On a generalization of Kelly’s combinatorial lemma

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Abstract: Kelly’s combinatorial lemma is a basic tool in the study of Ulam’s reconstruction conjecture. A generalization in terms of a family of $t$-element subsets of a $v$-element set was given by Pouzet. We consider a version of this generalization modulo a prime $p$. We give illustrations to graphs and tournaments.

Key words: Set, matrix, graph, tournament, isomorphism

1. Introduction

Kelly’s combinatorial lemma [24] is the assertion that the number $s(F,G)$ of induced subgraphs of a given graph $G$, isomorphic to $F$, is determined by the deck of $G$, provided that $|V(F)| < |V(G)|$, namely $s(F,G) = \frac{1}{|V(G)| - |V(F)|} \sum_{x \in V(G)} s(F,G-x)$ (where $G-x$ is the graph induced by $G$ on $V(G) \setminus \{x\}$).

In terms of a family $F$ of $t$-element subsets of a $v$-element set, it simply says that $|F| = \frac{1}{v-t} \sum_{x \in V(G)} |F-x|$ where $F-x := F \cap [V(G) \setminus \{x\}]^t$.

For sets $U,T$, we put $U(T) := \{F : T \subseteq F \in U\}$ and $U_{|K} := U \cap \mathcal{P}(K)$ (where $\mathcal{P}(K)$ is the set of subsets of $K$) so that $U_{|K}(T) := \{F : T \subseteq F \subseteq K, F \in U\}$ and $e(U) := |U|$. Pouzet [31, 32] gave the following extension of this result.

Lemma 1.1 (M.Pouzet [31]) Let $t$ and $r$ be integers, $V$ be a set of size $v \geq t+r$ elements, and $U$ and $U'$ be sets of $t$-element subsets $T$ of $V$. If for every subset $K$ of $k = t+r$ elements of $V$, $e(U_{|K}) = e(U'_{|K})$, then for all finite subsets $T'$ and $K'$ of $V$, such that $T'$ is contained in $K'$ and $K' \setminus T'$ has at least $t+r$ elements, $e(U'_{|K'}(T')) = e(U'_{|K'}(T'))$.

In particular, if $|V| \geq 2t+r = t+k$, we have this particular version of the combinatorial lemma of Pouzet:

Lemma 1.2 (M.Pouzet [31]) Let $v,t$, and $k$ be integers, $k \leq v$, $V$ be a set of $v$ elements with $t \leq \min(k,v-k)$, and $U$ and $U'$ be sets of $t$-element subsets $T$ of $V$. If for every $k$-element subset $K$ of $V$, $e(U_{|K}) = e(U'_{|K})$, then $U = U'$.

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Here we consider the case where \( e(U_K) \equiv e(U'_K) \mod p \) for every \( k \)-element subset \( K \) of \( V \); our main result, Theorem 1.3, is then a version, modulo a prime \( p \), of the particular version of the combinatorial lemma of Pouzet.

Kelly’s combinatorial lemma is a basic tool in the study of Ulam’s reconstruction conjecture. Pouzet’s combinatorial lemma has been used several times in reconstruction problems (see for example [1, 5, 6, 7, 11, 12]). Pouzet gave a proof of his lemma via a counting argument [32] and later by using linear algebra (related to incidence matrices) [31] (the paper was published earlier).

Let \( n, p \) be positive integers, the decomposition of \( n = \sum_{i=0}^{n(p)} n_ip^i \) in the basis \( p \) is also denoted \([n_0, n_1, \ldots, n_{n(p)}]_p\) where \( n_{n(p)} \neq 0 \) if and only if \( n \neq 0 \).

**Theorem 1.3** Let \( p \) be a prime number. Let \( v, t, \) and \( k \) be nonnegative integers, \( k \leq v, k = [k_0, k_1, \ldots, k_{k(p)}]_p, t = [t_0, t_1, \ldots, t_{t(p)}]_p \). Let \( V \) be a set of \( v \) elements with \( t \leq \min(k, v - k) \), and \( U \) and \( U' \) be sets of \( t \)-element subsets \( T \) of \( V \). We assume that \( e(U_K) \equiv e(U'_{K}) \mod p \) for every \( k \)-element subset \( K \) of \( V \).

