $x \in E$ and if $\langle x, x \rangle = 0$, then $x = 0$.

For each $x \in E$, we define $\|x\| = \|\langle x, x \rangle\|^\frac{1}{2}$ and $|x| = \langle x, x \rangle^\frac{1}{2}$. If $E$ is complete with $\|\cdot\|$, it is called a Hilbert $\mathfrak{A}$-module or a Hilbert $C^*$-module over $\mathfrak{A}$. Let $E$ and $F$ be Hilbert $\mathfrak{A}$-modules. An operator $T : E \rightarrow F$ is called adjointable if there exists an operator $T^* : F \rightarrow E$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$, for each $x \in E$ and $y \in F$. Every adjointable operator $T$ is bounded and $\mathfrak{A}$-linear (that is, $T(ax) = aT(x)$ for each $x \in E$ and $a \in \mathfrak{A}$). We denote the set of all adjointable operators from $E$ into $F$ by $\mathbb{L}(E,F)$. $\mathbb{L}(E,F)$ is a $C^*$-algebra and we denote it by $\mathbb{L}(E)$. Note that if $\{E_i : i \in I\}$ is a sequence of Hilbert $\mathfrak{A}$-modules, then $\oplus_{i \in I} E_i$, which is the set

$$\oplus_{i \in I} E_i = \left\{ \{x_i\}_{i \in I} : x_i \in E_i \text{ and } \sum_{i \in I} \langle x_i, x_i \rangle \text{ is norm convergent in } \mathfrak{A} \right\},$$

is a Hilbert $\mathfrak{A}$-module with pointwise operations and $\mathfrak{A}$-valued inner product $\langle x, y \rangle = \sum_{i \in I} \langle x_i, y_i \rangle$, where $x = \{x_i\}_{i \in I}$ and $y = \{y_i\}_{i \in I}$. For each $x = \{x_i\}_{i \in I} \in \oplus_{i \in I} E_i$, we define $\|\cdot\|_2$ by $\|x\|_2 = \|\sum_{i \in I} \langle x_i, x_i \rangle\|^\frac{1}{2}$. For more details about Hilbert $C^*$-modules, see [21].

In this paper we focus on finitely and countably generated Hilbert $C^*$-modules over unital $C^*$-algebras. A Hilbert $\mathfrak{A}$-module $E$ is finitely generated if there exists a finite set $\{x_1, \ldots, x_n\} \subseteq E$ such that every element $x \in E$ can be expressed as an $\mathfrak{A}$-linear combination $x = \sum_{i=1}^n a_i x_i, a_i \in \mathfrak{A}$. A Hilbert $\mathfrak{A}$-module $E$ is countably generated if there exists a countable set $\{x_i\}_{i \in I} \subseteq E$ such that $E$ equals the norm-closure of the $\mathfrak{A}$-linear hull of $\{x_i\}_{i \in I}$.

Let $E$ be a Hilbert $\mathfrak{A}$-module. A family $\{f_i\}_{i \in I} \subseteq E$ is a frame for $E$, if there exist real constants $0 < A \leq B < \infty$, such that for each $x \in E$,

$$A \langle x, x \rangle \leq \sum_{i \in I} \langle x, f_i \rangle \langle f_i, x \rangle \leq B \langle x, x \rangle. \quad (1)$$
The numbers $A$ and $B$ are called the lower and upper bound of the frame, respectively. In this case we call it an $(A, B)$ frame. The optimal lower frame bound is the supremum over all lower frame bounds and the optimal upper frame bound is the infimum over all upper frame bounds. If $A = B$, the frame is called tight ($A$-tight) and if $A = B = 1$, the frame is Parseval. If only the second inequality is required, we call it a Bessel sequence. If the sum in (1) converges in norm, the frame is called standard.

Let $\{E_i\}_{i \in I}$ be a sequence of Hilbert $\mathfrak{A}$-modules. A sequence $\Lambda = \{\Lambda_i \in \mathfrak{L}(E, E_i) : i \in I\}$ is called a g-frame for $E$ with respect to $\{E_i : i \in I\}$ if there exist real constants $A, B > 0$ such that for each $x \in E$,

$$A \langle x, x \rangle \leq \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle \leq B \langle x, x \rangle.$$  

$A$ and $B$ are g-frame bounds of $\Lambda$. In this case we call it an $(A, B)$ g-frame. The optimal bounds and tight and Parseval g-frames are defined similarly to frames. The g-frame is standard if for each $x \in E$, the sum converges in norm. If only the second-hand inequality is required, then $\Lambda$ is called a g-Bessel sequence.

For a standard g-Bessel sequence $\Lambda$, the operator $T_{\Lambda} : \oplus_{i \in I} E_i \longrightarrow E$ defined by $T_{\Lambda}(\{x_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^* x_i$ is called the synthesis operator of $\Lambda$. $T_{\Lambda}$ is adjointable and $T_{\Lambda}^* (x) = \{\Lambda_i x\}_{i \in I}$. Now we define the operator $S_{\Lambda}$ on $E$ by $S_{\Lambda} x = T_{\Lambda} T_{\Lambda}^* (x) = \sum_{i \in I} \Lambda_i^* \Lambda_i (x)$. If $\Lambda$ is a standard $(A, B)$ g-frame, then $A Id_E \leq S_{\Lambda} \leq B Id_E$.

Note that $\mathcal{F} = \{f_i\}_{i \in I}$ is a standard Bessel sequence (resp. frame) if and only if $\Lambda_{\mathcal{F}} = \{\Lambda_{f_i}\}_{i \in I}$ is a standard g-Bessel sequence (resp. g-frame), where $\Lambda_{f_i} (x) = \langle x, f_i \rangle$, for each $x \in E$ (see [16, Example 3.2]). This shows that each Bessel sequence (resp. frame) generates a g-Bessel sequence (resp. g-frame). For a standard Bessel sequence $\mathcal{F} = \{f_i\}_{i \in I}$, we denote $S_{\Lambda_{\mathcal{F}}}$ by $S_{\mathcal{F}}$.

Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be an (A,B) standard g-frame. We call $\Lambda^* = \{\Lambda_i S_{\Lambda}^{-1}\}_{i \in I}$ the canonical g-dual of $\Lambda$, which is a $(\frac{1}{B}, \frac{1}{A})$ standard g-frame. We denote the canonical dual of a standard frame $\mathcal{F} = \{f_i\}_{i \in I}$ by $\widehat{\mathcal{F}} = \{f_i\}_{i \in I}$, where $\tilde{f}_i = S_{\mathcal{F}}^{-1} f_i$. Recall that if $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Gamma = \{\Gamma_i\}_{i \in I}$ are standard g-Bessel sequences such that $\sum_{i \in I} \Gamma_i^* \Lambda_i x = x$ or equivalently $\sum_{i \in I} \Lambda_i^* \Gamma_i x = x$, for each $x \in E$, then $\Gamma$ (resp. $\Lambda$) is called a g-

dual of $\Lambda$ (resp. $\Gamma$). Also, duals for two standard Bessel sequences $\mathcal{F} = \{f_i\}_{i \in I}$ and $\mathcal{G} = \{g_i\}_{i \in I}$ can be defined by using the generated g-Bessel sequences, so $\mathcal{G}$ (resp. $\mathcal{F}$) is a dual of $\mathcal{F}$ (resp. $\mathcal{G}$) if $x = \sum_{i \in I} \langle x, f_i \rangle g_i$ or equivalently $x = \sum_{i \in I} \langle x, g_i \rangle f_i$, for each $x \in E$ (see [11, 13]). For more details about frames and g-frames in Hilbert $C^*$-modules, see [11, 2, 16, 28].

3. Approximate duals of g-frames in Hilbert $C^*$-modules

In this section all $C^*$-algebras are unital and all Hilbert $C^*$-modules are finitely or countably generated. All frames, g-frames, Bessel sequences, and g-Bessel sequences are standard. $\Lambda$ and $\Gamma$ denote $\{\Lambda_i \in \mathfrak{L}(E, E_i) : i \in I\}$ and $\{\Gamma_i \in \mathfrak{L}(E, E_i) : i \in I\}$, respectively. Also, $\mathcal{F} = \{f_i\}_{i \in I}$ and $\mathcal{G} = \{g_i\}_{i \in I}$ are subsets of a Hilbert $C^*$-module $E$.

For two standard g-Bessel sequences $\Lambda$ and $\Gamma$, the operator $S_{\Gamma \Lambda}$ is defined on $E$ by $S_{\Gamma \Lambda} = T_{\Gamma} T_{\Lambda}^*$. Since $S_{\Gamma \Lambda}^* S_{\Gamma \Lambda} = S_{\Lambda} \Gamma$, we have $\|Id_E - S_{\Gamma \Lambda}\| \leq \|Id_E - S_{\Lambda} \Gamma \|^* = \|Id_E - S_{\Lambda} \Gamma\|$.

Now we introduce approximate duals for g-Bessel sequences (and also for Bessel sequences by using the generated g-Bessel sequences) in Hilbert $C^*$-modules:
Definition 3.1 (i) Two standard $g$-Bessel sequences $\Lambda$ and $\Gamma$ are approximately dual $g$-frames if $\|Id_E - S_{\Gamma\Lambda}\| < 1$ or equivalently $\|Id_E - S_{\Lambda\Gamma}\| < 1$. In this case, we say that $\Gamma$ (resp. $\Lambda$) is an approximate $g$-dual of $\Lambda$ (resp. $\Gamma$).

(ii) Two standard Bessel sequences $\mathcal{F}$ and $\mathcal{G}$ are approximately dual frames if $\Lambda_{\mathcal{F}}$ and $\Lambda_{\mathcal{G}}$ are approximately dual $g$-frames, i.e. $\|Id_E - S_{\Lambda_{\mathcal{G}}\Lambda_{\mathcal{F}}}\| < 1$ or equivalently $\|Id_E - S_{\Lambda_{\mathcal{F}}\Lambda_{\mathcal{G}}}\| < 1$. In this case, we say that $\mathcal{G}$ (resp. $\mathcal{F}$) is an approximate dual of $\mathcal{F}$ (resp. $\mathcal{G}$). We denote $S_{\Lambda_{\mathcal{G}}\Lambda_{\mathcal{F}}}$ and $S_{\Lambda_{\mathcal{F}}\Lambda_{\mathcal{G}}}$ by $S_{\mathcal{G}\mathcal{F}}$ and $S_{\mathcal{F}\mathcal{G}}$, respectively.

It is clear that $S_{\Gamma\Lambda}(x) = \sum_{i \in I} \Gamma_i^* \Lambda_i(x)$ and $S_{\mathcal{G}\mathcal{F}}(x) = \sum_{i \in I} \langle x, f_i \rangle g_i$, for each $x \in E$. If $\Lambda$ and $\Gamma$ are $g$-duals, then they are approximately dual $g$-frames because $S_{\Lambda\Gamma} = Id_E$. Using the Neumann algorithm, we can see that $S_{\Lambda\Gamma}$ is invertible with $S_{\Lambda\Gamma}^{-1} = \sum_{n=0}^{\infty} (Id_E - S_{\Lambda\Gamma})^n$, so each $x \in E$ can be reconstructed as

$$x = \sum_{n=0}^{\infty} S_{\Lambda\Gamma}(Id_E - S_{\Lambda\Gamma})^n x, \quad x = \sum_{n=0}^{\infty} (Id_E - S_{\Lambda\Gamma})^n S_{\Gamma\Lambda} x.$$

Recall from [17] that a standard $g$-frame $\Lambda$ is a modular $g$-Riesz basis if it has the following property:

if $\sum_{i \in \Omega} \Lambda_i g_i = 0$, where $g_i \in E$ and $\Omega \subseteq I$, then $g_i = 0$, for each $i \in \Omega$.

A standard frame $\{f_i\}_{i \in I}$ for $E$ is a modular Riesz basis if it has the following property: if an $\mathcal{A}$-linear combination $\sum_{i \in \Omega} a_i f_i$ with coefficients $\{a_i : i \in \Omega\} \subseteq \mathcal{A}$ and $\Omega \subseteq I$ is equal to zero, then $a_i = 0$, for each $i \in \Omega$.

The following result is a generalization of Proposition 2.3 in [20] to Hilbert $C^*$-modules.

