Construction of self-reciprocal normal polynomials over finite fields of even characteristic

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Abstract: In this paper, a computationally simple and explicit construction of some sequences of normal polynomials and self-reciprocal normal polynomials over finite fields of even characteristic are presented.

Key words: Finite fields, normal polynomial, self-reciprocal

1. Introduction

Let \( F_q \), be the Galois field of order \( q = p^s \), where \( p \) is a prime and \( s \) is a natural number, and \( F_q^* \) be its multiplicative group. Let \( P(x) \) be a monic irreducible polynomial of degree \( n \) over \( F_q \) and \( \beta \) be a root of \( P(x) \). The field \( F_q(\beta) = F_{q^n} \) is an \( n \)-dimensional extension of \( F_q \) and can be considered as a vector space of dimension \( n \) over \( F_q \). The Galois group of \( F_{q^n} \) over \( F_q \) is cyclic and is generated by the Frobenius mapping \( \sigma(\alpha) = \alpha^q, \alpha \in F_{q^n} \). A normal basis of \( F_{q^n} \) over \( F_q \) is a basis of the form \( N = \{ \alpha, \alpha^q, \ldots, \alpha^{q^{n-1}} \} \), i.e. a basis that consists of the algebraic conjugates of a fixed element \( \alpha \in F_{q^n} \). Recall that an element \( \alpha \in F_{q^n} \) is said to generate a normal basis over \( F_q \) if its conjugates form a basis of \( F_{q^n} \) as a vector space over \( F_q \). For our convenience we call a generator of a normal basis a normal element. A monic irreducible polynomial \( F(x) \in F_q[x] \) is called normal polynomial or \( N \)-polynomial if its roots form a normal basis or, equivalently, if they are linearly independent over \( F_q \). The elements in a normal basis are exactly the roots of some \( N \)-polynomial. Hence, an \( N \)-polynomial is just another way of describing a normal basis. It is well known that such a basis always exists and any element of \( N \) is a generator of \( N \) (the normal basis theorem, see [4], Theorem 1.4.1).

The construction of \( N \)-polynomials over any finite field is a challenging mathematical problem. Interest in \( N \)-polynomials stems both from mathematical theory and practical applications such as coding theory and several cryptosystems using finite fields. The problem in general is: given an integer \( n \) and the ground field \( F_q \), construct a normal basis of \( F_{q^n} \) over \( F_q \), or, equivalently, construct an \( N \)-polynomial in \( F_q[x] \) of degree \( n \) by providing an efficient construction method.

Some results regarding constructions of special sequences \( (F_k(x))_{k \geq 0} \) of normal polynomials over \( F_q \) can be found in [2, 4, 6, 7, 9] and [10, 11]. All constructions are considered as computationally easy and explicit. Cohen [3] and McNay [8] gave iterative constructions of irreducible polynomials of 2-power degree over finite...
fields of odd characteristics. Meyn [10] and Chapman [2] showed that these polynomials are $N$-polynomials. Another family of $N$-polynomials of degree $2^k$ was suggested by Gao [4], who constructed specific sequences $(F_k(x))_{k \geq 0}$ of $N$-polynomials of degree $p^{k+2}$ over $\mathbb{F}_p$. In these constructions he used substitutions introduced earlier by Varshamov [13]. Kyuregyan in [6, 7] proposed a rather more general iterative technique of constructing sequences $(F_k(x))_{k \geq 0}$ of $N$-polynomials of degree $p^{k+2}$ over $\mathbb{F}_q$ compared with the ones given by Gao [4] and Scheerhorn [11]. While in the constructions of $N$-polynomials over $\mathbb{F}_{2^k}$ suggested by Gao [4] and Scheerhorn [11] the initial polynomial is a quadratic normal polynomial, in constructions suggested by Kyuregyan in [7] the initial polynomial is a normal polynomial of arbitrary degree.