1) If \( k_i = t_i \) for all \( i < t(p) \) and \( k_{t(p)} \geq t_{t(p)} \), then \( U = U' \).

2) If \( t = t_{t(p)}p^{t(p)} \) and \( k = \sum_{i=t(p)}^{k(p)} k_ip^i \), we have \( U = U' \), or one of the sets \( U, U' \) is the set of all \( t \)-element subsets of \( V \) and the other is empty, or (whenever \( p = 2 \)) for all \( t \)-element subsets \( T \) of \( V \), \( T \in U \) if and only if \( T \notin U' \).

We prove Theorem 1.3 in Section 3. We use Wilson’s theorem (Theorem 2.2) on incidence matrices.

In a reconstruction problem of graphs up to complementation [13], Wilson’s theorem yielded the following result:

**Theorem 1.4** ([13]) Let \( k \) be an integer, \( 2 \leq k \leq v - 2 \), \( k \equiv 0 (\mod 4) \). Let \( G \) and \( G' \) be 2 graphs on the same set \( V \) of \( v \) vertices (possibly infinite). We assume that \( e(G_K) \) has the same parity as \( e(G'_{K}) \) for all \( k \)-element subsets \( K \) of \( V \). Then \( G' = G \) or \( G' = \overline{G} \).

Here we look for similar results whenever \( e(G_K) \equiv e(G'_{K}) \mod p \). As an illustration of Theorem 1.3, we obtain the following result.

**Theorem 1.5** Let \( G \) and \( G' \) be 2 graphs on the same set \( V \) of \( v \) vertices (possibly infinite). Let \( p \) be a prime number and \( k \) be an integer, \( 2 \leq k \leq v - 2 \). We assume that for all \( k \)-element subsets \( K \) of \( V \), \( e(G_K) \equiv e(G'_{K}) \mod p \).

1) If \( p \geq 3, k \equiv 0, 1 (\mod p) \), then \( G' = G \).

2) If \( p \geq 3, k \equiv 0 (\mod p) \), then \( G' = G \), or one of the graphs \( G, G' \) is the complete graph and the other is the empty graph.

3) If \( p = 2, k \equiv 2 (\mod 4) \), then \( G' = G \).

We give other illustrations of Theorem 1.3, to graphs in section 4 and to tournaments in section 5.

2. Incidence matrices

We consider the matrix \( W_{t,k} \) defined as follows: Let \( V \) be a finite set, with \( v \) elements. Given nonnegative integers \( t, k \) with \( t \leq k \leq v \), let \( W_{t,k} \) be the \( \binom{v}{t} \) by \( \binom{v}{k} \) matrix of 0’s and 1’s, the rows of which are indexed
by the \( t \)-element subsets \( T \) of \( V \), the columns are indexed by the \( k \)-element subsets \( K \) of \( V \), and where the entry \( W_{t,k}(T,K) \) is 1 if \( T \subseteq K \) and is 0 otherwise. The matrix transpose of \( W_{t,k} \) is denoted \( tW_{t,k} \).

We say that a matrix \( D \) is a diagonal form for a matrix \( M \) when \( D \) is diagonal and there exist unimodular matrices (square integral matrices that have integral inverses) \( E \) and \( F \) such that \( D = EMF \). We do not require that \( M \) and \( D \) are square; here "diagonal" just means that the \((i,j)\) entry of \( D \) is 0 if \( i \neq j \). A fundamental result, due to R.M. Wilson [36], is the following.

