The prime tournaments $T$ with $|W_5(T)|=|T|-2$

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Received: 30.12.2014 • Accepted/Published Online: 20.05.2015 • Printed: 30.07.2015

Abstract: We consider a tournament $T = (V, A)$. For $X \subseteq V$, the subtournament of $T$ induced by $X$ is $T[X] = (X, A \cap (X \times X))$. A module of $T$ is a subset $X$ of $V$ such that for $a, b \in X$ and $x \in V \setminus X$, $(a, x) \in A$ if and only if $(b, x) \in A$. The trivial modules of $T$ are $\emptyset$, $\{x\} (x \in V)$, and $V$. A tournament is prime if all its modules are trivial. For $n \geq 2$, $W_{2n+1}$ denotes the unique prime tournament defined on $\{0, \ldots, 2n\}$ such that $W_{2n+1}[[0, \ldots, 2n-1]]$ is the usual total order. Given a prime tournament $T$, $W_5(T)$ denotes the set of $v \in V$ such that there is $W \subseteq V$ satisfying $v \in W$ and $T[W]$ is isomorphic to $W_5$. B.J. Latka characterized the prime tournaments $T$ such that $W_5(T) = \emptyset$. The authors proved that if $W_5(T) \neq \emptyset$, then $|W_5(T)| \geq |V| - 2$. In this article, we characterize the prime tournaments $T$ such that $|W_5(T)|=|V| - 2$.

Key words: Tournament, prime, embedding, critical, partially critical

1. Introduction
1.1. Preliminaries

A tournament $T = (V(T), A(T))$ (or $(V, A)$) consists of a finite set $V$ of vertices together with a set $A$ of ordered pairs of distinct vertices, called arcs, such that for all $x \neq y \in V$, $(x, y) \in A$ if and only if $(y, x) \not\in A$. The cardinality of $T$, denoted by $|T|$, is that of $V(T)$. Given a tournament $T = (V, A)$, with each subset $X$ of $V$ is associated the subtournament $T[X] = (X, A \cap (X \times X))$ of $T$ induced by $X$. For $X \subseteq V$ (resp. $x \in V$), the subtournament $T[V \setminus X]$ (resp. $T[V \setminus \{x\}]$) is denoted by $T - X$ (resp. $T - x$). Two tournaments $T = (V, A)$ and $T' = (V', A')$ are isomorphic, which is denoted by $T \simeq T'$, if there exists an isomorphism from $T$ onto $T'$, i.e. a bijection $f$ from $V$ onto $V'$ such that for all $x, y \in V$, $(x, y) \in A$ if and only if $(f(x), f(y)) \in A'$. We say that a tournament $T'$ embeds into $T$ if $T'$ is isomorphic to a subtournament of $T$. Otherwise, we say that $T$ omits $T'$. The tournament $T$ is said to be transitive if it omits the tournament $C_3 = (\{0, 1, 2\}, \{(0, 1), (1, 2), (2, 0)\})$. For a finite subset $V$ of $\mathbb{N}$, we denote by $\vec{V}$ the usual total order defined on $V$, i.e., the transitive tournament $(V, \{(i, j) : i < j\})$.

Some notations are needed. Let $T = (V, A)$ be a tournament. For two vertices $x \neq y \in V$, the notation $x \rightarrow y$ signifies that $(x, y) \in A$. Similarly, given $x \in V$ and $Y \subseteq V$, the notation $x \rightarrow Y$ (resp. $Y \rightarrow x$) means that $x \rightarrow y$ (resp. $y \rightarrow x$) for all $y \in Y$. Given $x \in V$, we set $N_T^+(x) = \{y \in V : x \rightarrow y\}$. For all

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2010 AMS Mathematics Subject Classification: 05C20, 05C60, 05C75.
n ∈ \mathbb{N} \setminus \{0\}, the set \{0, \ldots, n - 1\} is denoted by \mathbb{N}_n.

Let \( T = (V, A) \) be a tournament. A subset \( I \) of \( V \) is a *module* [11] (or a *clan* [7]) of \( T \) provided that for all \( x \in V \setminus I, x \rightarrow I \) or \( I \rightarrow x \). For example, \( \emptyset, \{x\} \), where \( x \in V \), and \( V \) are modules of \( T \), called *trivial* modules. A tournament is *prime* [4] (or *primitive* [7]) if all its modules are trivial. Notice that a tournament \( T = (V, A) \) and its *dual* \( T^* = (V, \{(x, y) : (y, x) \in A\}) \) share the same modules. Hence, \( T \) is prime if and only if \( T^* \) is.

For \( n \geq 2 \), we introduce the tournament \( W_{2n+1} \) defined on \( \mathbb{N}_{2n+1} \) as follows: \( W_{2n+1}[\mathbb{N}_{2n}] = \overrightarrow{\mathbb{N}_{2n}} \) and \( N_{W_{2n+1}}(2n) = \{2i : i \in \mathbb{N}_n\} \) (see Figure 1). In 2003, B.J. Latka [8] characterized the prime tournaments omitting the tournament \( W_5 \). In 2012, the authors were interested in the set \( W_5(T) \) of the vertices \( x \) of a prime tournament \( T = (V, A) \) for which there exists a subset \( X \) of \( V \) such that \( x \in X \) and \( T[X] \simeq W_5 \). They obtained the following.

**Theorem 1 ([1])** Let \( T \) be a prime tournament into which \( W_5 \) embeds. Then \( |W_5(T)| \geq |T| - 2 \). If, in addition, \( |T| \) is even, then \( |W_5(T)| \geq |T| - 1 \).

Our main result in this paper, presented in [3] without detailed proof, gives a characterization of the class \( \mathcal{T} \) of the prime tournaments \( T \) on at least 3 vertices such that \( |W_5(T)| = |T| - 2 \). This answers [1, Problem 4.4].

![Figure 1. \( W_{2n+1} \)](image)

1.2. Partially critical tournaments and the class \( \mathcal{T} \)

Our characterization of the tournaments of the class \( \mathcal{T} \) requires the study of their partial criticality structure, a notion introduced as a weakening of the notion of criticality defined in Section 2. These notions are defined in terms of critical vertices. A vertex \( x \) of a prime tournament \( T \) is *critical* [10] if \( T - x \) is not prime. The set of noncritical vertices of a prime tournament \( T \) was introduced in [9]. It is called the *support* of \( T \) and is denoted by \( \sigma(T) \). Let \( T \) be a prime tournament and let \( X \) be a subset of \( V(T) \) such that \( |X| \geq 3 \); we say that \( T \) is *partially critical according to \( T[X] \) (or \( T[X] \)-critical) [6] if \( T[X] \) is prime and if \( \sigma(T) \subseteq X \). We will see that: for \( T \in \mathcal{T} \), \( V(T) \setminus W_5(T) = \sigma(T) \). Partially critical tournaments are characterized by M.Y. Sayar in [9]. In order to recall this characterization, we first introduce the tools used to this end. Given a tournament \( T = (V, A) \), with each subset \( X \) of \( V \), such that \( |X| \geq 3 \) and \( T[X] \) is prime, are associated the following subsets of \( V \setminus X \):

- \( \langle X \rangle = \{x \in V \setminus X : x \rightarrow X \text{ or } X \rightarrow x\} \).
- For all \( u \in X \), \( X(u) = \{x \in V \setminus X : \{u, x\} \text{ is a module of } T[X \cup \{x\}]\} \).
- \( \text{Ext}(X) = \{x \in V \setminus X : T[X \cup \{x\}] \text{ is prime}\} \).
The family \( \{X(u) : u \in X\} \cup \{\text{Ext}(X), \langle X \rangle\} \) is denoted by \( p_X^T \).

