A note on m-embedded subgroups of finite groups

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Received: 27.02.2014 • Accepted/Published Online: 11.03.2015 • Printed: 30.07.2015

Abstract: Let $A$ be a subgroup of $G$. $A$ is m-embedded in $G$ if $G$ has a subnormal subgroup $T$ and a $\{1 \leq G\}$-embedded subgroup $C$ such that $G = AT$ and $T \cap A \leq C \leq A$. In this paper, we study the structure of finite groups by using m-embedded subgroups and obtain some new results about $p$-supersolvability and $p$-nilpotency of finite groups.

Key words: Sylow subgroup, $\{1 \leq G\}$-embedded, m-embedded subgroup, saturated formation, finite groups

1. Introduction

Throughout the paper, all groups are finite. Most of the notation is standard and can be found in [3, 6, 10, 11]. Let $\mathcal{F}$ be a class of groups. $\mathcal{F}$ is said to be a formation provided that (1) if $G \in \mathcal{F}$ and $H \leq G$, then $G/H \in \mathcal{F}$, and (2) if $G/M$ and $G/N$ are in $\mathcal{F}$, then $G/M \cap N$ is in $\mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if $G \in \mathcal{F}$ whenever $G = (G)$ in $\mathcal{F}$. It is well known that the class of all $p$-supersolvable groups and the class of all $p$-nilpotent groups are saturated formations. Let $A$ be a subgroup of $G$, $K \leq H \leq G$ and $p$ a prime. Then: (1) $A$ covers the pair $(K, H)$ if $AH = AK$; (2) $A$ avoids $(K, H)$ if $A \cap H = A \cap K$. Recall that a subgroup $A$ of $G$ is called a CAP-subgroup [3, A, Definition 10.8] if $A$ either covers or avoids each pair $(K, H)$, where $H/K$ is a chief factor of $G$. A subgroup $A$ is called a partial CAP-subgroup [1] or a semicover-avoiding subgroup [8] of $G$ if $A$ either covers or avoids each pair $(K, H)$, where $H/K$ is a factor of some fixed chief series of $G$. By using the CAP-subgroups and the semicover-avoiding subgroups, group theorists have obtained many interesting results (see, for example, [2, 4, 9]). Furthermore, if $E$ is a quasinormal subgroup of $G$, then for every maximal pair of $G$, that is, a pair $(K, H)$, where $K$ is a maximal subgroup of $H$, $E$ either covers or avoids $(K, H)$. Based on the definitions and properties above, Guo and Skiba presented a new concept as follows:

Definition 1.1 (7) Let $A$ be a subgroup of $G$ and $\Sigma = G_0 \leq G_1 \leq \ldots \leq G_n$ some subgroup series of $G$. Then $A$ is $\Sigma$-embedded in $G$ if $A$ either covers or avoids every maximal pair $(K, H)$ such that $G_{i-1} \leq K \leq H \leq G_i$, for some $i$.

Here we improve Theorem 4.1 of [7], and present a result of $p$-nilpotency of group $G$ with some “extra hypothesis”, where $p$ is an odd prime divisor of $|G|$. Meanwhile, we study the structure of $G$ under the

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2010 AMS Mathematics Subject Classification: 20D10, 20D20.
This research is supported by NSFC (Grant #11271016) and Qing Lan Project of Jiangsu Province and High-Level Personnel Support Program of Yangzhou University and 333 High-Level Personnel Training Project in Jiangsu Province.
assumption of \( G \) is \( p \)-solvable, where \( p \) is a prime divisor of \(|G|\).

**Theorem 1.2** Let \( p \) be an odd prime divisor of \(|G|\) and \( P \) be a Sylow \( p \)-subgroup of \( G \). Suppose that every maximal subgroup \( P_1 \) of \( P \) is \( m \)-embedded in \( G \). Then \( G \) is \( p \)-nilpotent if one of the following conditions holds:

1. \( N_G(P_1) \) is \( p \)-nilpotent for every maximal subgroup \( P_1 \) of \( P \).
2. \( N_G(P) \) is \( p \)-nilpotent.

**Theorem 1.3** Let \( G \) be a \( p \)-solvable group and \( P \) a Sylow \( p \)-subgroup of \( G \). Suppose that every maximal subgroup \( P_1 \) of \( P \) is \( m \)-embedded in \( G \). Then \( G \) is \( p \)-nilpotent if one of the following conditions holds:

1. \( N_G(P_1) \) is \( p \)-nilpotent for every maximal subgroup \( P_1 \) of \( P \).
2. \( N_G(P) \) is \( p \)-nilpotent.