Theorem 3.2 Let $\Lambda$ and $\Gamma$ be approximately dual $g$-frames with upper bounds $B$ and $D$, respectively. Then:

(i) $\Lambda$ and $\Gamma$ are $(\|S_{\Gamma\Lambda}^{-1}\|^{-2}, B)$ and $(\|S_{\Lambda\Gamma}^{-1}\|^{-2}, D)$ $g$-frames, respectively.

(ii) $\{\Gamma_i + \sum_{n=1}^{\infty} \Gamma_i(Id_E - S_{\Lambda\Gamma})^n\}_{i \in I}$ is a $g$-dual of $\Lambda$.

(iii) For each $N \in \mathbb{N}$, define $\psi_i^N = \Gamma_i + \sum_{n=1}^{N} \Gamma_i(Id_E - S_{\Lambda\Gamma})^n$. Then $\Psi_N = \{\psi_i^N\}_{i \in I}$ is an approximate $g$-dual of $\Lambda$ with $\|Id_E - S_{\Lambda\Psi_N}\| \leq \|Id_E - S_{\Lambda\Gamma}\|^{N+1} < 1$.

(iv) If $\Lambda$ is a modular $g$-Riesz basis, then $\widetilde{\Lambda}_i = \Gamma_i + \sum_{n=1}^{\infty} \Gamma_i(Id_E - S_{\Lambda\Gamma})^n = \lim_{N \rightarrow \infty} \psi_i^N$, for each $i \in I$.

Proof (i) Since $\Lambda$ and $\Gamma$ are approximately dual $g$-frames, $S_{\Gamma\Lambda}$ is invertible, so $\|S_{\Gamma\Lambda}^{-1}\|^{-1} \leq \|S_{\Gamma\Lambda}\|$, for each $x \in E$. Now by using the Cauchy–Schwarz inequality in Hilbert $C^*$-modules, we have

$$\|S_{\Gamma\Lambda}^{-1}\|^{-1} \|x\| \leq \|S_{\Gamma\Lambda}x\| = \sup_{\|y\| = 1} \|\langle S_{\Gamma\Lambda}x, y \rangle\| = \sup_{\|y\| = 1} \left\| \sum_{i \in I} \langle \Lambda_i x, \Gamma_i y \rangle \right\|$$

$$\leq \sup_{\|y\| = 1} \left\| \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle \right\|^\frac{1}{2} \left\| \sum_{i \in I} \langle \Gamma_i y, \Gamma_i y \rangle \right\|^\frac{1}{2}$$

$$\leq \sqrt{D} \left\| \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle \right\|^\frac{1}{2}.$$
Hence:

\[
\frac{\|S_\Gamma^{-1}\|^{-2}}{D} \leq \left\| \sum_{i \in I} \langle \Lambda_i x, \Lambda_i x \rangle \right\|
\]

and so by Theorem 3.1 in [28], \( \Lambda \) is a standard \( g \)-frame with the lower bound \( \frac{\|S_\Gamma^{-1}\|^{-2}}{D} \). Similarly, by considering \( S_\Gamma \) replaced by \( S_{\Gamma'} \) in the above conclusions, we obtain that \( \Gamma \) is a \( \left( \frac{\|S_{\Gamma'}^{-1}\|^{-2}}{B} \right) \) standard \( g \)-frame.

(ii) Since \( S_{\Gamma'}^{-1} = \sum_{n=0}^{\infty} (Id_E - S_{\Gamma'})^n \), we have \( \Gamma_i S_{\Gamma'}^{-1} = \Gamma_i + \sum_{n=1}^{\infty} \Gamma_i (Id_E - S_{\Gamma'})^n \) and it is easy to see that \( \{ \Gamma_i S_{\Gamma'}^{-1} \}_{i \in I} \) is a \( g \)-dual of \( \Lambda \).

(iii) For each \( n = 0, \ldots, N \), we have

\[
\left\| \sum_{i \in I} (\Gamma_i (Id_E - S_{\Gamma'})^n x, \Gamma_i (Id_E - S_{\Gamma'})^n x) \right\| \leq D \left\| (Id_E - S_{\Gamma'})^n \right\|^2 \|x\|^2,
\]

so \( \{ \Gamma_i (Id_E - S_{\Gamma'})^n \}_{i \in I} \) is a standard \( g \)-Bessel sequence by Theorem 3.1 in [28] and consequently \( \Psi_J \) is a standard \( g \)-Bessel sequence. Now the result can be obtained similar to the proof of Proposition 2.3 in [20].

(iv) Since \( \Lambda \) is a modular \( g \)-Riesz basis, Corollary 4.1 in [17] yields that \( \Lambda \) is the unique \( g \)-dual of \( \Lambda \). According to part (ii), \( \{ \Gamma_i + \sum_{n=1}^{\infty} \Gamma_i (Id_E - S_{\Gamma'})^n \}_{i \in I} \) is also a \( g \)-dual of \( \Lambda \), so \( \Lambda_i = \Gamma_i + \sum_{n=1}^{\infty} \Gamma_i (Id_E - S_{\Gamma'})^n = \lim_{N \to \infty} \psi_i^N \). \( \square \)

As a consequence of the above theorem and Example 3.2 in [16], we obtain the following result. Parts (ii) and (iii) of the following corollary are generalizations of Proposition 3.2 in [8] to Hilbert \( C^* \)-modules.

**Corollary 3.3** Let \( \mathcal{F} \) and \( \mathcal{G} \) be approximately dual frames with upper bounds \( B \) and \( D \), respectively. Then:

(i) \( \mathcal{F} \) and \( \mathcal{G} \) are \( \left( \frac{\|S_{\mathcal{F}}^{-1}\|^{-2}}{D}, B \right) \) and \( \left( \frac{\|S_{\mathcal{G}}^{-1}\|^{-2}}{B}, D \right) \) frames, respectively.

(ii) \( \{g_i + \sum_{n=1}^{\infty} (Id_E - S_{\mathcal{G}})^n g_i \}_{i \in I} \) is a dual of \( \mathcal{F} \).

(iii) For each \( N \in \mathbb{N} \), define \( h_i^N = g_i + \sum_{n=1}^{N} (Id_E - S_{\mathcal{G}})^n g_i \). Then \( h_N = \{h_i^N \}_{i \in I} \) is an approximate dual of \( \mathcal{F} \) with \( \|Id_E - S_{h_N^N} \| \leq \|Id_E - S_{\mathcal{G}} \|^N < 1 \).