**Theorem 2.1** (R.M. Wilson [36]) For \( t \leq \min(k,v-k) \), \( W_{t,k} \) has as a diagonal form the \( \binom{v}{i} \times \binom{v}{i} \) diagonal matrix with diagonal entries

\[
\binom{k-i}{t-i} \text{ with multiplicity } \binom{v}{i} - \binom{v}{i-1}, \quad i = 0, 1, \ldots, t.
\]

In this statement and in Theorem 2.2, \( \binom{v}{i} \) should be interpreted as zero.

Denote \( \text{rank}_Q W_{t,k} \) the rank of \( W_{t,k} \) over the field \( \mathbb{Q} \) of rational numbers, resp. \( \text{rank}_p W_{t,k} \) the rank of \( W_{t,k} \) over the \( p \)-element field \( \mathbb{F}_p \); similarly denote \( \text{Ker}_Q W_{t,k} \), \( \text{Ker}_p W_{t,k} \) the corresponding kernels. Clearly from Theorem 2.1, \( \text{rank}_Q W_{t,k} = \binom{v}{i} \). This yields Theorem 2.3 below due to D.H. Gottlieb [20] and independently W. Kantor [22]. On the other hand, from Theorem 2.1 follows \( \text{rank}_p W_{t,k} \), as given by Theorem 2.2.

**Theorem 2.2** (R.M. Wilson [36]) For \( t \leq \min(k,v-k) \), the rank of \( W_{t,k} \) modulo a prime \( p \) is

\[
\sum \binom{v}{i} - \binom{v}{i-1}
\]

where the sum is extended over those indices \( i \), \( 0 \leq i \leq t \), such that \( p \) does not divide the binomial coefficient \( \binom{k-i}{t-i} \).

This yields Theorem 2.3 below due to D.H. Gottlieb [20], and independently W. Kantor [22]. A simpler proof of Theorem 2.2 was obtained by P. Frankl [17]. Applications of Wilson’s theorem and its version modulo \( p \) have been considered by various authors, notably Chung and Graham [10] and Dammak et al. [13].

**Theorem 2.3** (D.H. Gottlieb [20], W. Kantor [22]) For \( t \leq \min(k,v-k) \), \( W_{t,k} \) has full row rank over the field \( \mathbb{Q} \) of rational numbers.

It is clear that \( t \leq \min(k,v-k) \) implies \( \binom{v}{i} \leq \binom{v}{k} \). Thus, from Theorem 2.3, we have the following result:

**Corollary 2.4** (W. Kantor [22]) For \( t \leq \min(k,v-k) \), \( \text{rank}_Q W_{t,k} = \binom{v}{i} \) and thus \( \text{Ker}_Q (tW_{t,k}) = \{0\} \).

If \( k := v-t \) then, up to a relabelling, \( W_{t,k} \) is the adjacency matrix \( A_{t,v} \) of the Kneser graph \( KG(t,v) \) [19], a graph whose vertices are the \( t \)-element subsets of \( V \), 2 subsets forming an edge if they are disjoint. The eigenvalues of Kneser graphs are computed in [19] (Theorem 9.4.3, page 200), and thus an equivalent form of Theorem 2.3 is:

**Theorem 2.5** \( A_{t,v} \) is nonsingular for \( t \leq \frac{v}{2} \).
We characterize values of $t$ and $k$ so that $\dim \ker_p(W_{t,k}) \in \{0,1\}$ and give a basis of $\ker_p(W_{t,k})$ that appears in the following result.

**Theorem 2.6** Let $p$ be a prime number. Let $v, t,$ and $k$ be nonnegative integers, $k \leq v,$ $k = \lfloor k_0, k_1, \ldots, k_{k(p)} \rfloor_p,$ $t = \lfloor t_0, t_1, \ldots, t_{k(p)} \rfloor_p,$ $t \leq \min(k, v - k).$ We have:

1) $j \leq t$ for all $j < t(p)$ and $k_{t(p)} \geq t_{t(p)}$ if and only if $\ker_p(W_{t,k}) = \{0\}.$

2) $t = t_{t(p)}p(p)$ and $k = \sum_{i=1}^{k(p)} t_i p_i$ if and only if $\dim \ker_p(W_{t,k}) = 1$ and $\{(1, 1, \cdots, 1)\}$ is a basis of $\ker_p(W_{t,k}).$

The proof of Theorem 2.6 uses Lucas’s theorem. The notation $a \mid b$ (resp. $a \nmid b$) means $a$ divides $b$ (resp. $a$ does not divide $b$).