**Lemma 1** ([7]) Let \( T = (V, A) \) be a tournament and let \( X \) be a subset of \( V \) such that \( |X| \geq 3 \) and \( T[X] \) is prime. The nonempty elements of \( p_X^T \) constitute a partition of \( V \setminus X \) and satisfy the following assertions:

- For \( u \in X, x \in X(u), \) and \( y \in V \setminus (X \cup X(u)) \), if \( T[X \cup \{x, y\}] \) is not prime, then \( \{u, x\} \) is a module of \( T[X \cup \{x, y\}] \).
- For \( x \in \langle X \rangle \) and \( y \in V \setminus (X \cup \langle X \rangle) \), if \( T[X \cup \{x, y\}] \) is not prime, then \( X \cup \{y\} \) is a module of \( T[X \cup \{x, y\}] \).
- For \( x \neq y \in \text{Ext}(X) \), if \( T[X \cup \{x, y\}] \) is not prime, then \( \{x, y\} \) is a module of \( T[X \cup \{x, y\}] \).

Furthermore, \( \langle X \rangle \) is divided into \( X^- = \{x \in \langle X \rangle : x \rightarrow X\} \) and \( X^+ = \{x \in \langle X \rangle : X \rightarrow x\} \). Similarly, for all \( u \in X \), \( X(u) \) is divided into \( X^-(u) = \{x \in X(u) : x \rightarrow u\} \) and \( X^+(u) = \{x \in X(u) : u \rightarrow x\} \). We then introduce the family \( q_X^T = \{\text{Ext}(X), X^-, X^+\} \cup \{X^-(u) : u \in X\} \cup \{X^+(u) : u \in X\} \).

A graph \( G = (V(G), E(G)) \) (or \( (V, E) \)) consists of a finite set \( V \) of vertices together with a set \( E \) of unordered pairs of distinct vertices, called edges. Given a vertex \( x \) of a graph \( G = (V, E) \), the set \( \{y \in V, \{x, y\} \in E\} \) is denoted by \( N_G(x) \). With each subset \( X \) of \( V \) is associated the subgraph \( G[X] = (X, E \cap \binom{X}{2}) \) of \( G \) induced by \( X \). An isomorphism from a graph \( G = (V, E) \) onto a graph \( G' = (V', E') \) is a bijection \( f \) from \( V \) onto \( V' \) such that for all \( x, y \in V, \{x, y\} \in E \) if and only if \( \{f(x), f(y)\} \in E' \). We now introduce the graph \( G_{2n} \) defined on \( \mathbb{N}_{2n} \), where \( n \geq 1 \), as follows. For all \( x, y \in \mathbb{N}_{2n}, \{x, y\} \in E(G_{2n}) \) if and only if \( |y - x| \geq n \) (see Figure 2).

**Figure 2.** \( G_{2n} \)

A graph \( G \) is connected if for all \( x \neq y \in V(G) \), there is a sequence \( x_0 = x, \ldots, x_m = y \) of vertices of \( G \) such that for all \( i \in \mathbb{N}_m, \{x_i, x_{i+1}\} \in E(G) \). For example, the graph \( G_{2n} \) is connected. A connected component of a graph \( G \) is a maximal subset \( X \) of \( V(G) \) (with respect to inclusion) such that \( G[X] \) is connected. The set of the connected components of \( G \) is a partition of \( V(G) \), denoted by \( \mathcal{C}(G) \). Let \( T = (V, A) \) be a prime tournament. With each subset \( X \) of \( V \) such that \( |X| \geq 3 \) and \( T[X] \) is prime, is associated its outside graph \( G_X^T \) defined by \( V(G_X^T) = V \setminus X \) and \( E(G_X^T) = \{\{x, y\} \in \binom{V \setminus X}{2} : T[X \cup \{x, y\}] \) is prime \}. We now present the characterization of partially critical tournaments.

**Theorem 2** ([9]) Consider a tournament \( T = (V, A) \) with a subset \( X \) of \( V \) such that \( |X| \geq 3 \) and \( T[X] \) is prime. The tournament \( T \) is \( T[X] \)-critical if and only if the assertions below hold.
1. \( \text{Ext}(X) = \emptyset \).

2. For all \( u \in X \), the tournaments \( T[X(u) \cup \{u\}] \) and \( T[(X) \cup \{u\}] \) are transitive.

3. For each \( Q \in \mathcal{C}(G_X^T) \), there is an isomorphism \( f \) from \( G_{2n} \) onto \( G_X^T | Q \) such that \( Q_1, Q_2 \in q_X^T \), where \( Q_1 = f(N_n) \) and \( Q_2 = f(N_{2n} \setminus N_n) \). Moreover, for all \( x \in Q_i \) (\( i = 1 \) or \( 2 \)), \( |N_{G_X^T}(x)| = |N^n_{T(Q_i)}(x)| + 1 \) (resp. \( n - |N^n_{T(Q_i)}(x)| \)) if \( Q_i = X^+ \) or \( X^-(u) \) (resp. \( Q_i = X^- \) or \( X^+(u) \)), where \( u \in X \).

The next corollary follows from Theorem 2 and Lemma 1.

**Corollary 1** Let \( T \) be a \( T[X] \)-critical tournament, \( T \) is entirely determined up to isomorphism by giving \( T[X] \), \( q_X^T \) and \( \mathcal{C}(G_X^T) \). Moreover, the tournament \( T \) is exactly determined by giving, in addition, either the graphs \( G_X^T | Q \) for any \( Q \in \mathcal{C}(G_X^T) \), or the transitive tournaments \( T[Y] \) for any \( Y \in q_X^T \).

We underline the importance of Theorem 2 and Corollary 1 in our description of the tournaments of the class \( T \). Indeed, these tournaments are introduced up to isomorphism as \( C_3 \)-critical tournaments \( T \) defined by giving \( \mathcal{C}(G_{N_3}^T) \) in terms of the nonempty elements of \( q_{N_3}^T \). Figure 3 illustrates a tournament obtained from such information. We refer to [10, Discussion] for more details about this purpose.

We now introduce the class \( \mathcal{H} \) (resp. \( \mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L} \)) of the \( C_3 \)-critical tournaments \( H \) (resp. \( I, J, K, L \)) such that:

- \( \mathcal{C}(G_{N_3}^H) = \{N_3^+(0) \cup N_3^-, N_3^+ \cup N_3^-(1)\} \) (see Figure 3);
- \( \mathcal{C}(G_{N_3}^I) = \{N_3^+(0) \cup N_3^+(2), N_3^+(1) \cup N_3^-(0)\};
- \( \mathcal{C}(G_{N_3}^J) = \{N_3^+(1) \cup N_3^-, N_3^+(1) \cup N_3^-(0)\};
- \( \mathcal{C}(G_{N_3}^K) = \{N_3^+(1) \cup N_3^+, N_3^+(0) \cup N_3^-(2)\};
- \( \mathcal{C}(G_{N_3}^L) = \{N_3^+(1) \cup N_3^+, N_3^+(0) \cup N_3^-(2), N_3^+ \cup N_3^-(0)\}.\)

Notice that for \( X = \mathcal{H}, \mathcal{I}, \mathcal{J} \) or \( \mathcal{K} \), \( \{|V(T)| : T \in X\} = \{2n + 1 : n \geq 3\} \) and \( \{|V(T)| : T \in \mathcal{L}\} = \{2n + 1 : n \geq 4\} \). We denote by \( \mathcal{H}^* \) (resp. \( \mathcal{I}^*, \mathcal{J}^*, \mathcal{K}^*, \mathcal{L}^* \)) the class of the tournaments \( T^* \), where \( T \in \mathcal{H} \) (resp. \( I, J, K, L \)).

