Existence of solutions for a first-order nonlocal boundary value problem with changing-sign nonlinearity

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Abstract: This work is concerned with the existence of positive solutions to a nonlinear nonlocal first-order multipoint problem. Here the nonlinearity is allowed to take on negative values, not only positive values.

Key words: Positive solution, nonlinear boundary condition, sign-changing problem

1. Introduction
In this paper, we are interested in the existence of positive solutions for the following first-order $m$-point nonlocal boundary value problem:

\begin{align}
    y'(t) + p(t)y(t) &= \sum_{i=1}^{n} f_i(t, y(t)), \quad t \in [0, 1], \\
    y(0) &= y(1) + \sum_{j=1}^{m} g_j(t_j, y(t_j)),
\end{align}

where $p : [0, 1] \to [0, \infty)$ is continuous, the nonlocal points satisfy $0 \leq t_1 < t_2 < \ldots < t_m \leq 1$, and the nonlinear functions $f_i : [0, 1] \times [0, \infty) \to (-\infty, \infty)$ and $g_j : [0, 1] \times [0, \infty) \to [0, \infty)$ are continuous.

First-order equations with various boundary conditions, including multipoint and nonlocal conditions, are of recent interest; see [3–11,13] and the references therein.

In [12], Zhao applied a monotone iteration method to the problem (if $T = \mathbb{R}$):

\begin{align}
    y'(t) + p(t)y(t) &= f(t, y(t)), \quad t \in [0, 1], \\
    y(0) &= g(x(1)),
\end{align}

where the functions $f$ and $g$ are positive-valued continuous functions.

In [2], using the Guo–Krasnosel’skii fixed point theorem, Anderson was interested in the existence of at least one positive solution to the problem (if $T = \mathbb{R}$):

\begin{align}
    y'(t) + p(t)y(t) &= \lambda f(t, y(t)), \quad t \in [0, 1], \\
    y(0) &= y(1) + \sum_{j=2}^{n} \gamma_j y(t_j),
\end{align}

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where the function $f : (0, 1) \times [0, \infty) \to (-\infty, \infty)$ is continuous.

In [3], using the Legget–Williams fixed point theorem, Anderson studied the existence of at least three positive solutions to the nonlinear first-order problem with a nonlinear nonlocal boundary condition given by

$$y'(t) - r(t)y(t) = \lambda \sum_{i=1}^{m} f_i(t, y(t)), \quad t \in [0, 1],$$

$$\lambda y(0) = y(1) + \sum_{j=1}^{n} \Lambda_j(\tau_j, y(\tau_j)), $$

where $r : [0, 1] \to [0, \infty)$ is continuous, and the nonlinear functions $f_i : [0, 1] \times [0, \infty) \to [0, \infty)$ and $\Lambda_j : [0, 1] \times [0, \infty) \to [0, \infty)$ are also continuous.

In [7], Goodrich considered the existence of at least one positive solution to the first-order semipositone discrete fractional boundary value problem:

$$\Delta\nu y(t) = \lambda f(t, y(t)), \quad t \in [0, T] \setminus \{0\},$$

$$y(0) - y(\nu) + \sum_{i=1}^{N} F(\tau_i, y(\tau_i)).$$

In a recent paper [8], using Krasnosel’skii’s fixed point theorem, Goodrich studied the existence of a positive solution to the first-order problem given by (if $T = \mathbb{R}$)

$$y'(t) + p(t)y(t) = \lambda f(t, y(t)), \quad t \in (a, b),$$

$$y(a) = y(b) + \int_{\tau_1}^{\tau_2} F(s, y(s))ds,$$

where $\tau_1, \tau_2 \in [a, b]$ with $\tau_1 < \tau_2$, $p$ and $F$ are nonnegative functions, and the nonlinearity $f$ can be negative for some values of $t$ and $y$.

Motivated greatly by the above-mentioned works, in this paper, we are interested in the existence and iteration of positive solutions for the nonlinear nonlocal first-order multipoint problem (1.1)–(1.2). By applying the monotone iteration method, we not only obtain the existence of positive solutions, but we also establish iterative schemes for approximating the solutions. The following monotone iteration method [1] is fundamental and important to the proof of our main result.