**Theorem 2.7** (Lucas’s theorem [29]) Let $p$ be a prime number, $t, k$ be positive integers, $t \leq k,$ $t = \lfloor t_0, t_1, \ldots, t_{k(p)} \rfloor_p$ and $k = \lfloor k_0, k_1, \ldots, k_{k(p)} \rfloor_p.$ Then

$${k \choose t} \equiv \prod_{i=0}^{t(p)} \frac{k_i}{t_i} \pmod{p}, \text{ where } \frac{k_i}{t_i} = 0 \text{ if } t_i > k_i.$$ \[\text{(mod } p)\]

For an elementary proof of Theorem 2.7, see Fine [15]. As a consequence of Theorem 2.7, we have the following result, which is very useful in this paper.

**Corollary 2.8** Let $p$ be a prime number, $t, k$ be positive integers, $t \leq k,$ $t = \lfloor t_0, t_1, \ldots, t_{k(p)} \rfloor_p$ and $k = \lfloor k_0, k_1, \ldots, k_{k(p)} \rfloor_p.$ Then

\[p | \left(\begin{array}{c}k \\ t\end{array}\right)\] if and only if there is $i \in \{0, 1, \ldots, t(p)\}$ such that $t_i > k_i.$

**Proof of Theorem 2.6.** 1) We prove that under the stated conditions $\left(\begin{array}{c}k-i \\ t-i\end{array}\right) \neq 0 \pmod{p}$ for every $i \in \{0, \ldots, t\}.$ From Theorem 2.1 it follows that $\ker_p(W_{t,k}) = \{0\}.$ Let $i \in \{0, \ldots, t\}$ then $i = \lfloor i_0, i_1, \ldots, i_{t(p)} \rfloor$ with $i_{t(p)} \leq t_{t(p)}.$ Since $j = t_j$ for all $j < t(p),$ then $(t-i)j = (k-i)j$ for all $j < t(p).$ As $k_{t(p)} \geq t_{t(p)} \geq i_{t(p)}$ then $(k-i)_{t(p)} \geq (t-i)_{t(p)}$; thus, by Corollary 2.8, $p \nmid \left(\begin{array}{c}k-i \\ t-i\end{array}\right)$ for all $i \in \{0, 1, \ldots, t\}.$ Now from Theorem 2.2,

\[\text{rank}_p W_{t,k} = \sum_{i=0}^{t(p)} \left(\begin{array}{c}i \\ t\end{array}\right) - \left(\begin{array}{c}v \\ i-1\end{array}\right) = \left(\begin{array}{c}v \\ t\end{array}\right).\] Then $\ker_p(W_{t,k}) = \{0\}.$

Now we prove the converse implication. From Theorem 2.1, $\ker_p(W_{t,k}) = \{0\}$ implies $p \nmid \left(\begin{array}{c}k-i \\ t-i\end{array}\right)$ for all $i \in \{0, 1, \ldots, t\},$ in particular $p \nmid \left(\begin{array}{c}k \\ t\end{array}\right).$ Then by Corollary 2.8, $k_j \geq t_j$ for all $j \leq t(p).$ We will prove that $k_j = t_j$ for all $j \leq t(p) - 1.$ By contradiction, let $s$ be the least integer in $\{0, 1, \ldots, t(p) - 1\},$ such that $k_s > t_s.$ We have $(t - (t_s + 1)p^s) s = p - 1, \ (k - (t_s + 1)p^s) s = k_s - t_s - 1$ and $p - 1 > k_s - t_s - 1.$ From Corollary 2.8, $p \mid \left(\begin{array}{c}k-(t_s+1)p^s \\ t-(t_s+1)p^s\end{array}\right),$ which is impossible.