**Remark 1** We have \( \mathcal{H}^* = \mathcal{H} \) and \( \mathcal{I}^* = \mathcal{I} \).

**Proof** Let \( T \in \mathcal{H} \). The permutation \( f \) of \( V(T) \) defined by \( f(1) = 0, f(0) = 1, \) and \( f(v) = v \) for all \( v \in V(T) \setminus \{0, 1\} \) is an isomorphism from \( T^* \) onto a tournament \( T' \) of the class \( \mathcal{H} \). Let now \( T \in I \) and let \( x \) be the unique vertex of \( N_3^+(2) \) such that \( |N^n_{T[N_3^+(2)]}(x)| = 0 \). The permutation \( g \) of \( V(T) \) defined by \( g(1) = 0, g(0) = 1, g(x) = 2, g(2) = x, \) and \( g(v) = v \) for \( v \in V(T) \setminus \{0, 1, 2, x\} \) is an isomorphism from \( T^* \) onto a tournament \( T' \) of the class \( \mathcal{I} \).

By setting \( \mathcal{M} = \mathcal{H} \cup \mathcal{I} \cup \mathcal{J} \cup \mathcal{J}^* \cup \mathcal{K} \cup \mathcal{K}^* \cup \mathcal{L} \cup \mathcal{L}^* \), we state our main result as follows.

**Theorem 3** Up to isomorphism, the tournaments of the class \( T \) are those of the class \( \mathcal{M} \). Moreover, for all \( T \in \mathcal{M} \), we have \( V(T) \setminus W_5(T) = \sigma(T) = \{0, 1\} \).
2. Critical tournaments and tournaments omitting $W_5$

We begin by recalling the characterization of the critical tournaments and some of their properties. A prime tournament $T = (V, A)$, with $|T| \geq 3$, is critical if $\sigma(T) = \emptyset$, i.e. if all its vertices are critical. In order to present the critical tournaments, characterized by J.H. Schmerl and W.T. Trotter in [10], we introduce the tournaments $T_{2n+1}$ and $U_{2n+1}$ defined on $N_{2n+1}$, where $n \geq 2$, as follows:

- $A(T_{2n+1}) = \{(i, j) : j - i \in \{1, \ldots, n\} \mod 2n + 1\}$ (see Figure 4).
- $A(T_{2n+1}) \setminus A(U_{2n+1}) = A(T_{2n+1}([n + 1, \ldots, 2n])$) (see Figure 5).

![Figure 4. $T_{2n+1}$](image)

![Figure 5. $U_{2n+1}$](image)

**Theorem 4 ([10])** Up to isomorphy, $T_{2n+1}$, $U_{2n+1}$, and $W_{2n+1}$, where $n \geq 2$, are the only critical tournaments.

Notice that a critical tournament is isomorphic to its dual. Moreover, as a tournament on 4 vertices is not prime, we have:

\[ T[N^+_3(0)] = N_{2k+1} \setminus N_{k+2}; \]
\[ T[N^+_3(0)] = (N_{k+2} \setminus N_3)^*; \]
\[ \text{for all } (i, j) \in N^+_3(0) \times N^-_3, \]
\[ i \rightarrow j \text{ if and only if } j - i \geq k - 1. \]
Fact 1 Up to isomorphy, $T_5$, $U_5$, and $W_5$ are the only prime tournaments on 5 vertices.

As mentioned in [2], the next remark follows from the definition of the critical tournaments.

Remark 2 Up to isomorphy, the prime subtournaments on at least 5 vertices of $T_{2n+1}$ (resp. $U_{2n+1}$, $W_{2n+1}$), where $n \geq 2$, are the tournaments $T_{2m+1}$ (resp. $U_{2m+1}$, $W_{2m+1}$), where $2 \leq m \leq n$.

To recall the characterization of the prime tournaments omitting $W_5$, we introduce the Paley tournament $P_7$ defined on $\mathbb{N}_7$ by $A(P_7) = \{(i, j) : j - i \in \{1, 2, 4\} \mod 7\}$. Notice that for all $x \neq y \in \mathbb{N}_7$, $P_7 - x \simeq P_7 - y$, and let $B_6 = P_7 - 6$.

Theorem 5 ([8]) Up to isomorphy, the prime tournaments on at least 5 vertices and omitting $W_5$ are the tournaments $B_6$, $P_7$, $T_{2n+1}$, and $U_{2n+1}$, where $n \geq 2$.

3. Some useful configurations

In this section, we introduce a number of configurations that occur in the proof of Theorem 3. These configurations involve mainly partially critical tournaments. We begin with the two following lemmas obtained in [2].

Lemma 2 ([2]) If $B_6$ embeds into a prime tournament $T$ on 7 vertices and if $T \not\simeq P_7$, then $|W_5(T)| = 7$.

Lemma 3 ([2]) Let $T$ be a $U_5$-critical tournament on 7 vertices. If $T \not\simeq U_7$, then $W_5(T) \cap \{3, 4\} \neq \emptyset$.

Lemma 4 specifies the $C_3$-critical tournaments with a connected outside graph. It follows from the examination of the different possible configurations obtained by using Theorem 2.

Lemma 4 Given a $C_3$-critical tournament $T$ on at least 5 vertices, if $G_{N_3}^T$ is connected, then $T$ is critical. More precisely, the different configurations are as follows where $i \in N_3$ and $i + 1$ is considered modulo 3.

1. If $C(G_{N_3}^T) = \{N_3^- (i) \cup N_3^+ (i + 1)\}$, then $T \simeq T_{2n+1}$ for some $n \geq 2$.

2. If $C(G_{N_3}^T) = \{N_3^- \cup N_3^+ (i)\}, \{N_3^- \cup N_3^- (i)\}, \{N_3^+ (i) \cup N_3^+ (i + 1)\}$, or $\{N_3^- (i) \cup N_3^- (i + 1)\}$, then $T \simeq U_{2n+1}$ for some $n \geq 2$.

3. If $C(G_{N_3}^T) = \{N_3^- \cup N_3^- (i)\}, \{N_3^+ \cup N_3^- (i)\},$ or $\{N_3^+ (i) \cup N_3^- (i + 1)\}$, then $T \simeq W_{2n+1}$ for some $n \geq 2$.

For a transitive tournament $T$, recall that $\min T$ denotes its smallest element and $\max T$ its largest.

Lemma 5 Given a $C_3$-critical tournament $T$ on at least 5 vertices, if $T[N_3 \cup e] \simeq T_5$ for all $e \in E(G_{N_3}^T)$, then $T \simeq T_{2n+1}$ for some $n \geq 2$.