**Theorem 1.1** Let $K$ be a cone in a Banach space $E$ and $v_0 \leq w_0$. Suppose that:

(i) $T : [v_0, w_0] \to E$ is completely continuous;

(ii) $T$ is monotone increasing on $[v_0, w_0]$;

(iii) $v_0$ is a lower solution of $T$, that is $v_0 \leq T v_0$;

(iv) $w_0$ is an upper solution of $T$, that is $T w_0 \leq w_0$.

Then the iterative sequences

$$v_n = T v_{n-1}, \quad w_n = T w_{n-1} (n = 1, 2, 3, \ldots)$$

satisfy
\[ v_0 \leq v_1 \leq \ldots \leq v_n \leq \ldots \leq w_n \leq \ldots \leq w_1 \leq w_0, \]

and converge to, respectively, \( v \) and \( w \) in \([v_0, w_0]\), which are fixed points of \( T \).

2. Main results

In this section we will state the sufficient conditions for the \( m \)-point nonlocal boundary value problem (1.1) – (1.2) to have positive solutions. For this purpose, first we will give some lemmas that will be used to give the main result.

Lemma 2.1 The function \( y(t) \) is a solution of the problem (1.1)-(1.2) if and only if

\[
y(t) = \sum_{i=1}^{n} \int_{0}^{1} G(t, s) f_i(s, y(s))ds + \frac{e^{-\int_{0}^{1} p(\xi)d\xi}}{1 + e^{-\int_{0}^{1} p(\xi)d\xi}} \sum_{j=1}^{m} g_j(t_j, y(t_j))
\]

where \( G(t, s) \) is defined by

\[
G(t, s) = \frac{e^{-\int_{1}^{s} p(\xi)d\xi}}{1 - e^{-\int_{0}^{1} p(\xi)d\xi}} \begin{cases} 1, & s < t, \\ e^{-\int_{0}^{s} p(\xi)d\xi}, & t \leq s. \end{cases}
\]

If we take the derivation of \( y(t) \), we can easily see that \( y(t) \) is the solution of the problem (1.1)-(1.2). In the proof of Theorem 2.2 in [3], a similar result was given. Therefore, we do not restate the proof here.

In [8], Goodrich found the upper and lower bounds for Green’s function on the general time scales in Lemma 2.4. Since the following lemma can be proven in a similar way, we give the lemma without the proof.

Lemma 2.2 Green’s function \( G(t, s) \) satisfies

\[
e^{-\int_{0}^{1} p(\xi)d\xi} G(s, s) \leq G(t, s) \leq e^{\int_{0}^{1} p(\xi)d\xi} G(s, s).
\]

The main result of this paper as follows.

Theorem 2.1 Assume that conditions \( f_i : [0, 1] \times [0, \infty) \to (-\infty, \infty), i = 1, 2, \ldots, n \) and \( g_j : [0, 1] \times [0, \infty) \to [0, \infty), j = 1, 2, \ldots, m \) are continuous and there exists a constant \( M > 0 \) such that \( f_i(t, y) > -M \) for all \((t, y) \in [0, 1] \times [0, \infty) \) and \( \int_{0}^{1} (f_i + M)ds > 0 \). If there exist positive constants \( r \) and \( R \) such that \( r > \frac{2M}{\gamma} \) and the following conditions are satisfied:

\[
(A_1) f_i(t, u) \leq f_i(t, v) \leq \frac{1 - e^{-\int_{0}^{1} p(\xi)d\xi}}{2n} R - M, \quad t \in [0, 1], \quad \frac{r}{2} \leq u \leq v \leq R,
\]

\[
(A_2) \frac{1 - e^{-\int_{0}^{1} p(\xi)d\xi}}{e^{-\int_{0}^{1} p(\xi)d\xi} m} \frac{r}{m} \leq g_j(t, u) \leq g_j(t, v) \leq \frac{1 - e^{-\int_{0}^{1} p(\xi)d\xi}}{2m} R, \quad t \in [0, 1], \quad \frac{r}{2} \leq u \leq v \leq R,
\]

where \( \gamma = \frac{1 - e^{-\int_{0}^{1} p(\xi)d\xi}}{1 + e^{\int_{0}^{1} p(\xi)d\xi}} e^{-\int_{0}^{1} p(\xi)d\xi} \), then the boundary value problem (1.1) – (1.2) has positive solutions.
First we consider the following boundary value problem:

\[ u'(t) + p(t)u(t) = \sum_{i=1}^{n} F_i(t, u_x(t)), \quad t \in [0, 1], \]
\[ u(0) = u(1) + \sum_{j=1}^{m} g_j(t_j, u_x(t_j)). \]