2) Set $n := t(p).$ We begin by the direct implication. Since $0 = k_n < t_n$ then, by Corollary 2.8, $p \mid \left(\begin{array}{c}k \\ t\end{array}\right).$ We will prove $p \nmid \left(\begin{array}{c}k-i \\ t-i\end{array}\right)$ for all $i \in \{i_0, i_1, \ldots, i_n\} \in \{1, 2, \ldots, t\}.$

Since $k_j = t_j = 0$ for all $j < n,$ then $(t-i)j = (k-i)j$ for all $j < n.$ From $t_n \geq i_n,$ we have $(t-i)_n \in \{t_n-i_n, t_n-i_n-1\}.$ Note that $(k-i)_n \in \{p-i_n-1, p-i_n\}$ and $p-i_n-1 \geq t_n-i_n$; thus $(k-i)_n \geq (t-i)_n.$
Therefore, for all \( j \leq n \), \((k-i)_j \geq (t-i)_j\). Then, by Corollary 2.8, \( p \nmid \binom{k-i}{t-i} \) for all \( i \in \{1,2,\ldots,t\} \).

Now from Theorem 2.2, \( \text{rank}_p W_{t,k} = \sum_{i=1}^{t} \binom{i}{t} - \binom{v}{t} = \binom{t}{t} - 1 \), and thus \( \dim \text{Ker}_p (W_{t,k}) = 1 \). Now \((1,1,\cdots,1)W_{t,k} = (\binom{t}{t}, \binom{t}{t}, \cdots, \binom{t}{t})\). Since \( p | \binom{t}{t} \), then \((1,1,\cdots,1)W_{t,k} \equiv 0 \pmod{p} \). Then \((1,1,\cdots,1)\) is a basis of \( \text{Ker}_p (W_{t,k}) \).

Now we prove the converse implication. Since \((1,1,\cdots,1)\) is a basis of \( \text{Ker}_p (W_{t,k}) \) and \((1,1,\cdots,1)W_{t,k} = (\binom{t}{t}, \binom{t}{t}, \cdots, \binom{t}{t})\), then \( p | \binom{t}{t} \). Since \( \dim \text{Ker}_p (W_{t,k}) = 1 \), then from Theorem 2.2, \( p \nmid \binom{k-i}{t-i} \) for all \( i \in \{1,2,\ldots,t\} \).

First, let us prove that \( t = t_n p^n \). Note that \( t_n \neq 0 \) since \( t \neq 0 \). Since \( p | \binom{t}{i} \), then, from Corollary 2.8, there is an integer \( j \in \{0,1,\ldots,n\} \) such that \( t_j > k_j \). Let \( A := \{ j < n : t_j \neq 0 \} \). By contradiction, assume \( A \neq \emptyset \).

Case 1. There is \( j \in A \) such that \( t_j > k_j \). We have \((t-p^n)_j = t_j, (k-p^n)_j = k_j \). Then from Corollary 2.8, we have \( p | \binom{k-p^n}{t-p^n} \), which is impossible.

Case 2. For all \( j \in A \), \( t_j \leq k_j \). Then \( t_n > k_n \). We have \((t-p^i)_n = t_n, (k-p^i)_n = k_n \). Then, from Corollary 2.8, we have \( p | \binom{k-p^i}{t-p^i} \), which is impossible.

From the above 2 cases, we deduce \( t = t_n p^n \).

Secondly, since \( p | \binom{t}{i} \), then by Corollary 2.8, \( t_n > k_n \). Let us show that \( k_n = 0 \). By contradiction, if \( k_n \neq 0 \) then \((t-p^n)_n = t_n - 1 > k_n - 1 = (k-p^n)_n \). From Corollary 2.8, \( p | \binom{k-p^n}{t-p^n} \), which is impossible.