Proof Let $T$ be a $C_3$-critical tournament on at least 5 vertices such that for all $e \in E(G_{N_3}^T)$, $T[N_3 \cup e] \simeq T_5$. Given $e \in E(G_{N_3}^T)$, by using Lemma 4 and Remark 2, $C = \{v, v'\}$, where $v \in N_3^- (i)$, $v' \in N_3^+ (i + 1)$, $i \in N_3$ and $i + 1$ is considered modulo 3. Then, by Theorem 2, the connected components of $T$ are the nonempty elements of the family $\{N_3^- (j) \cup N_3^+ (j + 1)\}_{j \in N_3}$, where $j + 1$ is considered modulo 3. The tournament $T$ is critical. Indeed, by using Theorem 2, for each $k \in N_3$, $\{\max T[N_3^+ (k + 1) \cup \{k + 1\}], \min T[N_3^- (k + 2) \cup \{k + 2\}]\}$, where
Lemma 6. Given a $U_5$-critical tournament, if $T[\mathbb{N}_5 \cup e] \simeq U_5$ for all $e \in E(G_5^T)$, then $T \simeq U_{2n+1}$ for some $n \geq 2$.

Proof. The subsets $X$ of $\mathbb{N}_7$ such that $U_7[X] \simeq U_5$ are the sets $\mathbb{N}_7 \setminus \{i, j\}$, where $\{i, j\} = \{0, 4\}, \{4, 1\}, \{1, 5\}, \{5, 2\}, \{2, 6\}$, or $\{6, 3\}$. By observing $d_X^{U_7}$ for such subsets $X$ and by Theorem 2, we deduce that the elements of $\mathcal{C}(G_5^T)$ are the nonempty elements among the following six sets: $\mathbb{N}_7^+ \cup \mathbb{N}_7^-(0)$, $\mathbb{N}_7^+(0) \cup \mathbb{N}_7^+(3)$, $\mathbb{N}_7^-(1) \cup \mathbb{N}_7^+(3)$, $\mathbb{N}_7^+(1) \cup \mathbb{N}_7^+(4)$, $\mathbb{N}_7^-(2) \cup \mathbb{N}_7^+(4)$, and $\mathbb{N}_7^- \cup \mathbb{N}_7^+(2)$. Suppose first that $|\mathcal{C}(G_5^T)| = 6$. The tournament $T$ is critical. Indeed, by using Theorem 2, $\{\min T[\mathbb{N}_7^+(3)], \max T[\mathbb{N}_7^+(3)]\}$ (resp. $\{\min T[\mathbb{N}_7^+(3)], \max T[\mathbb{N}_7^+(4)]\}$, $\{\min T[\mathbb{N}_7^-(4)], \max T[\mathbb{N}_7^+(4)]\}, \{\min T[\mathbb{N}_7^+(1)], \max T[\mathbb{N}_7^+(0)]\}$, $\{\min T[\mathbb{N}_7^+(2)], \max T[\mathbb{N}_7^+(1)]\}$) is a nontrivial module of $T - \{\}$ (resp. $T - 1, T - 2, T - 3, T - 4$). By Remark 2, $T \simeq U_{2n+1}$ for some $n \geq 8$. Suppose now that $|\mathcal{C}(G_5^T)| \leq 5$. Then $T$ embeds into a $U_5$-critical tournament $T'$ with $|\mathcal{C}(G_5^{T'})| = 6$. By the first case, $T' \simeq U_{2n+1}$ for some $n \geq 8$ and thus $T \simeq U_{2n+1}$ for some $n \geq 2$ by Remark 2.

Lemma 7. Let $T = (V, A)$ be a $T[X]$-critical tournament with $|V \setminus X| \geq 2$, let $Q = \mathbb{N}_2n$ be a connected component of $G_V^X$ such that $G_V^X[Q] = G_{2n}$, and let $e = \{i, i+n\}$, where $i \in \mathbb{N}_n$. Then the tournament $T - e$ is $T[X]$-critical. Moreover, $Q$ is included in any subset $Z$ of $V$ such that $T[Z] \simeq W_5$ and $Z \cap (V \setminus (Q \cup W_5(T-e))) \neq \emptyset$.

Proof. For $n \geq 2$, the function

$$f_1 : Q \setminus e \longrightarrow \mathbb{N}_2n$$

$$k \longmapsto \begin{cases} 
  k & \text{if } 0 \leq k \leq i - 1 \\
  k - 1 & \text{if } i + 1 \leq k \leq n + i - 1 \\
  k - 2 & \text{if } n + i + 1 \leq k \leq 2n - 1,
\end{cases}$$

is an isomorphism from $G_{2n} - e$ onto $G_{2n}$. It follows from Theorem 2 that $T - e$ is $T[X]$-critical. Now suppose that there is $Z \subseteq V$ such that $T[Z] \simeq W_5$ and $Z \cap (V \setminus (Q \cup W_5(T-e))) \neq \emptyset$. Therefore, we have $|Z \cap e| = 1$ or $e \subset Z$. Suppose for a contradiction that $|Z \cap e| = 1$, and set $\{z\} = Z \cap e$. As $\text{Ext}(V \setminus e) = \emptyset$, then by Lemma 1, either $z \in \langle V' \rangle$ or $z \in \langle V'(u) \rangle$, where $V' = V \setminus e$ and $u \in V'$. If $z \in \langle V' \rangle$, then $Z \setminus \{z\}$ is a nontrivial module of $T[Z]$, a contradiction. If $z \in \langle V'(u) \rangle$, then $u \not\in Z$, otherwise $\{u, z\}$ is a nontrivial module of $T[Z]$. Thus, $T[Z'] \simeq W_5$, where $Z' = (Z \setminus \{z\}) \cup \{u\} \subset V \setminus e$. A contradiction because $Z' \cap (V \setminus W_5(T-e)) \neq \emptyset$. Finally, for all $e' \in \{\{j, j+n\} : j \in \mathbb{N}_n\}$, the bijection $f$ from $V \setminus e$ onto $V \setminus e'$, defined by $f|_{V \setminus Q} = \text{Id}_{V \setminus Q}$ and $f|_{Q \setminus e} = f_1^{-1} \circ f_1$, is an isomorphism from $T - e$ onto $T - e'$. It follows that $V \setminus (Q \cup W_5(T-e')) = V \setminus (Q \cup W_5(T-e))$. Thus, as proved above, $e' \subset Z$, so that $Q \subset Z$.

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4. Proof of Theorem 3

We begin by establishing the partial criticality structure of the tournaments of the class \( T \). For this purpose, we use the notion of minimal tournaments for two vertices. Given a prime tournament \( T = (V, A) \) of cardinality \( \geq 3 \) and two distinct vertices \( x \neq y \in V \), \( T \) is said to be minimal for \( \{x, y\} \) (or \( \{x, y\} \)-minimal) when for all proper subset \( X \) of \( V \), if \( \{x, y\} \subset X \ (\mid X \mid \geq 3) \), then \( T[X] \) is not prime. These tournaments were introduced and characterized by A. Courner and P. Ille in \([5]\). From this characterization, the following fact, observed in \([1]\), is obtained by a simple and quick verification.

**Fact 2** \(([1, 5])\) Up to isomorphy, the tournaments \( C_3 \) and \( U_5 \) are the unique minimal tournaments for two vertices \( T \) such that \( |W_5(T)| \leq |T| - 2 \). Moreover, \( \{3, 4\} \) is the unique unordered pair of vertices for which \( U_5 \) is minimal.

**Proposition 1** Let \( T = (V, A) \) be a tournament of the class \( T \). Then the vertices of \( W_5(T) \) are critical and there exists \( z \in W_5(T) \) such that \( T[(V \setminus W_5(T)) \cup \{z\}] \simeq C_3 \). In particular, \( T \) is \( T[(V \setminus W_5(T)) \cup \{z\}] \)-critical.