(iv) If \( \mathcal{F} \) is a modular Riesz basis, then \( \tilde{f}_i = g_i + \sum_{n=1}^{\infty} (Id_E - S_{\mathcal{G}})^n g_i = \lim_{N \to \infty} h_i^N \), for each \( i \in I \).

We can get from the above theorem and corollary that a standard \( g \)-Bessel sequence (resp. Bessel sequence) is a standard \( g \)-frame (resp. frame) if and only if it has an approximate \( g \)-dual (resp. approximate dual).

Note that Theorem 2.5 in [20] shows that if \( \Lambda \) and \( \Gamma \) are two \( g \)-Bessel sequences in a Hilbert space \( H \), then a necessary and sufficient condition for \( \Lambda \) and \( \Gamma \) to be approximately dual \( g \)-frames is that there exist two Bessel sequences \( \mathcal{F} \) and \( \mathcal{G} \) in \( H \) that are approximately dual frames with \( S_{\Lambda \Gamma} = S_{\mathcal{F} \mathcal{G}} \). Now we have a similar result for approximate duals in Hilbert \( C^* \)-modules.

**Proposition 3.4** Let \( \Lambda \) and \( \Gamma \) be two \( g \)-Bessel sequences. Then \( \Lambda \) and \( \Gamma \) are approximately dual \( g \)-frames if and only if there exist two Bessel sequences \( \mathcal{F} \) and \( \mathcal{G} \) in \( E \) such that \( \mathcal{F} \) and \( \mathcal{G} \) are approximately dual frames with \( S_{\Lambda \Gamma} = S_{\mathcal{F} \mathcal{G}} \).
Proof Let $\Lambda$ and $\Gamma$ be approximately dual g-frames. As a result of Kasparov’s stabilization theorem, every finitely or countably generated Hilbert $C^*$-module has a standard Parseval frame (see [11, 22]). Let $\{f_{ij}\}_{i \in I}$ be a standard Parseval frame for $E_i$. It follows from Corollary 3.4 in [16] that $\mathcal{F} = \{\Lambda_i^*(f_{ij})\}_{i \in I, j \in J}$ and $\mathcal{G} = \{\Gamma_i^*(f_{ij})\}_{i \in I, j \in J}$ are standard Bessel sequences. Then for each $x \in E$, we have

$$S_{\mathcal{F}}g(x) = \sum_{i \in I} \sum_{j \in J} (x, \Gamma_i^*(f_{ij}))\Lambda_i^*(f_{ij}) = \sum_{i \in I} \Lambda_i^* \Gamma_i x = S_{\mathcal{G}}x,$$

so $\|S_{\mathcal{F}} - Id_E\| = \|S_{\mathcal{G}} - Id_E\| < 1$, and the result follows. The converse is clear. \hfill $\square$

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two $C^*$-algebras. Then $\mathfrak{A} \otimes \mathfrak{A}'$ is a $C^*$-algebra with the spatial norm and for each $a \in \mathfrak{A}$ and $a' \in \mathfrak{A}'$, we have $\|a \otimes a'\| = \|a\|\|a'\|$. The multiplication and involution on simple tensors are defined by $(a \otimes a')(b \otimes b') = ab \otimes a'b'$ and $(a \otimes a')^* = a^* \otimes a'^*$, respectively. As we know, if $a, a' \geq 0$, then $a \otimes a' \geq 0$.

Now let $E$ be a Hilbert $\mathfrak{A}$-module and $E'$ be a Hilbert $\mathfrak{A}'$-module. Then the (Hilbert $C^*$-module) tensor product $E \otimes E'$ is a Hilbert $\mathfrak{A} \otimes \mathfrak{A}'$-module. The module action and inner product for simple tensors are defined by $(a \otimes a')(x \otimes x') = (ax) \otimes (a'x')$ and $\langle x \otimes x', y \otimes y' \rangle = \langle x, y \rangle \otimes \langle x', y' \rangle$, respectively. Let $U$ and $U'$ be adjointable operators on $E$ and $E'$, respectively. Then the tensor product $U \otimes U'$ is an adjointable operator on $E \otimes E'$. Also, $(U \otimes U')^* = U^* \otimes U'^*$ and $\|U \otimes U'\| = \|U\|\|U'\|$. For more results about tensor products of $C^*$-algebras and Hilbert $C^*$-modules, see [23, 21].

Tensor products of frames and g-frames have been studied by some authors recently; see [15, 6, 16, 18].

It was proved in Proposition 3.2 in [19] that the direct sum of a countable number of g-duals (in Hilbert spaces) is a g-dual in the direct sum space but Example 2.9 in [20] shows that this is not necessarily true for approximate g-duals.

It was also shown in [20, Proposition 2.10] and [18, Corollary 3.8] (by using resolutions of the identity) that the tensor product of two g-duals (in Hilbert spaces) gives a g-dual in the tensor product space. In the following example, we show that the result does not necessarily hold for approximate g-duals:

Example 3.5 Let $H$ be a separable Hilbert space (as a special case of a Hilbert $C^*$-module) and $\Lambda = \{\Lambda_i\}_{i \in I}$ be an $A$-tight g-frame with $\sqrt{2} < A < 2$. It is easy to see that $\Lambda$ is an approximate g-dual of itself. Now the proof of Corollary 2.2 in [18] yields that $\Lambda \otimes \Lambda = \{\Lambda_i \otimes \Lambda_j\}_{i, j \in I}$ is an $A^2$-tight g-frame, so $S_{(\Lambda \otimes \Lambda)(\Lambda \otimes \Lambda)} = S_{(\Lambda \otimes \Lambda)} = A^2 \text{Id}_{(H \otimes H)}$. Thus, $\|S_{(\Lambda \otimes \Lambda)(\Lambda \otimes \Lambda)} - \text{Id}_{(H \otimes H)}\| = A^2 - 1 > 1$. This means that $\Lambda \otimes \Lambda$ is not an approximate g-dual of itself.

Now we consider tensor products of g-duals and approximate g-duals in Hilbert $C^*$-modules. In the following proposition $\Lambda' = \{\Lambda'_{i,j} \in \mathcal{L}(E', E'_j) : j \in J\}$, $\Gamma' = \{\Gamma'_{i,j} \in \mathcal{L}(E', E'_j) : j \in J\}$, $\mathcal{F}' = \{f'_{i,j}\}_{i \in I, j \in J}$ and $\mathcal{G}' = \{g'_{i,j}\}_{i \in I, j \in J}$.