Let \( s \in \{0,1,\ldots,n-1\} \); let us show that \( k_s = 0 \). By contradiction, if \( k_s \neq 0 \) then \((t-p^s)_s = p - 1, (k-p^s)_s = k_s - 1 \), thus \((t-p^s)_s > (k-p^s)_s \). Then, from Corollary 2.8, \( p | \binom{k-p^s}{t-p^s} \), which is impossible. \( \square \)

3. Proof of Theorem 1.3.

Let \( T_1, T_2, \cdots, T_{t(n)} \) be an enumeration of the \( t \)-element subsets of \( V \), let \( K_1, K_2, \cdots, K_{t(n)} \) be an enumeration of the \( k \)-element subsets of \( V \), and let \( W_{t,k} \) be the matrix of the \( t \)-element subsets versus the \( k \)-element subsets.

Let \( w_U \) be the row matrix \((u_1, u_2, \cdots, u_{t(n)})\) where \( u_i = 1 \) if \( T_i \in U \), 0 otherwise. We have

\[
w_U W_{t,k} = (\{T_i \in U : T_i \subseteq K_1\}), \cdots, \{T_i \in U : T_i \subseteq K_{t(n)}\}).
\]

\[
w_{U'} W_{t,k} = (\{T_i \in U' : T_i \subseteq K_1\}), \cdots, \{T_i \in U' : T_i \subseteq K_{t(n)}\}.
\]

Since for all \( j \in \{1,\ldots,t(n)\} \), \( e(U \cap K_j) \equiv e(U' \cap K_j) \pmod{p} \), then \((w_U - w_{U'})W_{t,k} = 0 \pmod{p} \), and \( w_U - w_{U'} \in \text{Ker}_p (W_{t,k}) \).

1) Assume \( k_i = t_i \) for all \( i < t(p) \) and \( k_{t(p)} \geq t_{t(p)} \). From 1) of Theorem 2.6, \( w_U - w_{U'} = 0 \), which gives \( U = U' \).

2) Assume \( t = t_{t(p)p^{t(p)}} \) and \( k = \sum_{i=t(p)+1}^{t(p)+k_{t(p)}} i p^i \). From 2) of Theorem 2.6, there is an integer \( \lambda \in [0, p-1] \) such that \( w_U - w_{U'} = \lambda(1,1,\cdots,1) \). It is clear that \( \lambda \in \{0,1,-1\} \). If \( \lambda = 0 \) then \( U = U' \). If \( \lambda = 1 \) and \( p \geq 3 \) then \( U = \{T_1, T_2, \cdots, T_{t(n)}\} \), \( U' = \emptyset \). If \( \lambda = 1 \) and \( p = 2 \) then \( U = \{T_1, T_2, \cdots, T_{t(n)}\} \), \( U' = \emptyset \), or \( T \in U \) if
and only if $T \not\in U'$. If $\lambda = -1$ and $p \geq 3$ then $U = \emptyset$, $U' = \{T_1, T_2, \ldots, T_{(\lambda)^k}\}$. If $\lambda = -1$ and $p = 2$ then $U' = \{T_1, T_2, \ldots, T_{(\lambda)^k}\}$, $U = \emptyset$, or $T \in U$ if and only if $T \not\in U'$. \hfill \square

4. Illustrations to graphs

Our notations and terminology follow [2]. A digraph $G = (V, E)$ or $G = (V(G), E(G))$ is formed by a finite set $V$ of vertices and a set $E$ of ordered pairs of distinct vertices, called arcs of $G$. The order (or cardinal) of $G$ is the number of its vertices. If $K$ is a subset of $V$, the restriction of $G$ to $K$, also called the induced subdigraph of $G$ on $K$, is the digraph $G|_K := (K, K^2 \cap E)$. If $K = V \setminus \{x\}$, we denote this digraph by $G_{-x}$.