In this paper, a computationally simple and explicit construction of sequences $(F_k(x))_{k \geq 0}$ of normal polynomials and $(F_k(x+1))_{k \geq 0}$ of self-reciprocal normal polynomials over $\mathbb{F}_{2^k}$ is presented. For this, we will show that all members of the sequence $(F_k(x))_{k \geq 0}$ defined by polynomials $F_k(x) \in \mathbb{F}_{2^k}[x]$ of degree $n2^k$ that are constructed by iterated application of the polynomial composition $F_k(x) = x^{n2^k}F_{k-1}(\frac{x^2+x+1}{x^2})$, $k \geq 0$, for a suitable chosen initial normal polynomial $F_0(x) \in \mathbb{F}_2[x]$ of degree $n$, for which the polynomial $F_0(x+1)$ is a self-reciprocal normal polynomial, are $N$-polynomials and the polynomials $F_k(x+1)$ are self-reciprocal normal polynomials over $\mathbb{F}_{2^k}$. Such a sequence of polynomials define a sequence of extension fields $\mathbb{F}_{2^{kn2^k}}$ whose union is denoted by $\mathbb{F}_{2^{kn2^k}} = \bigcup_{k \geq 0} \mathbb{F}_{2^{kn2^k}}$.

2. Preliminary notes

We need the following normality results for our further study.

Let $p$ denote the characteristic of $\mathbb{F}_q$ and let $n = n_1p^e = n_1t$, with $\text{gcd}(p, n_1) = 1$, and suppose that $x^n - 1$ has the following factorization in $\mathbb{F}_q[x]$

$$x^n - 1 = (x^{n_1} - 1)^t = (\phi_1(x)\phi_2(x)\cdots\phi_r(x))^t,$$

where $\phi_i(x) \in \mathbb{F}_q[x]$ are the distinct irreducible factors of $x^{n_1} - 1$. For $1 \leq i \leq r$, let

$$\phi_i(x) = \frac{x^n - 1}{\phi_i(x)}.$$

(2)

We assume that $\phi_i(x)$ has degree $m_i$ for $1 \leq i \leq r$. Furthermore, we will need Schwartz’s theorem in [12] (see also [9], Theorem 4.18), which allows us to check whether an irreducible polynomial is $N$-polynomial.

**Proposition 2.1** ([9], Theorem 4.18) Let $F(x)$ be an irreducible polynomial of degree $n$ over $\mathbb{F}_q$ and $\alpha$ be a root of $F(x)$. Let $x^n - 1$ factor as (1) and let $\phi_i(x)$ be as in (2). Then $F(x)$ is $N$-polynomial over $\mathbb{F}_q$ if and only if

$$L_{\phi_i}(\alpha) \neq 0 \text{ for } i = 1, 2, \ldots, r$$

where $L_{\phi_i}(x)$ is the linearized polynomial defined by

$$L_{\phi_i}(x) = \sum_{v=0}^{m_i} t_{iv}x^v \text{ if } \phi_i(x) = \sum_{v=0}^{m_i} t_{iv}x^v.$$
A result by Jungnickel in [5] states when an element of \( \mathbb{F}_q \) is a normal bases generator. We can restate it as follows.

**Lemma 2.2** Let \( f(x) = \sum_{i=0}^{n} c_i x^i \) be \( N \)-polynomial of degree \( n \) over \( \mathbb{F}_q \). Suppose \( g(x) = f(\frac{x-a}{b}) \), where \( a, b \in \mathbb{F}_q \) and \( b \neq 0 \). Then \( g(x) \) is \( N \)-polynomial if and only if \( na - b \frac{c_{n-1}}{c_n} \neq 0 \).

**Proof** Let \( n = n_1 p^e = n_1 t \), and then by (1), \( x^n - 1 \) has the following factorization in \( \mathbb{F}_q[x] \):

\[
x^n - 1 = (x^{n_1} - 1)^t = (\varphi_1(x)\varphi_2(x)\cdots\varphi_r(x))^t,
\]

where \( \varphi_1(x) = x - 1 \). Set for \( i = 2, 3, \ldots, r \)

\[
\phi_i(x) = \frac{x^n - 1}{\varphi_i(x)} = (x - 1)^t s_i(x) = (x - 1) s_i'(x),
\]

where

\[
s_i'(x) = (x - 1)^{t-1} s_i(x) = \sum_{v=0}^{m_i'} t_{iv} x^v.
\]

Hence,

\[
\phi_i(x) = \sum_{v=0}^{m_i'} t_{iv} x^{v+1} - \sum_{v=0}^{m_i'} t_{iv} x^v.
\]