(2.1) \hspace{2cm} (2.2)

where \( F_i(t, u_x(t)) = f_i(t, u_x(t)) + M \) and \( u_x(t) = \max\{(u - x)(t), 0\} \) such that \( x(t) = M\omega(t) \) and \( \omega(t) \) is the solution of

\[ y'(t) + p(t)y(t) = 1, \quad t \in [0, 1], \]
\[ y(0) = y(1). \]

Using Lemma 2.1 and Lemma 2.2 we can easily see that the unique solution \( \omega(t) \) of the above problem satisfies

\[ \omega(t) = \int_{0}^{t} G(t, s) ds \leq \int_{0}^{1} e^{\int_{0}^{s} p(\xi) d\xi} G(s, s) ds = \frac{1}{1 - e^{-\int_{0}^{s} p(\xi) d\xi}} \leq \frac{1}{1 - e^{-\int_{0}^{1} p(\xi) d\xi}} \leq \frac{\gamma}{\gamma} = \gamma. \]

Let \( E := \{y|\ y : [0, 1] \to R \text{ continuous }\} \) with the norm \( \|y\| = \max_{t \in [0, 1]} |y(t)| \). Denote \( K := \{y \in E : y(t) \geq \gamma\|y\|, t \in [0, 1]\} \). Then \( K \) is a normal cone of \( E \). Now we define an operator \( T : K \to K \) by

\[ Tu(t) := \int_{0}^{1} G(t, s) \sum_{i=1}^{n} F_i(s, u_x(s)) ds + \frac{e^{-\int_{0}^{s} p(\xi) d\xi}}{1 + e^{-\int_{0}^{s} p(\xi) d\xi}} \sum_{j=1}^{m} g_j(t_j, u_x(t_j)), \]

and then it is easy to see that fixed points of \( T \) are nonnegative solutions of the BVP (2.3)-(2.4).

Let \( u_0(t) = r \) and \( w_0(t) = R \) for \( t \in [0, 1] \).

Now we will verify that \( T : [v_0, w_0] \to K \) is completely continuous.

First, \( T \) is continuous. Let \( u_n(n = 1, 2, ...) \), \( u \in [v_0, w_0] \) and \( \lim_{n \to \infty} u_n = u \). Then,

\[ r \leq u_n \leq R, \quad r \leq u \leq R, \quad t \in [0, 1]. \]

For any given \( \epsilon > 0 \), since \( f_i \) are uniformly continuous on \( [0, 1] \times [\frac{r}{2}, R] \), there exists \( \delta_1 > 0 \) such that for any \( u_1, u_2 \in [\frac{r}{2}, R] \) with \( |u_1 - u_2| < \delta_1 \)

\[ |f_i(s, u_1) - f_i(s, u_2)| < \frac{\epsilon}{2n} \left(1 - e^{-\int_{0}^{s} p(\xi) d\xi}\right), \quad s \in [0, 1]. \]

On the other hand, since \( g_j \) are uniformly continuous on \( [0, 1] \times [\frac{r}{2}, R] \), there exists \( \delta_2 > 0 \) such that for any \( u_1, u_2 \in [\frac{r}{2}, R] \) with \( |u_1 - u_2| < \delta_2 \)

\[ |g_j(s, u_1) - g_j(s, u_2)| < \frac{\epsilon}{2m}, \quad s \in [0, 1]. \]

Let \( \delta = \min\{\delta_1, \delta_2\} \). Then it follows from \( \lim_{n \to \infty} u_n = u \) that there exists a positive \( N \) such that for any \( n > N \),

\[ |u_n(s) - u(s)| < \delta, \quad s \in [0, 1]. \]
We can easily see that $|u_n - u| < \delta$ implies $|u_{n_k} - u_x| < \delta$.