Let $G = (V, E)$ and $G' = (V', E')$ be 2 digraphs. A one-to-one correspondence $f$ from $V$ onto $V'$ is an isomorphism from $G$ onto $G'$ provided that for $x, y \in V$, $(x, y) \in E$ if and only if $(f(x), f(y)) \in E'$. The digraphs $G$ and $G'$ are then said to be isomorphic, which is denoted by $G \cong G'$ if there is an isomorphism from one of them onto the other. A subset $I$ of $V$ is an interval [16, 21, 34] (or an autonomous subset [23] or a clan [14], or an homogeneous subset [18] or a module [35]) of $G$ provided that for all $a, b \in I$ and $x \in V \setminus I$, $(a, x) \in E(G)$ if and only if $(b, x) \in E(G)$, and the same for $(x, a)$ and $(x, b)$. For example, $\emptyset$, $\{x\}$ where $x \in V$, and $V$ are intervals of $G$, called trivial intervals. A digraph is then said to be indecomposable [34] (or primitive [14]) if all its intervals are trivial; otherwise it is said to be decomposable.

We say that $G$ is a graph (resp. tournament) when for all distinct vertices $x, y$ of $V$, $(x, y) \in E$ if and only if $(y, x) \in E$ (resp. $(x, y) \in E$ if and only if $(y, x) \not\in E$); we say that $\{x, y\}$ is an edge of the graph $G$ if $(x, y) \in E$, $x$, $y \in V$, and $E$ is identified with a subset of $[V]^2$, the set of pairs $\{x, y\}$ of distinct elements of $V$.

Let $G = (V, E)$ be a graph, the complement of $G$ is the graph $\overline{G} := (V, [V]^2 \setminus E)$. We denote by $e(G) := |E(G)|$ the number of edges of $G$. The degree of a vertex $x$ of $G$, denoted $d_G(x)$, is the number of edges that contain $x$. A 3-element subset $T$ of $V$ such that all pairs belong to $E(G)$ is a triangle of $G$. Let $T(G)$ be the set of triangles of $G$ and let $t(G) := |T(G)|$. A 3-element subset of $V$ that is a triangle of $G$ or of $\overline{G}$ is a 3-homogeneous subset of $G$. We set $H^{(3)}(G) := T(G) \cup T(\overline{G})$, the set of 3-homogeneous subsets of $G$, and $h^{(3)}(G) := |H^{(3)}(G)|$.

Another proof of Theorem 1.4 using Theorem 1.3. Here $p = 2$, $t = 2 = [0, 1]_p$, and $k = [0, 0, k_2, \ldots]_p$.

From 2) of Theorem 1.3, $U = U'$, or one of the sets $U, U'$ is the set of all 2-element subsets of $V$ and the other is empty, or for all 2-element subsets $T$ of $V$, $T \in U$ if and only if $T \not\in U'$. Thus $G' = G$ or $G' = \overline{G}$. \hfill $\square$

Proof of Theorem 1.5. We may suppose $V$ finite. We set $U := E(G)$, $U' := E(G')$. For all $K \subseteq V$ with $|K| = k$, we have: $\{\{x, y\} \subseteq K : \{x, y\} \subseteq U\} = E(G|_K)$ and $\{\{x, y\} \subseteq K : \{x, y\} \subseteq U'\} = E(G'|_K)$.

Since $e(G|_K) \equiv e(G'|_K) \mod p$, then $|\{\{x, y\} \subseteq K : \{x, y\} \subseteq U\}| \equiv |\{\{x, y\} \subseteq K : \{x, y\} \subseteq U'\}| \mod p$.

1) $p \geq 3$, $t := 2 = [2]_p$, and $k_0 \geq 2$. From 1) of Theorem 1.3, $U = U'$; thus $G = G'$.

2) $p \geq 3$, $t := 2 = [2]_p$, and $k_0 = 0$. From 2) of Theorem 1.3, we have $U = U'$ or one of $U, U'$ is the set of all 2-element subsets of $V$ and the other is empty. Then $G = G'$ or one of the graphs $G, G'$ is the complete graph and the other is the empty graph.