Since \( f(x) \) is \( N \)-polynomial, then by Proposition 2.1 we have \( L_{\phi_i}(\alpha) \neq 0 \) for each \( i = 1, 2, \ldots, r \), where \( \alpha \) is a root of \( f(x) \). We need to show that \( L_{\phi_i}(a + b\alpha) \neq 0 \) is also true for each \( i = 2, 3, \ldots, r \), where \( a + b\alpha \) is a root of \( g(x) \). Since

\[
L_{\phi_i}(a + b\alpha) = \sum_{v=0}^{m_i'} t_{iv}(a + b\alpha)^{q^v} - \sum_{v=0}^{m_i'} t_{iv}(a + b\alpha)^{q^v},
\]

we have

\[
L_{\phi_i}(a + b\alpha) = a \sum_{v=0}^{m_i'} t_{iv} + b \sum_{v=0}^{m_i'} t_{iv} \alpha^{q^v} - a \sum_{v=0}^{m_i'} t_{iv} - b \sum_{v=0}^{m_i'} t_{iv} \alpha^{q^v},
\]

\[
= b(\sum_{v=0}^{m_i'} t_{iv} \alpha^{q^v} - \sum_{v=0}^{m_i'} t_{iv} \alpha^{q^v})
\]

\[
= b L_{\phi_i}(\alpha) \neq 0,
\]

and hence, for \( g(x) \) to be an \( N \)-polynomial, it suffices to solve the condition \( L_{\phi_i}(a + b\alpha) \neq 0 \). On the other hand, we have

\[
\phi_1(x) = \frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + x + 1 = \sum_{i=0}^{n-1} x^i.
\]

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So:

\[ L_{\phi_1}(a+b\alpha) = \sum_{i=0}^{n-1} (a+b\alpha)^i \]

\[ = \sum_{i=0}^{n-1} a + b \sum_{i=0}^{n-1} \alpha^i \]

\[ = na + b Tr_{q^n|q}(\alpha) = na - b \frac{c_n-1}{e_n}, \]

(4)

which is nonzero by hypothesis. This completes the proof. \(\Box\)

In the following propositions a family of irreducible polynomials of degree \(n2^k\) over \(\mathbb{F}_{2^s}\) is suggested. We will use them in the proof of our results.

**Proposition 2.3 ([1], Theorem 2.2)** Recalling the definitions of \(P^*\) and \(P^*\), let \(P(x) = \sum_{i=0}^{n} c_i x^i\) be an irreducible polynomial over \(\mathbb{F}_{2^s}\) of degree \(n\). Then

\[ F(x) = x^{2n} P \left( \frac{x^2 + \delta_0 x + \delta_1}{x^2} \right) \quad \delta_0, \delta_1 \in \mathbb{F}_{2^s} \]

is an irreducible polynomial of degree \(2n\) over \(\mathbb{F}_{2^s}\) if and only if

\[ Tr_{2^s|2}(\delta_1 \left( \frac{P^{*'}(0)}{P^*(0)} + n \right)) \neq 0. \]

**Proposition 2.4 ([1], Theorem 3.1)** Let \(P(x)\) be an irreducible polynomial of degree \(n\) over \(\mathbb{F}_{2^s}\). Define

\[ F_0(x) = P(x), \]

\[ F_k(x) = x^{n2^k} F_{k-1} \left( \frac{x^2 + x + 1}{x^2} \right) \quad k \geq 1. \]

(5)

Suppose that

\[ Tr_{2^s|2}(\frac{P'(1)}{P(1)}) \cdot Tr_{2^s|2}(\frac{P^{*'}(0)}{P^*(0)} + n) \neq 0. \]

Then \((F_k(x))_{k \geq 1}\) is a sequence of irreducible polynomials over \(\mathbb{F}_{2^s}\) of degree \(n2^k\).

3. Construction of \(N\)-polynomials over finite fields

In this section we establish theorems that will show how Propositions 2.3 and 2.4 can be applied to produce \(N\)-polynomials over \(\mathbb{F}_{2^s}\).