Thus, using Lemma 2.2 and the continuity of the functions $f_i (i = 1, 2, ..., n)$ and $g_j (j = 1, 2, ..., m)$, we can easily get

$$|Tu_n(t) - Tu(t)| \leq \int_0^1 G(t, s) \sum_{i=1}^n |f_i(s, u_{n_k}(s)) - f_i(s, u_x(s))| ds + \sum_{j=1}^m |g_j(t, u_{n_k}(t)) - g_j(t, u_x(t))|$$

$$\leq \int_0^1 e^{\int_0^t p(\xi)d\xi} G(s, s) \sum_{i=1}^n \frac{\epsilon}{2n} \left(1 - e^{\int_0^t p(\xi)d\xi} \right) ds + \sum_{j=1}^m \frac{\epsilon}{2m}$$

$$= \frac{1}{1 - e^{-f_0^t p(\xi)d\xi}} \left(1 - e^{\int_0^t p(\xi)d\xi} \right) + \frac{\epsilon}{2m} = \epsilon,$$

which indicates that $\lim_{n \to \infty} Tu_n = Tu$. So $T : [v_0, w_0] \to K$ is continuous.

Next, we will show that $T : [v_0, w_0] \to K$ is compact. Let $A \subset [v_0, w_0]$ be a bounded set. Define $Q := \max_{[0,1] \times [t, R]} f_1(t, u(t))$ and $S := \max_{[0,1] \times [t, R]} g_1(t, u(t))$. If $u \in [r, R]$ we can see easily that $u_x = \max((u - x)(t), 0) = u(t) - x(t) = u(t) - Mw(t) \geq u(t) - M = R - \frac{r}{2} = R$ and $u_x \leq u \leq R$. It shows that $u_x \in [\frac{r}{2}, R]$.

For $u \in A$ and $t \in [0, 1]$, using Lemma 2.2,

$$Tu(t) \leq \frac{1}{1 - e^{-f_0^t p(\xi)d\xi}} \int_0^1 \sum_{i=1}^n \left(f_i(s, u_x(s)) + M\right) ds + \frac{1}{1 - e^{-f_0^t p(\xi)d\xi}} \sum_{j=1}^m g_j(t, u_x(t))$$

$$\leq \frac{1}{1 - e^{-f_0^t p(\xi)d\xi}} \{(Q + M)n + Sm\},$$

which shows that $T(A)$ is uniformly bounded.

On the other hand, for any $u \in A$ and $t_1, t_2 \in [0, 1]$ with $t_1 \geq t_2$ and $c \in (t_2, t_1)$, we have

$$|Tu(t_1) - Tu(t_2)| \leq \left| \int_0^1 (G(t_1, s) - G(t_2, s)) \sum_{i=1}^n \left(f_i(s, u_x(s)) + M\right) ds \right|$$

$$+ \left| \frac{e^{-f_0^t p(\xi)d\xi} - e^{-f_0^t p(\xi)d\xi}}{1 - e^{-f_0^t p(\xi)d\xi}} \sum_{j=1}^m g_j(t, u_x(t)) \right|$$

$$\leq \int_0^1 \left| e^{-f_0^t p(\xi)d\xi} - e^{-f_0^t p(\xi)d\xi} \right| \sum_{i=1}^n \left(f_i(s, u_x(s)) + M\right) ds$$

$$+ \left| \frac{e^{-f_0^t p(\xi)d\xi} - e^{-f_0^t p(\xi)d\xi}}{1 - e^{-f_0^t p(\xi)d\xi}} \sum_{j=1}^m g_j(t, u_x(t)) \right|$$

$$\leq \frac{1}{1 - e^{-f_0^t p(\xi)d\xi}} |p(c)||t_1 - t_2| \left\{ \int_0^1 \sum_{i=1}^n \left(f_i(s, u_x(s)) + M\right) ds + \sum_{j=1}^m g_j(t, u_x(t)) \right\}$$

$$\leq \frac{1}{1 - e^{-f_0^t p(\xi)d\xi}} |p(c)||t_1 - t_2| \{(Q + M)n + Sm\},$$
which implies that \( T(A) \) is equi-continuous.