**Theorem 1.4** Let \( G \) be a \( p \)-solvable group and \( p \) a prime divisor of \(|G|\). If every maximal subgroup of \( F_p(G) \) containing \( O_{p'}(G) \) is \( m \)-embedded in \( G \), then \( G \) is \( p \)-supersolvable.

2. Preliminaries

For the sake of convenience, we first list here some known results that will be useful in the sequel.

**Lemma 2.1** (7, Lemma 2.13) Let \( K \) and \( H \) be subgroups of \( G \). Suppose that \( K \) is \( m \)-embedded in \( G \) and \( H \) is normal in \( G \). Then

1. If \( H \triangleleft K \), then \( K = H \) is \( m \)-embedded in \( G / H \).
2. If \( K \triangleleft E \triangleleft G \), then \( K \) is \( m \)-embedded in \( E \).
3. If \( |H|, |K| = 1 \), then \( HK/H \) is \( m \)-embedded in \( G / H \).
4. Suppose that \( K \) is a \( p \)-subgroup for some prime \( p \), \( K \) is \( m \)-embedded in \( G \), and \( K \) is not \( \{1 \leq G\} \)-embedded in \( G \). Then \( G \) has a normal subgroup \( M \) such that \(|G : M| = p \) and \( G = KM \).

**Lemma 2.2** (7, Lemma 2.14) Let \( P \) be a normal nonidentity \( p \)-subgroup of \( G \) with \(|P| = p^n \) and \( P \cap \Phi(G) = 1 \). Suppose that there is an integer \( k \) such that \( 1 \leq k < n \) and the subgroups of \( P \) of order \( p^k \) are \( m \)-embedded in \( G \), then some maximal subgroup of \( P \) is normal in \( G \).

**Lemma 2.3** (7, Lemma 2.5) Every \( \{1 \leq G\} \)-embedded subgroup of \( G \) is subnormal in \( G \).

3. The proofs

**Proof of Theorem 1.1** Assume that the assertion is false and choose \( G \) to be a counterexample of minimal order. We will divide the proof into the following steps.

1. \( O_{p'}(G) = 1 \).

   In fact, if \( O_{p'}(G) \neq 1 \), then we consider the quotient group \( G/O_{p'}(G) \). If \( N_G(P_1) \) is \( p \)-nilpotent, then

   \[
   N_{G/O_{p'}(G)}(P_1O_{p'}(G)/O_{p'}(G)) = N_G(P_1)O_{p'}(G)/O_{p'}(G)
   \]

   is \( p \)-nilpotent. By Lemma 2.1(3), \( G/O_{p'}(G) \) satisfies the conditions of the theorem, and the minimal choice of \( G \) implies that \( G/O_{p'}(G) \) is \( p \)-nilpotent. Hence \( G \) is \( p \)-nilpotent, a contradiction. Similarly, if \( N_G(P) \) is \( p \)-nilpotent, then we have \( G/O_{p'}(G) \) is \( p \)-nilpotent also, a contradiction.

2. If \( S \) is a proper subgroup of \( G \) containing \( P \), then \( S \) is \( p \)-nilpotent.
If \( N_G(P_1) \) is \( p \)-nilpotent, clearly, \( N_S(P_1) \leq N_G(P_1) \) and then \( N_S(P_1) \) is \( p \)-nilpotent. Applying Lemma 2.1(2), we find that \( S \) satisfies the hypothesis of our theorem. Now, the minimal choice of \( G \) implies that \( S \) is \( p \)-nilpotent. If \( N_G(P) \) is \( p \)-nilpotent, then we still obtain that \( S \) is \( p \)-nilpotent since \( N_S(P) \leq N_G(P) \).

(3) \( O_p(G) \neq 1 \) and \( G/N \) is \( p \)-nilpotent, where \( N = O_p(G) \) is the unique minimal normal subgroup of \( G \).

Case I. \( N_G(P_1) \) is \( p \)-nilpotent.

Since \( G \) is not \( p \)-nilpotent, \( N_G(Z(J(P))) \) is not \( p \)-nilpotent by the Glauberman–Thompson Theorem, where \( J(P) \) is the Thompson subgroup of \( P \). Then \( P \leq N_G(Z(J(P))) \). By (2), we have \( N_G(Z(J(P))) = G \) and hence \( O_p(G) \neq 1 \). Let \( N \) be a minimal normal subgroup of \( G \) contained in \( O_p(G) \).