**Theorem 3.1** Let \(P(x) = \sum_{i=0}^{n} c_i x^i\), with \(P(x) \neq x\) an \(N\)-polynomial of degree \(n\) over \(\mathbb{F}_{2^s}\) such that \(P(x+1)\) is a self-reciprocal polynomial over \(\mathbb{F}_{2^s}\). Also let

\[ F(x) = x^{2n} P \left( \frac{x^2 + x + 1}{x^2} \right). \]

(6)
Then $F(x)$ is an $N$-polynomial of degree $2n$ over $\mathbb{F}_{2^s}$, if and only if

$$Tr_{2^s|2}(c_n - 1 + n) \neq 0.$$  

**Proof**  
Recall the definition of $Ord_{\alpha, \sigma}$. Since $P(x)$ is an irreducible polynomial over $\mathbb{F}_{2^s}$, Proposition 2.3 and the hypothesis imply that $F(x)$ is irreducible over $\mathbb{F}_{2^s}$. Let $\alpha \in \mathbb{F}_{2^n}$ be a root of $P(x)$. Since $P(x)$ is an $N$-polynomial of degree $n$ over $\mathbb{F}_{2^s}$ by the hypothesis, $\alpha \in \mathbb{F}_{2^n}$ is a normal element over $\mathbb{F}_{2^s}$ and hence has order $Ord_{\alpha, \sigma}(x) = x^n - 1$.

Let $n = n_1 2^e$, where $n_1$ is a nonnegative integer with $gcd(n_1, 2) = 1$ and $e \geq 0$. For convenience we denote $2^e$ by $t$. Let $x^n - 1$ have the following factorization in $\mathbb{F}_{2^s}[x]$:  

$$x^n - 1 = (\varphi_1(x)\varphi_2(x)\cdots\varphi_r(x))^t,$$  

(7)  

where the polynomials $\varphi_i(x) \in \mathbb{F}_q[x]$ are the distinct irreducible factors of $x^{n_1} - 1$. Set  

$$\phi_i(x) = \frac{x^n - 1}{\varphi_i(x)} = \sum_{v=0}^{m_i} t_i v x^v, i = 1, 2, \ldots, r.$$  

(8)  

By the hypothesis $\frac{c_n - 1}{c_n} + n \neq 0$, and so by Lemma 2.2, $P(x + 1)$ is a normal polynomial.

Now we proceed by proving that $F(x)$ is a normal polynomial. Let $\alpha_1$ be a root of $F(x)$.

We only need to show that the $\sigma$-order of $\alpha_1$ is  

$$Ord_{\alpha_1, \sigma}(x) = x^{2n} - 1.$$  

Note that by (7) the polynomial $x^{2n} - 1$ has the following factorization in $\mathbb{F}_{2^s}[x]$:  

$$x^{2n} - 1 = (\varphi_1(x) \varphi_2(x) \cdots \varphi_r(x))^t,$$  

where $\varphi_i(x) \in \mathbb{F}_{2^s}[x]$ are distinct irreducible factors of $x^{n_1} - 1$. Let  

$$H_i(x) = \frac{x^{2n} - 1}{\varphi_i(x)},$$  

or  

$$H_i(x) = \frac{x^{2n} - 1}{\varphi_i(x)} = (x^n + 1) \cdot \frac{x^n - 1}{\varphi_i(x)}.$$  

By (8) we obtain  

$$H_i(x) = (x^n + 1) \cdot \phi_i(x).$$  

Hence, since $\phi_i(x) = \sum_{v=0}^{m_i} t_i v x^v, i = 1, 2, \ldots, r$, we have  

$$H_i(x) = \sum_{v=0}^{m_i} t_i v (x^{n+v} + x^v).$$  

It follows that  

$$L_{H_i}(\alpha_1) = \sum_{v=0}^{m_i} t_i v (\alpha_1^{2x^n} + \alpha_1^{2x^v}).$$  

(9)
Note that, according to Proposition 2.1, to complete the proof of the theorem we only need to show that

\[ L_{H_i}(\alpha_1) \neq 0 \text{ for each } i = 1, 2, \ldots, r. \]