Consequently, \( T : [v_0, w_0] \to K \) is compact.

Now we will show that \( T \) is monotone increasing on \([v_0, w_0]\).

Suppose that \( u, v \in [v_0, w_0] \) and \( u \leq v \). Then \( r \leq u(t) \leq v(t) \leq R \) for \( t \in [0,1] \). As we have shown previously for \( u_x \), we easily have \( v_x \in \left[\gamma, R\right] \), and so we get \( \frac{r}{2} \leq u_x(t) \leq v_x(t) \leq R \). Thus, by (A1) and (A2), for \( t \in [0,1] \), we have

\[
T u(t) = \int_0^1 G(t, s) \sum_{i=1}^{n} (f_i(s, u_x(s)) + M) ds + \frac{e^{-\int_0^1 p(\xi) d\xi}}{1 - e^{-\int_0^1 p(\xi) d\xi}} \sum_{j=1}^{m} g_j(t_j, u_x(t_j))
\]

\[
\leq \int_0^1 G(t, s) \sum_{i=1}^{n} (f_i(s, v_x(s)) + M) ds + \frac{e^{-\int_0^1 p(\xi) d\xi}}{1 - e^{-\int_0^1 p(\xi) d\xi}} \sum_{j=1}^{m} g_j(t_j, v_x(t_j)) = T v(t),
\]

which shows that \( T u \leq T v \).

Now we will prove that \( v_0 = r \) is a lower solution of \( T \). For any \( t \in [0,1] \), since \( v_{0_x} \in \left[\gamma, R\right] \), it is obvious that

\[
T v_0(t) = \int_0^1 G(t, s) \sum_{i=1}^{n} (f_i(s, v_0_x(s)) + M) ds + \frac{e^{-\int_0^1 p(\xi) d\xi}}{1 - e^{-\int_0^1 p(\xi) d\xi}} \sum_{j=1}^{m} g_j(t_j, v_0_x(t_j))
\]

\[
\geq \frac{e^{-\int_0^1 p(\xi) d\xi}}{1 - e^{-\int_0^1 p(\xi) d\xi}} \sum_{j=1}^{m} 1 - \frac{e^{-\int_0^1 p(\xi) d\xi}}{1 - e^{-\int_0^1 p(\xi) d\xi}} = r = v_0(t),
\]

which implies that \( v_0 \leq T v_0 \).

We show that \( w_0 = R \) is an upper solution of \( T \). In view of (A1) and (A2) and using \( w_{0_x} = \max\{w_0 - x, 0\} \leq w_0 = R \), we have

\[
T w_0(t) = \int_0^1 G(t, s) \sum_{i=1}^{n} (f_i(s, w_{0_x}(s)) + M) ds + \frac{e^{-\int_0^1 p(\xi) d\xi}}{1 - e^{-\int_0^1 p(\xi) d\xi}} \sum_{j=1}^{m} g_j(t_j, w_{0_x}(t_j))
\]

\[
\leq \int_0^1 G(t, s) \sum_{i=1}^{n} (f_i(s, R) + M) ds + \frac{e^{-\int_0^1 p(\xi) d\xi}}{1 - e^{-\int_0^1 p(\xi) d\xi}} \sum_{j=1}^{m} g_j(t_j, R)
\]

\[
\leq \int_0^1 \frac{1}{1 - e^{-\int_0^1 p(\xi) d\xi}} \sum_{i=1}^{n} \left( \frac{R}{2n} \right) \left( 1 - e^{-\int_0^1 p(\xi) d\xi} \right) ds + \frac{1}{1 - e^{-\int_0^1 p(\xi) d\xi}} \sum_{j=1}^{m} \left( \frac{R}{2m} \right) \left( 1 - e^{-\int_0^1 p(\xi) d\xi} \right)
\]

\[
= \frac{1}{1 - e^{-\int_0^1 p(\xi) d\xi}} \left( \frac{R}{2} + \frac{R}{2} \right) = R = w_0(t),
\]

which implies that \( T w_0 \leq w_0 \).