Proposition 3.6 (i) Let $\Gamma$ be an approximate dual g-dual (resp. a g-dual) of $\Lambda$. If $\Gamma'$ is a g-dual of $\Lambda'$, then $\Gamma \otimes \Gamma' = \{\Gamma_{i,j} \otimes \Gamma'_{i,j}\}_{i \in I, j \in J}$ is an approximate g-dual (resp. a g-dual) of $\Lambda \otimes \Lambda' = \{\Lambda_{i,j}\}_{i \in I, j \in J}$.

(ii) Let $\mathcal{G}$ be an approximate dual (resp. a dual) of $\mathcal{F}$. If $\mathcal{G}'$ is a dual of $\mathcal{F}'$, then $\mathcal{G} \otimes \mathcal{G}' = \{g_i \otimes g'_{i,j}\}_{i \in I, j \in J}$ is an approximate dual (resp. a dual) of $\mathcal{F} \otimes \mathcal{F}' = \{f_i \otimes f'_{i,j}\}_{i \in I, j \in J}$.
Hence, the **Hilbert Schwarz inequality** in the above proposition.

Note that Proposition 2.10, Corollary 2.11 in [23] imply that \( 0 \leq \langle (S_A \otimes S_{A'})z, z \rangle \leq BB'(z, z), \) for each \( z \in E \otimes E' \). Now it is easy to obtain that \( \sum_{(i,j) \in I \times J} \|((A_i \otimes A'_j)z, (A_i \otimes A'_j)z)\| \) converges in norm and

\[
\left\| \sum_{(i,j) \in I \times J} \|((A_i \otimes A'_j)z, (A_i \otimes A'_j)z)\| \right\| \leq BB\|z\|^2.
\]

so \( A \otimes A' \) is a standard g-Bessel sequence by Theorem 3.1 in [28] (also, see [16, Section 5]). Similarly, we can get that \( \Gamma \otimes \Gamma' \) is a standard **g-Bessel sequence**. It is also easy to see that

\[
S_{(\Gamma \otimes \Gamma')/(A \otimes A')}((x \otimes x')) = (S_{(\Gamma \otimes \Gamma')}(x) \otimes S_{(\Gamma \otimes \Gamma')}(x')) = (S_{(\Gamma \otimes \Gamma')} \otimes I_d)(x \otimes x'),
\]

for each \( x \otimes x' \in E \otimes E' \), and since the operators are bounded, we have \( S_{(\Gamma \otimes \Gamma')/(A \otimes A')} = S_{(\Gamma \otimes \Gamma')} \otimes I_d \). Therefore

\[
\|S_{(\Gamma \otimes \Gamma')/(A \otimes A')} - I_d\| = \|(S_{(\Gamma \otimes \Gamma')} - I_d) \otimes I_d\| = \|S_{(\Gamma \otimes \Gamma')} - I_d\| < 1.
\]

This means that \( \Gamma \otimes \Gamma' \) is an approximate g-dual of \( A \otimes A' \). It is clear that if \( \Gamma \) and \( \Gamma' \) are g-duals of \( A \) and \( A' \), respectively, then \( S_{(\Gamma \otimes \Gamma')/(A \otimes A')} = I_d(E \otimes E') \), so \( \Gamma \otimes \Gamma' \) is a g-dual of \( A \otimes A' \).

(ii) We can get the result by using Example 3.2 in [16] and part (i). \( \square \)

Note that Proposition 2.10, Corollary 2.11 in [20], and part (ii) of Corollary 3.8 in [18] are special cases of the above proposition.

Now we show that approximate duals in Hilbert \( C^* \)-modules are stable under small perturbations. The following result is analogous to part (i) of Theorem 3.1 in [20] that we need in the next section.

**Proposition 3.7** Let \( A \) be a g-Bessel sequence and \( \Psi = \{\psi_i\}_{i \in I} \) be an approximate g-dual (resp. a g-dual) of \( A \) with upper bound \( C \). If \( \Gamma \) is a sequence such that \( \Gamma - \Lambda \) is a g-Bessel sequence with upper bound \( K \) and \( CK < (1 - \|I_d - S_{\Psi(A)}\|)^2 \) (resp. \( CK < 1 \)), then \( \Gamma \) and \( \Psi \) are approximately dual g-frames.

**Proof** Let \( \Omega \) be a finite subset of \( I \) and \( B \) be an upper bound for \( A \). Then

\[
\left\| \sum_{i \in I} \langle \Gamma_i x, \Gamma_i x \rangle \right\| \leq \left\| \{\Gamma_i x\}_{i \in \Omega} \right\|_2 + \left\| \{\Gamma_i x - \Lambda_i x\}_{i \in \Omega} \right\|_2 \leq (\sqrt{B} + \sqrt{K})\|x\|,
\]

for each \( x \in E \). Thus, by Theorem 3.1 in [28], \( \Gamma \) is a standard g-Bessel sequence. Now by using the Cauchy–Schwarz inequality in Hilbert \( C^* \)-modules, for each \( x \in E \), we have

\[
\| (I_d - S_{\Psi(A)} x) \| \leq \| (I_d - S_{\Psi(A)} x) \| + \| (S_{\Psi(A)} - S_{\Psi}) x) \|
\]

\[
\leq \| (I_d - S_{\Psi(A)} x) \| + \sup_{\|y\| = 1} \left\{ \left\| \sum_{i \in I} \langle \Lambda_i - \Gamma_i \rangle x \| \right\|^2 \left\| \sum_{i \in I} |\psi_i y|\|^2 \right\} \right\}
\]

\[
\leq \| (I_d - S_{\Psi(A)} x) \| + \sqrt{CK} \|x\| \leq (\|I_d - S_{\Psi(A)}\| + \sqrt{CK})\|x\|.
\]

Hence, \( \|I_d - S_{\Psi(A)}\| \leq \|I_d - S_{\Psi(A)}\| + \sqrt{CK} < 1 \). Also, if \( A \) and \( \Psi \) are g-duals, then \( S_{\Psi(A)} = I_d \) and we have

\[
\|I_d - S_{\Psi(A)}\| \leq \sqrt{CK} < 1.
\] \( \square \)

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The following result is a generalization of Proposition 3.10 in [20] to Hilbert $C^*$-modules.