From (6), if \( \alpha_1 \) is a zero of \( F(x) \), then \( \frac{\alpha_1^2 + \alpha_1 + 1}{\alpha_1^2} \) is a zero of \( P(x) \). It may thus be assumed that

\[ \alpha = \frac{\alpha_1^2 + \alpha_1 + 1}{\alpha_1^2}, \]

where \( \alpha \) is a root of \( P(x) \). It follows that

\[ \alpha + 1 = \frac{\alpha_1 + 1}{\alpha_1^2} = \frac{1}{\alpha_1} + \frac{1}{\alpha_1^2}. \tag{10} \]

Now, by (10) and observing that \( P(x) \) is an irreducible polynomial of degree \( n \) over \( \mathbb{F}_{2^n} \), we obtain

\[ \alpha + 1 = (\alpha + 1)^{2^n} = \left( \frac{1}{\alpha_1} + \frac{1}{\alpha_1^2} \right)^{2^n}. \tag{11} \]

It follows from (10) and (11) that

\[ \left( \frac{1}{\alpha_1^2} + \left( \frac{1}{\alpha_1} \right)^{2^n} \right) = \left( \frac{1}{\alpha_1} + \left( \frac{1}{\alpha_1} \right)^{2^n} \right). \tag{12} \]

It is clear that \( \left( \frac{1}{\alpha_1} + \left( \frac{1}{\alpha_1} \right)^{2^n} \right) \neq 0 \).

Hence, it follows from (12) that \( \frac{1}{\alpha_1} + \left( \frac{1}{\alpha_1} \right)^{2^n} = 1 \). Therefore,

\[ \alpha_1^{2^n} = \frac{\alpha_1}{1 + \alpha_1}. \tag{13} \]

Now by (10) and (13), we can obtain

\[ \alpha_1^{2^n} + \alpha_1 = \frac{1}{\alpha + 1}. \tag{14} \]

Thus, by (9) and (14), we have

\[ L_{H_i}(\alpha_1) = \sum_{v=0}^{m_v} t_{iv} \left( \frac{1}{\alpha + 1} \right)^{2^v}. \tag{15} \]

Since \( \frac{1}{\alpha + 1} \) is a zero of the normal polynomial \( (P(x+1))^* \), therefore \( L_{H_i}(\alpha_1) \neq 0 \). Hence, \( F(x) \) is a normal polynomial of degree \( 2n \) over \( \mathbb{F}_{2^n} \), and the proof is completed. \( \square \)

4. Recurrent methods for constructing normal polynomials

In this section we describe a computationally simple and explicit recurrent method for constructing higher degree normal polynomials over finite fields \( \mathbb{F}_{2^n} \) starting from a normal polynomial. We begin by establishing the following theorem.
Theorem 4.1 Let $P(x) = \sum_{i=0}^{n} c_i x^i$, with $P(x) \neq x$ an $N$-polynomial of degree $n$ over $\mathbb{F}_{2^s}$ such that $P(x+1)$ is a self-reciprocal polynomial over $\mathbb{F}_{2^s}$. Define

$$F_0(x) = P(x),$$

$$F_k(x) = x^{n2^k} F_{k-1}(\frac{x^2 + x + 1}{x^2}) \quad k \geq 1. \quad (16)$$

Then $(F_k(x))_{k \geq 0}$ and $(F_k(x+1))_{k \geq 0}$ are the sequences of $N$-polynomials and self-reciprocal $N$-polynomials of degree $n2^k$ over $\mathbb{F}_{2^s}$, respectively, if and only if

$$\text{Tr}_{2^s|2}(\frac{P'(1)}{P(1)}) \cdot \text{Tr}_{2^s|2}(\frac{c_n-1}{c_n} + n) \neq 0,$$

where $P'(1)$ is the formal derivative of $P(x)$ at point 1.

Proof It is easy to check that the polynomial $F_k(x+1)$, for each $k \geq 1$, is self-reciprocal by using the definitions. According to Proposition 2.4 for each $k \geq 1$, $F_k(x)$ is an irreducible polynomial over $\mathbb{F}_{2^s}$. Consequently, $(F_k(x+1))_{k \geq 0}$ is a sequence of irreducible polynomials over $\mathbb{F}_{2^s}$. The proof of normality of the irreducible polynomial $F_k(x)$ for each $k \geq 1$ is done by mathematical induction on $k$.