**Proof** By Theorem 1, \( |T| \) is odd and \( \geq 7 \). First, suppose by contradiction that there is \( \alpha \in W_5(T) \) such that \( T - \alpha \) is prime. Since \( |T - \alpha| \) is even and \( \geq 6 \) with \( |V(T - \alpha) \setminus W_5(T - \alpha)| \geq 2 \), then by Theorems 1 and 5, \( T - \alpha \simeq B_6 \) and \( T \not\simeq P_r \). By contradiction by Lemma 2. Second, let \( X \) be a minimal subset of \( V \) such that \( V \setminus W_5(T) \subset X \ (\mid X \mid \geq 3) \) and \( T[X] \) is prime, so that \( T[X] \) is \( (V \setminus W_5(T)) \)-minimal. By Fact 2, \( T[X] \simeq C_3 \) or \( U_5 \). Suppose, toward a contradiction that \( T[X] \simeq U_5 \) and take \( T[X] = U_5 \). By Fact 2, \( V \setminus W_5(T) = \{3, 4\} \). As \( T \) is \( U_5 \)-critical, then by Lemma 6 and Theorem 5, there exists \( e \in E(G^T_X) \) such that \( T[X \cup e] \) is prime and not isomorphic to \( U_7 \). It follows from Lemma 3, that there exists a subset \( Z \) of \( X \cup e \) such that \( T[Z] \simeq W_5 \) and \( Z \cap (V \setminus W_5(T)) \neq \emptyset \), a contradiction. 

Now, we prove Theorem 3 for tournaments on 7 vertices.

**Proposition 2** Up to isomorphy, the class \( \mathcal{M} \) and the class \( T \) have the same tournaments on 7 vertices. Moreover, for each tournament \( T \) on 7 vertices of the class \( \mathcal{M} \), we have \( V(T) \setminus W_5(T) = \sigma(T) = \{0, 1\} \).

**Proof** Let \( T = (V, A) \) be a tournament on 7 vertices of the class \( \mathcal{M} \). \( T \in \mathcal{M} \setminus (\mathcal{L} \cup \mathcal{L}^\ast) \) because the tournaments of the class \( \mathcal{L} \) have at least 9 vertices. Let \( e \in E(G^T_N) \). By Lemma 4, \( T - e \simeq U_5 \) or \( T_5 \). By Lemma 7, if there exists a subset \( Z \subset V \) such that \( T[Z] \simeq W_5 \), then \( e \subset Z \). It follows that \( V \setminus N_3 \subset Z \). Thus \( V \setminus W_5(T) = \{0, 1\} \) by verifying that \( T - \{1, 2\} \not\simeq W_5 \), \( T - \{0, 2\} \not\simeq W_5 \) and \( T - \{0, 1\} \simeq W_5 \). As \( T \) is \( C_3 \)-critical, \( \sigma(T) = \{0, 1\} \) from the following. First, \( T - 2 \) is not prime because \( \{0\} \cup N_3^- \cup N_3^+(0) \) (resp. \( \{1\} \cup N_3^+(0), \{0, 1\} \cup N_3^+(0) \cup N_3^-(1), \{1\} \cup N_3^-(0)) \) is a nontrivial module of \( T - 2 \) if \( T \in \mathcal{H} \) (resp. \( \mathcal{L}, \mathcal{J}, \mathcal{K} \)). Second, by Lemma 1, we have \( \text{Ext}(X) = \{0, 1\} \), where \( X = V \setminus \{0, 1\} \), because \( \{0, 1\} \cap \langle X \rangle = \emptyset \), and for all \( u \in X \), \( \{0, 1\} \cap X(u) = \emptyset \) because \( V \setminus W_5(T) = \{0, 1\} \).

Conversely, let \( T \) be a tournament on 7 vertices of the class \( \mathcal{T} \). By Proposition 1, we can assume that \( T \) is \( C_3 \)-critical with \( V(T) \setminus W_5(T) \subset N_3 \). By Lemma 4 and Theorem 5, \( |\mathcal{L}(G^T_N)| = 2 \). We distinguish the following cases.

- \( N_3^+ \neq \emptyset \) and \( N_3^- \neq \emptyset \). By Theorem 2, \( |N_3^-| = |N_3^+| = 1 \). Therefore, we can assume that \( N_3(0) \neq \emptyset \) and \( N_3(2) = \emptyset \). It suffices to verify that \( |N_3(0)| = |N_3^+(0)| = 1 \) because, in this case, by using Theorem 2 and Lemma 4, \( T \in \mathcal{H} \). By using again Theorem 2 and Lemma 4, we verify the following. First, if \( |N_3(0)| = 2 \),
then \( C(G_{N_3}^T) = \{N_3^+ \cup N_3^-(0), N_3^- \cup N_3^+(0)\} \). Therefore, \( T - \{0,1\} \simeq T - \{0,2\} \simeq W_5 \), a contradiction. Second, if \( |N_3^-(0)| = 1 \), then \( C(G_{N_3}^T) = \{N_3^+ \cup N_3^-(0), N_3^- \cup N_3^+(1)\} \). Therefore, \( T \simeq U_7 \), a contradiction by Theorem 5.

- \( \langle N_3 \rangle = \emptyset \). By Theorem 2, we can assume that \( |N_3^-(0)| = |N_3^+(0)| = 1 \). We have \( |N_3(1)| = 1 \). Otherwise, by Theorem 2 and Lemma 4, we can suppose that \( C(G_{N_3}^T) = \{N_3^+(0) \cup N_3^+(1), N_3^-(0) \cup N_3^-(1)\} \). Therefore, \( T - \{1,2\} \simeq T - \{0,2\} \simeq W_5 \), a contradiction. We have also \( C(G_{N_3}^T) = \{N_3^+(0) \cup N_3(2), N_3^-(0) \cup N_3(1)\} \). Otherwise, again by Theorem 2 and Lemma 4, \( C(G_{N_3}^T) = \{N_3^+(0) \cup N_3^+(1), N_3^-(0) \cup N_3(1)\} \), so that \( T \simeq U_7 \), a contradiction by Theorem 5. Thus, we distinguish four cases. If \( |N_3^-(2)| = |N_3^+(1)| = 1 \), then \( T \simeq T_7 \), which contradicts Theorem 5. If \( |N_3^+(2)| = |N_3^+(1)| = 1 \), then \( T - \{0,2\} \simeq T - \{0,1\} \simeq W_5 \), a contradiction. If \( |N_3^-(2)| = |N_3^+(1)| = 1 \), then \( T \in \mathcal{I} \). If \( |N_3^-(2)| = |N_3^+(1)| = 1 \), then \( T \) is isomorphic to a tournament of the class \( \mathcal{I} \) with \( V(T) \setminus W_5(T) = \{0,2\} \).

- \( \emptyset \neq \langle N_3 \rangle \in q_3^T \). By interchanging \( T \) and \( T^* \), we can suppose that \( \langle N_3 \rangle = N_3^- \). In this case, \( |N_3^-| = 1 \) by Theorem 2. First, suppose that \( |N_3(0)| = 2 \) and \( |N_3(1)| = 1 \). By Theorem 2 and Lemma 4, \( C(G_{N_3}^T) = \{N_3^+(0) \cup N_3^-(0) \cup N_3(1)\} \). We have \( |N_3^+(1)| = 1 \), otherwise \( T \simeq U_7 \), a contradiction by Theorem 5. Thus, \( T \) is isomorphic to a tournament of the class \( \mathcal{K} \) with \( V(T) \setminus W_5(T) = \{0,2\} \). Second, suppose that \( |N_3(0)| = 1 \) and \( |N_3(1)| = 2 \). Again by Theorem 2 and Lemma 4, \( C(G_{N_3}^T) = \{N_3^+(1) \cup N_3^-, N_3^-(1) \cup N_3^-(0)\} \), so that \( T \in \mathcal{J} \). Lastly, suppose that \( |N_3(0)| = |N_3(1)| = 1 \). By Theorem 2 and Lemma 4, we can suppose that \( C(G_{N_3}^T) = \{N_3^+(1) \cup N_3^-, N_3(0) \cup N_3(2)\} \). By Lemma 4, we distinguish only three cases. If \( |N_3^-(2)| = |N_3^-(0)| = 1 \), then \( T - \{0,1\} \simeq T - \{1,2\} \simeq W_5 \), a contradiction. If \( |N_3^+(0)| = |N_3^+(2)| = 1 \), then \( T \simeq U_7 \), which contradicts Theorem 5. If \( |N_3^+(2)| = |N_3^-(0)| = 1 \), then \( T \in \mathcal{K} \).