**Proposition 3.8** Let $0 \leq \lambda_1, \lambda_2 < 1$, $A, B, \varepsilon > 0$, and $K = \lambda_1 + \frac{\varepsilon}{\sqrt{\lambda_1}} + \frac{\lambda_2(1+\lambda_1)\sqrt{\lambda_1+\varepsilon}}{\sqrt{\lambda_1(1-\lambda_2)}}$.

(i) If $\Lambda$ is an $(A, B)$ $g$-frame and $\Gamma$ is a sequence satisfying

$$\left\| \sum_{i \in \Omega} (\Lambda_i^* - \Gamma_i^*) f_i \right\| \leq \lambda_1 \left\| \sum_{i \in \Omega} \Lambda_i^* f_i \right\| + \lambda_2 \left\| \sum_{i \in \Omega} \Gamma_i^* f_i \right\| + \varepsilon \left\| \sum_{i \in \Omega} |f_i|^2 \right\|^{\frac{1}{2}}, \tag{2}$$

for each finite subset $\Omega \subseteq I$, $f_i \in E_i$ with $K < 1$, then $\bar{\Lambda}$ is an approximate $g$-dual of $\Gamma$ and $\Gamma$ is a $g$-frame.

(ii) If $\mathcal{F} = \{f_i\}_{i \in I}$ is an $(A, B)$ frame and $\mathcal{G} = \{g_i\}_{i \in I}$ is a sequence satisfying

$$\left\| \sum_{i \in \Omega} a_i f_i - \sum_{i \in \Omega} a_i g_i \right\| \leq \lambda_1 \left\| \sum_{i \in \Omega} a_i f_i \right\| + \lambda_2 \left\| \sum_{i \in \Omega} a_i g_i \right\| + \varepsilon \left\| \sum_{i \in \Omega} |a_i|^2 \right\|^{\frac{1}{2}},$$

for each finite subset $\Omega \subseteq I$, $a_i \in E_i \subseteq A$ with $K < 1$, then $\bar{\mathcal{F}}$ is an approximate dual of $\mathcal{G}$ and $\mathcal{G}$ is a frame.

**Proof**

(i) Suppose that $\{e_{i,k}\}_{k \in J_i}$ is a Parseval frame for $E_i$ and $\{e_{i,k}\}_{i \in \Omega, k \in J_i}$ is a finite subset of $\mathbb{A}$, where $\Omega$ and $\Omega_i$'s are finite index sets. Since $\Lambda$ is an $(A, B)$ standard $g$-frame, Corollary 3.4 in [16] yields that $\{u_{i,k} = \Lambda_i^*(e_{i,k}) : i \in I, k \in J_i\}$ is an $(A, B)$ standard frame. Now for $v_{i,k} = \Gamma_i^*(e_{i,k})$, we have

$$\left\| \sum_{i \in \Omega} \sum_{k \in J_i} c_{i,k} (u_{i,k} - v_{i,k}) \right\| = \left\| \sum_{i \in \Omega} \sum_{k \in J_i} (\Lambda_i^* - \Gamma_i^*)(c_{i,k} e_{i,k}) \right\| \leq \lambda_1 \left\| \sum_{i \in \Omega} \sum_{k \in J_i} c_{i,k} u_{i,k} \right\| + \lambda_2 \left\| \sum_{i \in \Omega} \sum_{k \in J_i} c_{i,k} v_{i,k} \right\| + \varepsilon \left\| \sum_{i \in \Omega} \sum_{k \in J_i} |c_{i,k}|^2 \right\|^{\frac{1}{2}}.$$

Hence, Theorem 3.2 in [14] implies that $\{v_{i,k} = \Gamma_i^*(e_{i,k}) : i \in I, k \in J_i\}$ is a standard Bessel sequence with upper bound $\frac{[(1+\lambda_1)\sqrt{\lambda_1+\varepsilon}]}{(1-\lambda_2)^2}$, so (by [16, Theorem 3.3]) $\Gamma$ is a standard $g$-Bessel sequence with upper bound $\frac{[(1+\lambda_1)\sqrt{\lambda_1+\varepsilon}]}{(1-\lambda_2)^2}$. Thus, for each $\{f_i\}_{i \in I} \in \oplus_{i \in I} E_i$, the series $\sum_{i \in I} \Gamma_i^* f_i$ converges in $E$ and from (2), we can get

$$\left\| \sum_{i \in I} (\Lambda_i^* - \Gamma_i^*) f_i \right\| \leq \lambda_1 \left\| \sum_{i \in I} \Lambda_i^* f_i \right\| + \lambda_2 \left\| \sum_{i \in I} \Gamma_i^* f_i \right\| + \varepsilon \left\| \sum_{i \in I} |f_i|^2 \right\|^{\frac{1}{2}}. \tag{3}$$

Since for each $x \in E$, $\{f_i = \bar{\Lambda}_i x\}_{i \in I} \in \oplus_{i \in I} E_i$ and $\frac{1}{A}$ is an upper bound for $\bar{\Lambda}$, by using (3), we have

$$\|S_\Lambda x - \bar{x} - x\| \leq \|S_\Lambda x - \bar{x}\| \leq (\lambda_1 + \frac{\varepsilon}{\sqrt{A}}) \|x\| + \lambda_2 \|S_\Gamma x\|.$$

Now we can obtain a result similar to the proof of Proposition 3.10 in [20].
(ii) Since $A_f^*(a) = af_i$ and $A_g^*(a) = ag_i$, for each $a \in \mathfrak{A}$, the result follows from part (i) for $A_i = A_{f_i}$ and $\Gamma_i = A_{g_i}$.

4. Approximate duals and $\varepsilon$-nearly Parseval frames

In this section, we consider $\varepsilon$-nearly Parseval frames in Hilbert spaces, which are useful in applications (see [5]). We obtain some results for approximate duals of $\varepsilon$-nearly Parseval and $\varepsilon$-close frames (since Hilbert spaces are special cases of Hilbert $C^*$-modules, we do not state the definitions of frames, g-frames, and approximate duals in Hilbert spaces separately).

$\varepsilon$-nearly Parseval frames were defined in [5] and we have the following definition:

**Definition 4.1** Suppose that $H$ is a separable Hilbert space and $\{H_i\}_{i \in I}$ is a sequence of separable Hilbert spaces.