For $k = 1$, $F_1(x)$ is a normal polynomial according to Theorem 3.1.

For $k \geq 2$, we show that $F_k(x)$ is also a normal polynomial. To this end we need to show that the hypothesis of Theorem 3.1 is satisfied. However, by induction hypothesis, we have $F_{k-1}(x)$ as a normal polynomial and $F_{k-1}(x+1)$ as a self-reciprocal polynomial. Thus, by Theorem 3.1, $F_k(x)$ is a normal polynomial if and only if

$$\text{Tr}_{2^s|2}(\frac{F_{k-1}'(0)}{F_{k-1}'(0)} + 2^{k-1}n) \neq 0,$$

or

$$\text{Tr}_{2^s|2}(\frac{F_{k-1}'(0)}{F_{k-1}'(0)}) \neq 0.$$

However, from (16), we have

$$F_{k-1}'(x) = x^{n2^{k-1}} F_{k-1}(\frac{1}{x})$$

$$= x^{n2^{k-1}} \left( \frac{1}{x} \right)^{n2^{k-1}} F_{k-2}(\frac{\frac{1}{2}}{x} + \frac{\frac{1}{2}}{x^2} + 1)$$

$$= F_{k-2}(x^2 + x + 1). \quad (17)$$

So

$$F_{k-1}'(0) = F_{k-2}(1) \quad (18)$$

and

$$F_{k-1}'(0) = F_{k-2}'(1). \quad (19)$$
On the other hand:

\[ F'_{k-1}(x) = x^{n2^{(k-1)} - 2} F_{k-2}' \left( \frac{x^2 + x + 1}{x^2} \right). \]  

(20)

So

\[ F_{k-1}(1) = F_{k-2}'(1). \]  

(21)

Using (19) and (21), we get

\[ F'_{k-1}(0) = P'(1). \]  

(22)

Obviously by (16)

\[ F_{k-1}(1) = F_{k-2}(1). \]  

(23)

So (18) and (23) imply that

\[ F_{k-1}'(0) = P(1). \]  

(24)

Hence, by (22) and (24) we obtain

\[ Tr_{2^2} \left( \frac{F'_{k-1}(0)}{P(1)} \right) = Tr_{2^2} \left( \frac{P'(1)}{P(1)} \right), \]

(25)

which is not equal to zero by the hypothesis of the theorem and so \((F_k(x))_{k \geq 0}\) is a sequence of \(N\)-polynomials of degree \(n2^k\) over \(\mathbb{F}_{2^2}\). Finally, we note that by Lemma 2.2, for every \(k \geq 1\), \(F_k(x+1)\) is an \(N\)-polynomial if and only if \(F_k'(0) \neq 0\). Thus, (22) and the hypothesis of the theorem imply that \((F_k(x+1))_{k \geq 0}\) is a sequence of self-reciprocal \(N\)-polynomials of degree \(n2^k\) over \(\mathbb{F}_{2^2}\). The theorem is proved.

\[ \square \]

**Example 4.2** Consider the normal polynomial \(P(x) = x^2 + x + 1\) over \(\mathbb{F}_2\). It is easy to see that the assumptions of Theorem 4.1 are fulfilled. Therefore, the composite polynomials

\[ F_1(x) = x^4 P \left( \frac{x^2 + x + 1}{x^2} \right) = x^4 + x^3 + 1 \]

and

\[ F_2(x) = x^8 F_1 \left( \frac{x^2 + x + 1}{x^2} \right) \]

\[ = x^8 + x^7 + x^5 + x^4 + x^3 + x^2 + 1 \]

are normal polynomials over \(\mathbb{F}_2\). Furthermore, the polynomials

\[ F_1(x+1) = x^4 + x^3 + x^2 + x + 1 \]

and

\[ F_2(x+1) = x^8 + x^7 + x^6 + x^4 + x^2 + x + 1 \]

are self-reciprocal normal polynomials over \(\mathbb{F}_2\). Obviously, Theorem 4.1 describes a computationally simple and explicit recurrent method for constructing normal and self-reciprocal normal polynomials, so computing the normal and self-reciprocal normal polynomials \(F_k(x)\) and \(F_k(x+1)\), respectively, for \(k \geq 3\) is not a complex problem.
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References