If \( N = P \), then \( G/N \) is \( p \)-nilpotent. If \( |P : N| = p \), then \( G = N_G(N) \) is \( p \)-nilpotent, a contradiction. Now we may assume that \( |P : N| > p \). For every maximal subgroup \( P_1/N \) of \( P/N \),

\[
N_{G/N}(P_1/N) = N_G(P_1N)/N = N_G(P_1)/N
\]

is \( p \)-nilpotent and \( P_1/N \) is \( m \)-embedded in \( G/N \) by Lemma 2.1(1). Therefore \( G/N \) satisfies the hypothesis of the theorem, and hence \( G/N \) is \( p \)-nilpotent. Obviously, \( N \) is the unique minimal normal subgroup of \( G \) contained in \( O_p(G) \) and \( \Phi(G) = 1 \). Then we obtain that \( N = O_p(G) \) is an elementary abelian \( p \)-group.

Case II. \( N_G(P) \) is \( p \)-nilpotent.

Since \( G \) is not \( p \)-nilpotent, by Corollary of [12], there exists a characteristic subgroup \( H \) of \( P \) such that \( N_G(H) \) is not \( p \)-nilpotent. Since \( N_G(P) \) is \( p \)-nilpotent, we may choose a characteristic subgroup \( H \) of \( P \) such that \( N_G(H) \) is not \( p \)-nilpotent, but \( N_G(K) \) is \( p \)-nilpotent for any characteristic subgroup \( K \) of \( P \) with \( H < K \leq P \). Since \( P \leq N_G(H) \) and \( N_G(K) \) is not \( p \)-nilpotent, we have \( N_G(H) = G \) by (2). This leads to \( O_p(G) \neq 1 \) and \( N_G(K) \) is \( p \)-nilpotent for any characteristic subgroup \( K \) of \( P \) such that \( O_p(G) < K \leq P \). Now by using Corollary of [12] again, we see that \( G/O_p(G) \) is \( p \)-nilpotent and \( |P : O_p(G)| > p \). Let \( N \) be a minimal normal subgroup of \( G \) contained in \( O_p(G) \).

Since \( |P : N| > p \), \( P/N \) is a Sylow \( p \)-subgroup of \( G/N \), and

\[
N_{G/N}(P/N) = N_G(PN)/N = N_G(P)/N
\]

is \( p \)-nilpotent and every maximal subgroup \( P_1/N \) of \( P/N \) is \( m \)-embedded in \( G/N \) by Lemma 2.1(1). Therefore \( G/N \) satisfies the hypothesis of the theorem, and hence \( G/N \) is \( p \)-nilpotent. Obviously, \( N \) is the unique minimal normal subgroup of \( G \) contained in \( O_p(G) \) and \( \Phi(G) = 1 \). Then we obtain that \( N = O_p(G) \) is an elementary abelian \( p \)-group.

(4) \( G = PQ \), where \( Q \) is a Sylow \( q \)-subgroup of \( G \) and \( q \neq p \) is a prime divisor of \( |G| \).

By (3), immediately we obtain that \( G \) is \( p \)-solvable, and then by (1) \( C_G(N) = N \) since \( N \leq C_G(N) \leq N \). For any \( q \in \pi(G) \) with \( q \neq p \), Theorem 6.3.5 of [5] implies that there exists a Sylow \( q \)-subgroup \( Q \) of \( G \) such that \( G_1 = PQ \) is a subgroup of \( G \). If \( G_1 < G \), then \( G_1 \) is \( p \)-nilpotent by (2). This leads to \( Q \leq C_G(N) \leq N \), a contradiction. Thus \( G = PQ \).

(5) The final contradiction.

Since \( N \not\leq \Phi(G) \), there exists a maximal subgroup \( M \) of \( G \) such that \( G = NM \) and \( N \cap M = 1 \). Let \( M_p \) be Sylow \( p \)-subgroup of \( M \). Firstly, we may assume that \( M_p \neq 1 \). Otherwise, \( M_p = 1 \) and then \( P = N \). If \( N_G(P) \) is \( p \)-nilpotent, then \( G \) is \( p \)-nilpotent, a contradiction. If \( N_G(P_1) \) is \( p \)-nilpotent, then there exists a
maximal subgroup $P_1$ of $P$ such that $P_1$ is normal in $G$ by Lemma 2.2. Therefore $G = N_G(P_1)$ is $p$-nilpotent, a contradiction. Now we may obtain the final contradiction as follows.

Now we pick a maximal subgroup $P_1$ of $P$ such that $M_p \leq P_1$. By hypothesis, $P_1$ is m-embedded in $G$, that is, $G$ has a subnormal subgroup $T$ and a $\{1 \leq G\}$-embedded subgroup $C$ such that $G = P_1 T$ and $P_1 \cap T \leq C \leq P_1$. Applying Lemma 2.3, we obtain that $C \leq O_p(G) = N$.