(i) Let $A_i$ be a bounded operator from $H$ into $H_i$ and $\varepsilon < 1$. We say that $\Lambda = \{A_i\}_{i \in I}$ is an $\varepsilon$-nearly Parseval g-frame if for each $f \in H$

$$(1 - \varepsilon)\|f\|^2 \leq \sum_{i \in I} \|A_if\|^2 \leq (1 + \varepsilon)\|f\|^2.$$ 

(ii) Let $\{f_i\}_{i \in I}$ be a sequence in $H$ and $\varepsilon < 1$. We say that $\{f_i\}_{i \in I}$ is an $\varepsilon$-nearly Parseval frame if for each $f \in H$

$$(1 - \varepsilon)\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq (1 + \varepsilon)\|f\|^2.$$ 

It is clear that if $\varepsilon = 0$, then an $\varepsilon$-nearly Parseval g-frame is a Parseval g-frame and so it is a g-dual of itself. Now we have the following result for approximate duals:

**Theorem 4.2**

(i) If $\Lambda$ is an $\varepsilon$-nearly Parseval g-frame, then it is an approximate g-dual of itself.

(ii) If $\{f_i\}_{i \in I}$ is an $\varepsilon$-nearly Parseval frame, then it is an approximate dual of itself.

**Proof**

(i) Since $\Lambda$ is an $\varepsilon$-nearly Parseval g-frame, we have $(1 - \varepsilon).Id_H \leq S_\Lambda \leq (1 + \varepsilon).Id_H$, so $-\varepsilon.Id_H \leq (S_\Lambda - Id_H) \leq \varepsilon.Id_H$. Because $S_{\Lambda \Lambda} = S_\Lambda$, we obtain that $\|S_{\Lambda \Lambda} - Id_H\| \leq \varepsilon < 1$. This means that $\Lambda$ is an approximate g-dual of itself.

(ii) Since frames are special cases of g-frames, we get the result from part (i). \qed

Note that if $\{A_i\}_{i \in I}$ is a g-Bessel sequence and $J \subset I$, then we denote the optimal upper bound of $\{A_i\}_{i \in J^c}$ by $B(J^c)$.

It was shown in Theorem 3.1 in [20] that if $\{A_i\}_{i \in I}$ is a Parseval g-frame and $B(J^c) < 1$, then $\{A_i\}_{i \in J}$ is an approximate g-dual of itself. Now we have the following result for $\varepsilon$-nearly Parseval g-frames:

**Proposition 4.3**

(i) Let $\Lambda$ be an $\varepsilon$-nearly Parseval g-frame and $J \subset I$ such that $B(J^c) < 1 - \varepsilon$. Then $\{A_i\}_{i \in J}$ is an approximate g-dual of itself.
(ii) Let \( \{f_i\}_{i \in I} \) be an \( \varepsilon \)-nearly Parseval frame and \( J \subset I \) such that \( B(J^c) < 1 - \varepsilon \). Then \( \{f_i\}_{i \in J} \) is an approximate dual of itself.

**Proof**  
(i) We have

\[
\sum_{i \in J} \|A_i f\|^2 = \sum_{i \in I} \|A_i f\|^2 - \sum_{i \notin J} \|A_i f\|^2 \\
\geq \sum_{i \in I} \|A_i f\|^2 - B(J^c)\|f\|^2 \geq (1 - \varepsilon - B(J^c))\|f\|^2.
\]

Hence, \( (1 - (\varepsilon + B(J^c))) \) is a lower bound for \( \{A_i\}_{i \in J} \). Therefore, \( \{A_i\}_{i \in J} \) is an \( \varepsilon' \)-nearly Parseval frame, where \( \varepsilon' = (\varepsilon + B(J^c)) \). Now by Theorem 4.2, \( \{A_i\}_{i \in J} \) is an approximate g-dual of itself.

(ii) The result follows from part (i).

Recall that two sequences \( \{f_i\}_{i \in I} \) and \( \{g_i\}_{i \in I} \) in \( H \) are \( \varepsilon \)-close if \( \sum_{i \in I} \|f_i - g_i\|^2 \leq \varepsilon^2 \) (see [5, Definition 2.4]).

**Example 4.4** Let \( H = \mathbb{C}^2 \), \( \{e_1, e_2\} \) be the standard orthonormal basis for \( H \) and \( \frac{2}{3} < \varepsilon < 1 \). For \( F = \{\sqrt{2}e_1, e_2\} \) and \( G = \{0, e_2\} \), it is easy to see that \( F \) is an \( \varepsilon \)-nearly Parseval frame that is \( \varepsilon \)-close to \( G \), but \( F \) and \( G \) are not approximately dual frames because \( G \) is not a frame.

Now we have the following result:

**Proposition 4.5**  
(i) Let \( F = \{f_i\}_{i \in I} \) be an \( \varepsilon \)-nearly Parseval frame with upper bound \( A \). If \( \{g_i\}_{i \in I} \) and \( \{f_i\}_{i \in I} \) are \( \varepsilon \)-close with \( \sqrt{A}\varepsilon < 1 - \|\text{Id}_H - S_F\| \), then \( F \) is an approximate dual of \( \{g_i\}_{i \in I} \).

(ii) If \( F \) in part (i) is also an \( A \)-tight frame with \( \sqrt{A}\varepsilon < 1 - |1 - A| \), then \( F \) is an approximate dual of \( \{g_i\}_{i \in I} \).

**Proof**  
(i) Since \( F \) is an \( \varepsilon \)-nearly Parseval frame, it is an approximate dual of itself by Theorem 4.2. Also, for each \( f \in H \), we have

\[
\sum_{i \in I} |\langle f, f_i - g_i \rangle|^2 \leq \|f\|^2 \sum_{i \in I} \|f_i - g_i\|^2 \leq \varepsilon^2 \|f\|^2.
\]

By considering \( C = A \) and \( K = \varepsilon^2 \), we obtain that \( \{f_i - g_i\}_{i \in I} \) is a Bessel sequence with upper bound \( K \) and \( \sqrt{CK} = \varepsilon \sqrt{A} < 1 - \|\text{Id}_H - S_F\| \). Now the result follows from Proposition 3.7 (or [20, Theorem 3.1]).