If we construct sequences \( \{v_n\}_{n=1}^{\infty} \) and \( \{w_n\}_{n=1}^{\infty} \) as follows:

\[
v_n = T v_{n-1}, \quad w_n = T w_{n-1}, \quad n = 1, 2, 3, \ldots,
\]

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then it follows from Theorem 1 that
\[ v_0 \leq v_1 \leq \ldots \leq v_n \leq \ldots \leq w_n \leq \ldots \leq w_1 \leq w_0, \]
and \( \{v_n\} \) and \( \{w_n\} \) converge to, respectively, \( v \) and \( w \), which are the solutions of the problem (2.1)-(2.2).

It follows from
\[
Tv_0(1) = \int_0^1 G(1,s) \sum_{i=1}^n (f_i(s,v_0(s)) + M) ds + \frac{e^{-\int_0^1 p(\xi) d\xi}}{1 - e^{-\int_0^1 p(\xi) d\xi}} \sum_{j=1}^m g_j(t_j,v_0(t_j))
\]
\[
\geq \frac{e^{-\int_0^1 p(\xi) d\xi}}{1 - e^{-\int_0^1 p(\xi) d\xi}} \left\{ \sum_{i=1}^n \int_0^1 (f_i(s,v_0(s)) + M) ds + \sum_{j=1}^m g_j(t_j,v_0(t_j)) \right\} > 0
\]
that
\[ 0 < (Tv_0)(1) \leq (Tv)(1) = v(1) \leq w(1), \]
which shows that \( v \) and \( w \) are positive solutions of the problem (2.1)-(2.2).

Moreover, we get
\[ v(t) \geq \gamma \|v\| \geq \gamma r \geq 2M\gamma^{-1} \text{ and } w(t) \geq \gamma \|w\| \geq 2M\gamma^{-1}. \]

Hence, for \( t \in [0,1] \)
\[ y(t) = v(t) - x(t) \geq \frac{2M}{\gamma} - \frac{M}{\gamma} = \frac{M}{\gamma} > 0 \text{ and } \tilde{y}(t) = w(t) - x(t) > 0, \]
and it can be easily seen that \( y \) and \( \tilde{y} \) are the positive solutions of the problem (1.1)-(1.2). \( \square \)

**Example 2.1** We consider the following first-order m-point nonlocal boundary value problem:

\[
y'(t) + \sqrt{7}y(t) = f_1(t,y) + f_2(t,y) + f_3(t,y), \quad t \in [0,1],
\]
\[ y(0) = y(1) + g_1(t,y), \]
\[ (2.3) \]
\[ (2.4) \]

where \( f_1(t,y) = \frac{1}{20}t \sqrt[3]{y(t)+3}, \quad f_2(t,y) = \frac{1}{4}((t-2)^3 + \sqrt{y(t)}) \), \( f_3(t,y) = \frac{1}{200}(y(t)-2) \) and

\[ g_1(t,y) = \begin{cases} \frac{1}{2} \left( 145 + e^{-\frac{1}{2}} \left( y\left( \frac{1}{2} \right) - 32 \right) \right), & y > 32; \\ \frac{145}{2}, & y \leq 32. \end{cases} \]

Since \( f_1 \leq 0, f_2 \leq -2 \) and \( f_3 \leq -\frac{1}{200} \), then \( M = 2 \) and we can calculate \( \gamma \approx 0.08 \). If we choose \( r = 64 \) and \( R = 280 \), then all the conditions of Theorem 2.1 are fulfilled. Thus, the problem (2.5)-(2.6) has positive solutions \( y \) and \( \tilde{y} \). Furthermore, if we construct sequences \( \{v_n\} \) and \( \{w_n\} \) such that \( v_n = T v_{n-1} \) and \( w_n = T w_{n-1} \), \( n = 1,2,... \) where \( v_0(t) = 32 \) and \( v_0(t) = 280 \), then \( \lim_{n \to \infty} v_n - x = v - x = y \) and \( \lim_{n \to \infty} w_n - x = w - x = \tilde{y} \).
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