Assume that $C \neq 1$. If $C < N$, then for $N \cap M = 1$, we obtain $C$ neither covers nor avoids maximal pair $(M, G)$, a contradiction. Hence we may assume that $C = N$, i.e. $N \leq P_1$ and then $P = NM_p \leq P_1 < P$, a contradiction.

Assume that $C = 1$. The Sylow $p$-subgroup of $T$ is cyclic with order $p$. It follows from $N \leq O^p(G) \leq T$ that $|N| = p$. Therefore $M \cong G/N = N_G(N)/C_G(N)$ is isomorphic to a subgroup of $Aut(N)$, and then $M$ is cyclic with order $q^\alpha$ by (4), that is, $M_p = 1$, a contradiction.

The final contradiction completes our proof.

**Proof of Theorem 1.2** Assume that the assertion is false and choose $G$ to be a counterexample of minimal order. Furthermore, we have that

(1) $O_{p'}(G) = 1$.

If $L = O_{p'}(G) \neq 1$, we consider $G/L$. Clearly, $P_1 L/L$ is a maximal subgroup of Sylow $p$-subgroup of $G/L$ where $P_1$ is a maximal subgroup of $P$. Since $P_1$ is m-embedded in $G$, we have $P_1 L/L$ is also m-embedded in $G/L$ by Lemma 2.1(3). Therefore $G/L$ satisfies the condition of the theorem. The minimal choice of $G$ implies that $G/L$ is $p$-supersolvable, and hence $G$ is $p$-supersolvable, a contradiction.

(2) $O_p(G) \neq 1$.

Since $G$ is $p$-solvable and $O_{p'}(G) = 1$, we have that a minimal normal subgroup of $G$ is an abelian $p$-group and hence $O_p(G) \neq 1$.

(3) Final contradiction.

By (2), we may pick a minimal normal subgroup $N$ of $G$ contained in $O_p(G)$. If $N = P$ then $G/N$ is $p$-supersolvable. If $N = P_1$, where $P_1$ is a maximal subgroup of $P$, then $G/N$ is $p$-supersolvable. Now we may assume that $|P : N| > p$. By Lemma 2.1(1), we know that $G/N$ satisfies the condition of the theorem, and hence the minimality of $G$ implies that $G/N$ is $p$-supersolvable; on the other hand, since the class of all $p$-supersolvable groups is a saturated formation, we have $N'$ is the unique minimal normal subgroup of $G$ and $O_p(G) = N \not\leq \Phi(G)$. If $O_p(G) = P$, then by Lemma 2.2, some maximal subgroup of $P$ is normal in $G$, a contradiction. Now we may assume that $N \leq P$.

Clearly, there exists a maximal subgroup $M$ of $G$ such that $G = NM$ with $N \cap M = 1$ and $P = NM_p$ with $M_p \neq 1$. Now we choose a maximal subgroup $P_1$ with $M_p \leq P_1$. By hypothesis, $P_1$ is m-embedded in $G$. Therefore $G$ has a subnormal subgroup $T$ and a $\{1 \leq G\}$-embedded subgroup $C$ such that $G = P_1 T$ and $P_1 \cap T \leq C \leq P_1$. On the other hand, we know that $C \leq O_p(G)$. Therefore $C \leq N$. If $1 < C < N$, then for $N \cap M = 1$, we have $C$ neither covers nor avoids maximal pair $(M, G)$. Now we may assume that either $C = N$ or $C = 1$. By the choice of $P_1$, we immediately have $P_1 \cap T = 1$ and then the Sylow $p$-subgroup of $T$ is cyclic with order $p$. It follows from $N \leq O^p(G) \leq T$ that $|N| = p$. Therefore $G$ is $p$-supersolvable since $G/N$ $p$-supersolvable, a contradiction.

The final contradiction completes our proof.
Proof of Theorem 1.3. Assume that the assertion is false and choose $G$ to be a counterexample of minimal order. Furthermore, we have that

(1) $O_{p'}(G) = 1$.

If $T = O_{p'}(G) \neq 1$, we consider $G/T$. Firstly, $F_p(G/T) = F_p(G)/T$. Let $M/T$ be a maximal subgroup of $F_p(G/T)$. Then $M$ is a maximal subgroup of $F_p(G)$ containing $O_{p'}(G)$. Since $M$ is m-embedded in $G$, then $M/T$ is m-embedded in $G/T$ by Lemma 2.1(3). Thus $G/T$ satisfies the hypothesis of the theorem. The minimality of $G$ implies that $G/T$ is $p$-supersolvable and so is $G$, a contradiction.