(ii) The result follows from part (i) by considering \( S_F = A\text{Id}_H \).

**Remark 4.6** We can obtain from part (ii) of the above proposition that if \( F = \{f_i\}_{i \in I} \) is an \( A \)-tight frame with \( A \leq 1 \), then each sequence \( \{g_i\}_{i \in I} \) that is \( \varepsilon \)-close to \( F \) with \( \varepsilon < \sqrt{A} \) is an approximate dual of \( F \). Especially if \( F \) is a Parseval frame that is \( \varepsilon \)-close to \( G = \{g_i\}_{i \in I} \), then \( F \) and \( G \) are approximately dual frames.

Let \( F = \{f_i\}_{i=1}^n \) be an \( \varepsilon \)-nearly Parseval frame for a d-dimensional Hilbert space \( H \). As we know (see [7, Theorem 5.3.4] and [3]), \( \{S_F^{-1} f_i\}_{i=1}^n \) is the closest Parseval frame to \( F \). It was proved in [5, Proposition 3.1 and
Remark 3.2] that the relation \( \sum_{i=1}^{n} \|S_{F_i}^{-1} f_i - f_i\|^2 = d(2 - \varepsilon - 2\sqrt{1 - \varepsilon}) \leq \frac{d^2}{8} \) holds if \( \{\frac{1}{\sqrt{1 - \varepsilon}} f_i\}_{i=1}^{n} \) is a Parseval frame (or equivalently \( S_{F_i}^{-1} = \frac{1}{\sqrt{1 - \varepsilon}} Id_{H} \), so we have \( \sum_{i=1}^{n} \|S_{F_i}^{-1} f_i - f_i\|^2 = d(2 - \varepsilon - 2\sqrt{1 - \varepsilon}) \leq \frac{d^2}{8} \)). It was also shown in Example 2.4 in [24] that if \( \mathcal{F} = \{f_i\}_{i=1}^{n} \) is an \( \varepsilon \)-nearly Parseval frame for a finite-dimensional Hilbert space \( H \), then \( \{\frac{1}{2}(f_i + S_{F_i}^{-1} f_i)\}_{i=1}^{n} \) is a \( (1, 1 + \frac{4}{2}) \) frame, which is much closer to \( \mathcal{F} \) than \( \{S_{F_i}^{-1} f_i\}_{i=1}^{n} \) with better frame bounds compared with the bounds of \( \mathcal{F} \). Therefore, the frames of the forms \( \{\frac{1}{\sqrt{1 - \varepsilon}} f_i\}_{i\in I} \) and \( \{\frac{1}{2}(f_i + S_{F_i}^{-1} f_i)\}_{i\in I} \) are useful in applications (also, see [1, Section 3]). Now we have the following results:

**Proposition 4.7** (i) Let \( \Lambda = \{\Lambda_i\}_{i\in I} \) be an \( \varepsilon \)-nearly Parseval g-frame with \( \varepsilon < \frac{1}{3} \). Then \( \{\frac{1}{\sqrt{1 - \varepsilon}} \Lambda_i\}_{i\in I} \) is an approximate g-dual of itself.

(ii) Let \( \{f_i\}_{i\in I} \) be an \( \varepsilon \)-nearly Parseval frame with \( \varepsilon < \frac{1}{3} \). Then \( \{\frac{1}{\sqrt{1 - \varepsilon}} f_i\}_{i\in I} \) is an approximate dual of itself.

**Proof** (i) We have \((1 - \varepsilon)Id_{H} \leq S_{\Lambda} \leq (1 + \varepsilon)Id_{H}, \) so \( 0 \leq \frac{S_{\Lambda}}{1 - \varepsilon} - Id_{H} \leq \frac{2\varepsilon}{1 - \varepsilon} Id_{H} \). Therefore, \( \|\frac{S_{\Lambda}}{1 - \varepsilon} - Id_{H}\| \leq \frac{2\varepsilon}{1 - \varepsilon} < 1 \). This means that \( \{\frac{1}{\sqrt{1 - \varepsilon}} \Lambda_i\}_{i\in I} \) and \( \{\frac{1}{\sqrt{1 - \varepsilon}} g_i\}_{i\in I} \) are approximately dual g-frames.

(ii) The result follows from part (i).

**Proposition 4.8** Let \( \mathcal{F} = \{f_i\}_{i\in I} \) be a frame and \( \mathcal{G} = \{g_i\}_{i\in I} \), where \( g_i = \frac{1}{2}(f_i + S_{F_i}^{-1} f_i) \).

(i) If \( \|S_{\mathcal{F}} - Id_{H}\| < 2 \), then \( \mathcal{F} \) and \( \mathcal{G} \) are approximately dual frames.

(ii) If \( \mathcal{F} \) is an \( \varepsilon \)-nearly Parseval frame, then \( \mathcal{F} \) and \( \mathcal{G} \) are approximately dual frames.

(iii) If \( \mathcal{F} = \{f_i\}_{i\in I} \) is an A-tight frame with \( A < 3 \), then \( \mathcal{F} \) and \( \mathcal{G} \) are approximately dual frames.

**Proof** (i) For each \( f \in H \), we have

\[
S_{\mathcal{F}\mathcal{G}}(f) = \frac{1}{2} \left( \sum_{i\in I} \langle f, f_i \rangle f_i + \sum_{i\in I} \langle f, S_{F_i}^{-1} f_i \rangle f_i \right) = \frac{1}{2} \langle S_{\mathcal{F}} f + f \rangle.
\]

Hence:

\[
\|S_{\mathcal{F}\mathcal{G}} - Id_{H}\| = \frac{1}{2} \|S_{\mathcal{F}} - Id_{H}\| < 1.
\]

This means that \( \mathcal{F} \) and \( \mathcal{G} \) are approximately dual frames.

(ii) Since \( \mathcal{F} \) is an \( \varepsilon \)-nearly Parseval frame, \( \mathcal{F} \) is an approximate dual of itself by Theorem 4.2, and so \( \|S_{\mathcal{F}} - Id_{H}\| < 1 \). Now we get the result from part (i).

(iii) Since \( \mathcal{F} \) is A-tight, we have \( S_{\mathcal{F}} = A.Id_{H} \), so \( \|S_{\mathcal{F}} - Id_{H}\| = |A - 1| < 2 \), and the result follows from part (i).
References