We complete our structural study of the tournaments of the class \( \mathcal{T} \) by the following two corollaries.

**Corollary 2** Let \( T \) be a \( C_3 \)-critical tournament such that \( V(T) \setminus W_5(T) = \{0,1\} \). Then there exist \( Q \neq Q' \in C(G_{N_3}^T) \) and a tournament \( R \) on 7 vertices of the class \( \mathcal{M} \) such that for all \( e \in E(G_{N_3}^T[Q]) \) and for all \( e' \in E(G_{N_3}^T[Q']) \), there exists an isomorphism \( f \) from \( R \) onto \( T[N_3 \cup e \cup e'] \). Moreover, \( f(0) = 0 \), \( f(1) = 1 \) and we have:

1. If \( R \in \mathcal{H} \cup \mathcal{J} \cup \mathcal{J}^* \), then \( f(2) = 2 \);

2. If \( R \in \mathcal{I} \cup \mathcal{K} \cup \mathcal{K}^* \), then \( f(2) = 2 \) or \( N_3(2) = \{f(2)\} \).

**Proof** To begin, notice the following remark: given a \( D[X] \)-critical tournament \( D \), for any edges \( a \) and \( b \) belonging to a same connected component of \( G_{N_3}^T \), we have \( D[X \cup a] \simeq D[X \cup b] \). Therefore, by Fact 1, Lemma 5, and Theorem 5, there exists \( Q \in C(G_{N_3}^T) \) such that for all \( a \in E(G_{N_3}^T[Q]) \), \( T[N_3 \cup a] \simeq U_5 \). By Lemma 4 and Remark 2, the tournament \( T[N_3 \cup Q] \) is isomorphic to \( U_{2n+1} \), for some \( n \geq 2 \), and does not admit a prime subtournament on 7 vertices other than \( U_7 \). Therefore, by Lemma 6, Theorem 5, and the remark above, there exists \( Q' \in C(G_{N_3}^T) \setminus \{Q\} \) such that for all \( e \in E(G_{N_3}^T[Q]) \) and for all \( e' \in E(G_{N_3}^T[Q']) \), \( T[N_3 \cup e \cup e'] \) is prime and not isomorphic to \( U_7 \). Moreover, \( T[N_3 \cup e \cup e'] \neq P_7 \) because the vertices of \( P_7 \) are all noncritical.
Likewise, $T[N_3 \cup e \cup e'] \not\cong T_7$ by Remark 2. It follows from Theorem 5 and Proposition 2 that there exists an isomorphism $f$ from a tournament $R$ on 7 vertices of the class $\mathcal{M}$ onto $T[N_3 \cup e \cup e']$. As $(0,1) \in A(R) \cap A(T)$ and $V(R) \setminus W_5(R) = V(T) \setminus W_5(T) = \{0,1\}$ by Proposition 2, then $f$ fixes 0 and 1. If $R \in \mathcal{H} \cup \mathcal{J} \cup \mathcal{J}^*$, then $f$ fixes 2 because 2 is the unique vertex $x$ of $R$ such that $R[\{0,1,x\}] \cong C_3$. If $R \in \mathcal{I} \cup \mathcal{K} \cup \mathcal{K}^*$, then $|\{x \in V(R) : R[\{0,1,x\}] \cong C_3\}| = 2$. Therefore, $f(2) = 2$ or $\alpha$, where $\alpha$ is the unique vertex of $N_3(2)$ in the tournament $T[N_3 \cup e \cup e']$.

\[\square\]

**Corollary 3** For all $T \in \mathcal{T}$, we have $V(T) \setminus W_5(T) = \sigma(T)$.

**Proof** Let $T$ be a tournament of the class $\mathcal{T}$ such that $V(T) \setminus W_5(T) = \{0,1\}$. By Proposition 1, we can assume that $T$ is $C_3$-critical. By the same proposition, it suffices to prove that $\{0,1\} \subseteq \sigma(T)$. By Corollary 2, there is a subset $X$ of $V(T)$ such that $N_3 \subseteq X$ and $T[X]$ is isomorphic to a tournament on 7 vertices of the class $\mathcal{M}$. Suppose for a contradiction that $T$ admits a critical vertex $i \in \{0,1\}$, and let $Y = X \setminus \{i\}$. By Proposition 2, $T[Y]$ is prime. As $T$ is $T[Y]$-critical, then $i \notin \text{Ext}(Y)$ by Theorem 2. This is a contradiction because $T[X]$ is prime.

Now, we prove that $\mathcal{M} \subseteq \mathcal{T}$. More precisely:

**Proposition 3** For all tournament $T$ of the class $\mathcal{M}$, we have $V(T) \setminus W_5(T) = \sigma(T) = \{0,1\}$.

**Proof** Let $T$ be a tournament on $(2n + 1)$ vertices of the class $\mathcal{M}$ for some $n \geq 3$. By Corollary 3, it suffices to prove that $V(T) \setminus W_5(T) = \{0,1\}$. We proceed by induction on $n$. By Proposition 2, the statement is satisfied for $n = 3$. Let now $n \geq 4$. Therefore, either $T$ is a tournament on 9 vertices of the class $\mathcal{L} \cup \mathcal{L}^*$ or there is $Q \in \mathcal{C}(G_{N_3}^T)$ such that $|Q| \geq 4$. In the first case, for all $e \in E(G_{N_3}^T)$, $T - e$ is isomorphic to $U_7$ or to a tournament on 7 vertices of the class $\mathcal{K} \cup \mathcal{K}^*$. Therefore, if there exists a subset $Z$ of $V(T)$ such that $Z \setminus \{0,1\} \neq \emptyset$ and $T[Z] \cong W_5$, then, for all $e \in E(G_{N_3}^T)$, $e \subseteq Z$ by Lemma 7. Thus, $V(T) \setminus N_3 \subseteq Z$, a contradiction. As, furthermore, $W_5$ embeds into $T$, then $V(T) \setminus W_5(T) = \{0,1\}$ by Theorem 1. In the second case, let $Q \in \mathcal{C}(G_{N_3}^T)$ such that $|Q| \geq 4$. Let $\mathcal{X} = \mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K}$, or $\mathcal{L}$. For $T \in \mathcal{X}$, by Lemma 7, there is $e \in E(G_{N_3}^T[Q])$ such that $T - e$ is $C_3$-critical. Moreover, $T - e$ is isomorphic to a tournament of the class $\mathcal{X}$ because $\mathcal{C}(G_{N_3}^T)$ is as described in the same class. By induction hypothesis, $W_5$ embeds into $T - e$, and thus into $T$. By Theorem 1, it suffices to verify that $\{0,1\} \subseteq V(T) \setminus W_5(T)$. Therefore, suppose that there exists $Z \subseteq V(T)$ such that $Z \setminus \{0,1\} \neq \emptyset$ and $T[Z] \cong W_5$. By induction hypothesis and by Lemma 7, $Q \subseteq Z$, so that $Z \subseteq Q \cup N_3$. This is a contradiction by Theorem 5, because $T[N_3 \cup Q] \cong U_{Q+3}$ or $T_{Q+3}$ by Lemma 4. \[\square\]