(2) $\Phi(G) = 1$ and $F_p(G) = F(G) = O_p(G)$.

If not, then $L = \Phi(G) \neq 1$. We consider $G/L$. Since $O_{p'}(G) = 1$, it is easy to show that $F_p(G) = F(G) = O_p(G)$. This implies that $F_p(G/L) = O_p(G/L) = O_p(G)/L = F_p(G)/L$. If $P_1/L$ is a maximal subgroup of $F_p(G/L)$, then $P_1$ is a maximal subgroup of $F_p(G)$. Since $P_1$ is m-embedded in $G$ and hence $P_1/L$ is m-embedded in $G/L$ by Lemma 2.1(1). Thus $G/L$ satisfies the hypothesis of the theorem. The minimal choice of $G$ implies that $G/L$ is $p$-supersolvable and so is $G$, since the class of all $p$-supersolvable groups is a saturated formation, a contradiction.

(3) Every minimal normal subgroup of $G$ contained in $F(G)$ is cyclic of order $p$.

By (2), $P = F(G) = R_1 \times \cdots \times R_t$, where $R_i$ ($i = 1, 2, \ldots, t$) is a minimal normal subgroup of $G$ contained in $F(G)$. At the same time, Lemma 2.2 implies that $t \geq 2$. Since $G$ is $p$-solvable and $O_{p'}(G) = 1$, we have $C_G(O_p(G)) \leq O_p(G)$. Thus $C_G(F(G)) = F(G)$. Suppose that there exists $R_i$ such that $|R_i| > p$. Without loss of generality, let $i = 1$ and $R = R_2 \times \cdots \times R_t$. Obviously, we may assume that $P/R \cap \Phi(G)/R = 1$, in fact, if $P/R \cap \Phi(G)/R \neq 1$, then $P/R \leq \Phi(G)/R$ since $R_1 \cong P/R$ is a chief factor of $G$. Therefore $P \leq \Phi(G)/R$ and then $P = P \cap \Phi(G)/R = R(P \cap \Phi(G)) = R$, a contradiction. Applying Lemma 2.1(1), $G/R$ satisfies the hypothesis of the theorem and we have that some maximal subgroup of $P/R$ is normal in $G/R$ by Lemma 2.2, which contradicts the minimality of $R_1$. Therefore every $R_i$ is of order $p$.

(4) The final contradiction.

By (3), $P = F(G) = R_1 \times \cdots \times R_t$, where $R_i$ is a minimal normal subgroup of $G$ of order $p$. For each $i$ the quotient $G/C_G(R_i)$ is a subgroup of $\text{Aut}(R_i)$ and hence is abelian. Since the class of all $p$-supersolvable groups is a formation, we have $G/\bigcap_{i=1}^t(C_G(R_i))$ is $p$-supersolvable, and thus $G/F(G)$ is $p$-supersolvable because $\bigcap_{i=1}^t(C_G(R_i)) = C_G(F(G)) = F(G)$. Actually, all chief factors of $G$ below $F(G)$ are cyclic groups of order $p$; therefore $G$ is $p$-supersolvable.

The final contradiction completes our proof.

4. Applications

Obviously, if $H$ is $\{1 \leq G\}$-embedded in $G$, then $H$ is m-embedded in $G$. Therefore we have the following corollaries.

Corollary 4.1 Let $p$ be an odd prime divisor of $|G|$ and $P$ be a Sylow $p$-subgroup of $G$. If every maximal subgroup $P_1$ of $P$ is $\{1 \leq G\}$-embedded in $G$ and $N_G(P_1)$ is $p$-nilpotent, then $G$ is $p$-nilpotent.

Corollary 4.2 Let $p$ be an odd prime divisor of $|G|$ and $P$ be a Sylow $p$-subgroup of $G$. If every maximal subgroup $P_1$ of $P$ is $\{1 \leq G\}$-embedded in $G$ and $N_G(P)$ is $p$-nilpotent, then $G$ is $p$-nilpotent.
Corollary 4.3 Let $G$ be a $p$-solvable group. If every maximal subgroup of a Sylow subgroup of $G$ is $\{1 \leq G\}$-embedded in $G$, then $G$ is $p$-supersolvable.

Corollary 4.4 Let $G$ be a $p$-solvable group and $p$ a prime divisor of $|G|$. If every maximal subgroup of $F_p(G)$ containing $O_p(G)$ is $\{1 \leq G\}$-embedded in $G$, then $G$ is $p$-supersolvable.

Acknowledgment

We thank the referee for his/her careful reading of the manuscript and for his/her suggestions, which have helped to improve our original version.

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