We are now ready to construct the tournaments of the class $\mathcal{T}$. We partition these tournaments $T$ according to the following invariant $c(T)$. For $T \in \mathcal{T}$, $c(T)$ is the minimum of $|\mathcal{C}(G_{\sigma(T) \cup \{x\}}^T)|$, the minimum being taken over all the vertices $x$ of $W_5(T)$ such that $T[\sigma(T) \cup \{x\}] \cong C_3$. Notice that $c(T) = c(T^*)$. As $T$ is $T[\sigma(T) \cup \{x\}]$-critical by Proposition 1, then $c(T) \leq 4$. Moreover, $c(T) \geq 2$ by Lemma 4. Proposition 1 leads us to classify the tournaments $T$ of the class $\mathcal{T}$ according to the different values of $c(T)$. We will see that $c(T) = 2$ or $3$. Theorem 3 results from Propositions 3, 4, 5, and 6.

**Proposition 4** Up to isomorphism, the tournaments $T$ of the class $\mathcal{T}$ such that $c(T) = 2$ are those of the class $\mathcal{M} \setminus (\mathcal{L} \cup \mathcal{L}^*)$.
Proof. For all $T \in \mathcal{M} \setminus (\mathcal{L} \cup \mathcal{L}^*)$, we have $T \in \mathcal{T}$ by Proposition 3, and $c(T) = 2$ by Lemma 4. Now let $T$ be a tournament on $(2n + 1)$ vertices of the class $\mathcal{T}$ such that $c(T) = 2$. By Proposition 1, we can assume that $T$ is $C_3$-critical with $V(T) \setminus W_5(T) = \{0, 1\}$ and $|\mathcal{C}(G_{R_3}^T)| = 3$. By Corollary 2 and by interchanging $T$ and $T^*$, there is a tournament $R$ on 7 vertices of the class $\mathcal{H} \cup \mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$ such that for all $e \in E(G_{R_3}(Q))$ and for all $e' \in E(G_{R_3}(Q'))$, there exists an isomorphism $f$, fixing 0 and 1, from $R$ onto $T|\mathcal{N}_3 \cup e \cup e'|$, where $Q$ and $Q'$ are the two different connected components of $G_{R_3}^T$. If $f(2) = 2$, then, by Theorem 2, $T$ and $R$ are in the same class $\mathcal{H}, \mathcal{I}, \mathcal{J}$, or $\mathcal{K}$. Suppose now that $f(2) \neq 2$. By Corollary 2, $R \in \mathcal{I} \cup \mathcal{K}$. If $R \in \mathcal{I}$ (resp. $\mathcal{K}$), then $T|\mathcal{N}_3 \cup e \cup e'|$ is a tournament on 7 vertices of the class $\mathcal{T}'$ (resp. $\mathcal{K}'$) of the $C_3$-critical tournaments $Z$ such that $|\mathcal{C}(G_{Z_3}(T))| = \{N_3^-(0) \cup N_3^+(1), N_3^-(1) \cup N_3^+(2)\}$ (resp. $\mathcal{C}(G_{Z_3}(T)) = \{N_3^-(1) \cup N_3^-(2), N_3^+(1) \cup N_3^+(2)\}$). By Theorem 2, $T \in \mathcal{T}'$ (resp. $\mathcal{K}'$). Moreover, by considering the vertex $\alpha = \min T|\mathcal{N}_3^-(2)$ (resp. $\max T|\mathcal{N}_3^+(2)$) and by using Corollary 3, $T$ is also $T[\{0, 1, \alpha\}]$-critical with $\mathcal{C}(G_{Z_3}(T)) = \{(0, 1, \alpha)^{-}(0) \cup \{0, 1, \alpha\}^{+}(1), (0, 1, \alpha)^{-}(0) \cup \{0, 1, \alpha\}^{+}(\alpha)\}$ (resp. $\mathcal{C}(G_{Z_3}(T)) = \{(0, 1, \alpha)^{-}(1) \cup \{0, 1, \alpha\}^{-}, (0, 1, \alpha)^{-}(0) \cup \{0, 1, \alpha\}^{-}(\alpha)\}$). It follows that $T$ is isomorphic to a tournament of the class $\mathcal{I}$ (resp. $\mathcal{K}$).

Proposition 5. Up to isomorphy, the tournaments $T$ of the class $\mathcal{T}$ such that $c(T) = 3$ are those of the class $\mathcal{L} \cup \mathcal{L}^*$.

Proof. Let $T$ be a tournament of the class $\mathcal{L} \cup \mathcal{L}^*$. $T \in \mathcal{T}$ by Proposition 3. Moreover, $c(T) = 3$ by Theorem 2. Indeed, it suffices to observe that for all $x \in \{i \in V(T) \setminus \mathcal{N}_3 : T[\{0, 1, i\}] \simeq C_3\} = \mathcal{N}_3^2(2)$, we have $\max T|\mathcal{N}_3^2(1)| \in X^+(1)$, $\min T|\mathcal{N}_3^2| \in X^-$, $\min T|\mathcal{N}_3^2| \in X^+$, $\max T|\mathcal{N}_3^2(0)| \in X^-(0)$ and $2 \in X^+(x)$, where $X = \{0, 1, x\}$.

Now let $T$ be a tournament on $(2n + 1)$ vertices of $\mathcal{T}$ such that $c(T) = 3$. By Proposition 1, we can assume that $T$ is $C_3$-critical with $V(T) \setminus W_5(T) = \{0, 1\}$ and $|\mathcal{C}(G_{R_3}^T)| = 3$. By Corollary 2 and by interchanging $T$ and $T^*$, there is a tournament $R$ on 7 vertices of the class $\mathcal{H} \cup \mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$ such that for all $e \in E(G_{R_3}(Q))$ and $e' \in E(G_{R_3}(Q'))$, there exists an isomorphism $f$, which fixes 0 and 1, from $R$ onto $T|\mathcal{N}_3 \cup e \cup e'|$, where $Q \neq Q' \in \mathcal{C}(G_{R_3}^T)$. Take $e'' \in (G_{R_3}(Q'))$, where $Q'' = \mathcal{C}(G_{R_3}^T) \setminus \{Q, Q'\}$. Suppose, toward a contradiction, that $R \in \mathcal{H} \cup \mathcal{J}$. By Theorem 2 and by Corollary 2, if $R \in \mathcal{H}$ (resp. $R \in \mathcal{J}$), then $\{Q, Q'\} = \{N_3^+(0) \cup N_3^-(1), N_3^-(0) \cup N_3^+(1)\}$ (resp. $\{N_3^+(1) \cup N_3^-, N_3^-(0) \cup N_3^-(1)\}$). Therefore, by Lemma 4, $Q'' = \{N_3^+(1) \cup N_3^+(2)\}$, $\{N_3^+(0) \cup N_3^+(1)\}$ or $\{N_3^-(0) \cup N_3^-(2)\}$ (resp. $\{N_3^+(0) \cup N_3^+(2)\}$ or $\{N_3^-(1) \cup N_3^+(2)\}$). We verify that in each of these cases, either $T[\{0 \cup e \cup e''\}]$, $T[\{0 \cup e' \cup e''\}]$, $T[\{1 \cup e \cup e''\}]$ or $T[\{1 \cup e' \cup e''\}]$ is isomorphic to $W_5$, a contradiction. Therefore, $R \in \mathcal{I} \cup \mathcal{K}$. By Corollary 2, $f(2) = 2$ or $\alpha$, where $\alpha$ is the unique vertex of $\mathcal{N}_3(2)$ in $T|\mathcal{N}_3 \cup e \cup e'|$.

Suppose, again by contradiction, that $R \in \mathcal{I}$. We begin by the case where $f(2) = 2$. By Theorem 2, we can suppose that $Q = \{N_3^+(0) \cup N_3^+(2)\}$ and $Q' = \{N_3^-(0) \cup N_3^+(1)\}$. By Lemma 4, $Q'' = \{N_3^+(1) \cup N_3^+(2)\}$ or $\{N_3^-(0) \cup N_3^+(2)\}$ or $\{N_3^-(1) \cup N_3^+(2)\}$. If $Q'' = \{N_3^-(1) \cup N_3^+(2)\}$ (resp. $\{N_3^-(0) \cup N_3^+(2)\}$), then $T[\{0 \cup e \cup e''\}] \simeq W_5$ (resp. $T[\{1 \cup e' \cup e''\}] \simeq W_5$), a contradiction. If $Q'' = \{N_3^-(1) \cup N_3^+(2)\}$, then, by taking $X = \{0, 1, x\}$, where $x = \min T|\mathcal{N}_3^+(2)|$, we obtain a contradiction because, by Corollary 3, $T[\{x\}]$ is isomorphic to $2$. Indeed, $\mathcal{C}(G_{R_3}^T) = \{X^{-}(0) \cup X^+(1), X^+(0) \cup X^+(x)\}$, with $X^{-}(0) = \mathcal{N}_3^-(0)$, $X^+(1) = \mathcal{N}_3^+(1)$, $X^+(0) = \mathcal{N}_3^+(0) \cup \mathcal{N}_3^+(1)$,
and $X^+(x) = N_3^+(2) \cup \{2\} \cup (N_3^-(2) \setminus \{x\})$. Now if $f(2) = \alpha$, then we obtain again a contradiction. Indeed, by replacing $T$ by $T^*$ and by interchanging the vertices 0 and 1. \{Q, Q'\} = \{N_3^+(0) \cup N_3^-(2), N_3^-(0) \cup N_3^+(1)\}$ as in the case where $f(2) = 2$.

At present, $R \in K$. We begin by the case where $f(2) = 2$. By Theorem 2, we can suppose that \[Q = \{N_3^-(0) \cup N_3^+(1)\} \text{ and } Q' = \{N_3^+(0) \cup N_3^-(2)\}.\] By Lemma 4, $Q'' = \{N_3^+(0) \cup N_3^-(1)\}$, $\{N_3^-(0) \cup N_3^+(1)\}$ or $\{N_3^+(1) \cup N_3^-(2)\}$. If $Q'' = \{N_3^-(1) \cup N_3^+(2)\}$ (resp. $\{N_3^+(1) \cup N_3^-(0)\}$), then $T[\{0\} \cup e \cup e''] \simeq W_5$ (resp. $T[\{1\} \cup e' \cup e'''] \simeq W_5$), a contradiction. If $Q'' = \{N_3^+(1) \cup N_3^-(2)\}$, then, by taking $X = \{0, 1, x\}$, where $x = \max T[N_3^+(2)]$, we have a contradiction because, by Corollary 3, $T$ is $T[X]$-critical with $|C(G_T^X)| = 2$. Indeed, $C(G_T^X) = \{X^- \cup X^+(1), X^+(0) \cup X^-(x)\}$, with $X^- = N_3^-(0)$, $X^+(1) = N_3^+(1)$, $X^+(0) = N_3^-(1) \cup N_3^+(3)\}$ and $X^-(x) = N_3^-(2) \cup \{2\} \cup (N_3^+(2) \setminus \{x\})$. If $Q'' = \{N_3^-(0) \cup N_3^+(1)\}$, then $T \in \mathcal{L}$. Now suppose that $f(2) = \alpha$. By Theorem 2, we can suppose that $Q = \{N_3^+(1) \cup N_3^-\}$ and $Q' = \{N_3^-(1) \cup N_3^+(2)\}$. By Lemma 4, $Q'' = \{N_3^+(0) \cup N_3^-\}$, $\{N_3^+(2) \cup N_3^-(0)\}$, or $\{N_3^-(2) \cup N_3^+(0)\}$. If $Q'' = \{N_3^+(0) \cup N_3^-\}$ or $\{N_3^-(2) \cup N_3^+(0)\}$, then $T[\{0\} \cup e \cup e'''] \simeq W_5$, a contradiction. If $Q'' = \{N_3^+(0) \cup N_3^-(2)\}$, then we obtain the same configuration giving $|C(G_T^X)| = 2$ in the case where $f(2) = 2$. If $Q'' = \{N_3^-(0) \cup N_3^+(1)\}$, then $T$ is isomorphic to a tournament of the class $\mathcal{L}^*$. \[\square\]

**Proposition 6**  *For any tournament $T$ of the class $\mathcal{T}$, we have $c(T) = 2$ or 3.***

**Proof**  Let $T$ be a tournament on $(2n+1)$ vertices of the class $\mathcal{T}$ for some $n \geq 3$. We proceed by induction on $n$. By Propositions 4 and 5, the statement is satisfied for $n = 3$ and for $n = 4$. Let $n \geq 5$. By Proposition 1, we can assume that $T$ is $C_3$-critical with $V(T) \setminus W_5(T) = \{0, 1\}$. By Theorem 2 and Lemma 4, $2 \leq c(T) \leq 4$. Therefore, we only consider the case where $|C(G_T^N_{N_3})| = 4$. By Corollary 2, there exist $Q \neq Q' \in C(G_T^N_{N_3})$ and a tournament $R$ on 7 vertices of the class $\mathcal{M}$, such that for all $e \in E(G_T^N_{N_3}[Q])$ and for all $e' \in E(G_T^N_{N_3}[Q'])$, $T[N_3 \cup e \cup e'] \simeq R$. By Lemma 7, there exists $e'' \in E(G_T^N_{N_3}[Q'\prime\prime])$, where $Q'\prime\prime \in C(G_T^N_{N_3}) \setminus \{Q, Q'\}$, such that $T - e''$ is $C_3$-critical. As $W_5$ embeds into $T - e''$, then $V(T - e''') \setminus W_5(T - e''') = \{0, 1\}$ by Theorem 1. Therefore, $T - e'' \in \mathcal{T}$. By induction hypothesis, $c(T - e''') = 2$ or 3. By Theorem 2, if $c(T - e''') = 2$, then $c(T) = 2$ or 3. Therefore, suppose that $c(T - e''') = 3$. By Proposition 5 and by interchanging $T$ and $T^*$, we can assume that $T - e'' \in \mathcal{L}$. By Theorem 2 and by taking $e'' = \{x, x'\}$, we can assume that $x \in N_3^-(1)$ and $x' \in N_3^+(2)$. Thus, for $X = \{0, 1, x'\}$, we have $T[X] \simeq C_3$ and $X^+(x') = \emptyset$. It follows from Theorem 2 that $c(T) < 4$. \[\square\]

**References**