3) $p = 2$, $t = 2 = [0, 1]_p$, and $k = [0, 1, k_2, \ldots]_p$. From 1) of Theorem 1.3, we have $U = U'$; thus $G = G'$.

The following result concerns graphs $G$ and $G'$ such that $h^{(3)}(G|_K) \equiv h^{(3)}(G'|_K)$ modulo a prime $p$, for all $k$-element subsets $K$ of $V$.  

954
Theorem 4.1 Let $G$ and $G'$ be 2 graphs on the same set $V$ of $v$ vertices (possibly infinite). Let $p$ be a prime number and $k$ be an integer, $3 \leq k \leq v - 3$.

1) If $h^3(G_{1|K}) = h^3(G'_{1|K})$ for all $k$-element subsets $K$ of $V$ then $G$ and $G'$ have the same $3$-element homogeneous subsets.

2) Assume $p \geq 5$. If $k \not\equiv 1, 2 \pmod{p}$ and $h^3(G_{1|K}) \equiv h^3(G'_{1|K}) \pmod{p}$ for all $k$-element subsets $K$ of $V$, then $G$ and $G'$ have the same $3$-element homogeneous subsets.

3) If $(p = 2$ and $k \equiv 3 \pmod{4})$ or $(p = 3$ and $3 \mid k)$, and $h^3(G_{1|K}) \equiv h^3(G'_{1|K}) \pmod{p}$ for all $k$-element subsets $K$ of $V$, then $G$ and $G'$ have the same $3$-element homogeneous subsets.

Proof We may suppose $V$ finite.

We have $H^3(G) = \{\{a, b, c\} : G_{1\{a,b,c\}}$ is a $3$-element homogeneous subset$\}$.

We set $U := H^3(G)$ and $U' := H^3(G')$. For all $K \subseteq V$ with $|K| = k$, we have: $\{T \subseteq K : T \in U\} = H^3_{G_{1|K}}$ and $\{T \subseteq K : T \in U'\} = H^3_{G'_{1|K}}$. Set $t := |T| = 3$.

1) Since $h^3(G_{1|K}) = h^3(G'_{1|K})$ for all $k$-element subsets $K$ of $V$ then $|\{T \subseteq K : T \in U\}| = |\{T \subseteq K : T \in U'\}|$. From Lemma 1.2 it follows that $U = U'$; then $G$ and $G'$ have the same $3$-element homogeneous subsets.

2) Since $h^3(G_{1|K}) \equiv h^3(G'_{1|K}) \pmod{p}$ for all $k$-element subsets $K$ of $V$ then $|\{T \subseteq K : T \in U\}| \equiv |\{T \subseteq K : T \in U'\}| \pmod{p}$.

Case 1. $k_0 \geq 3$. Then $p \geq 5$, $t := 3 = [3]_p$, and $t_0 = 3 \leq k_0$. From 1) of Theorem 1.3 we have $U = U'$; thus $G$ and $G'$ have the same $3$-element homogeneous subsets.

Case 2. $k_0 = 0$. Then $p \geq 5$, $t := 3 = [3]_p$. By Ramsey’s theorem [33], every graph with at least 6 vertices contains a $3$-element homogeneous subset. Then $U$ and $U'$ are nonempty and so from 2) of Theorem 1.3, $U = U'$; thus $G$ and $G'$ have the same $3$-element homogeneous subsets.

3) Since $h^3(G_{1|K}) \equiv h^3(G'_{1|K}) \pmod{p}$ for all $k$-element subsets $K$ of $V$ then $|\{T \subseteq K : T \in U\}| \equiv |\{T \subseteq K : T \in U'\}| \pmod{p}$.

Case 1. $p = 2$ and $k \equiv 3 \pmod{4}$. Let $t := 3 = [1, 1]_p$. In this case, $k = [1, 1, k_2, \ldots]_p$; then from 1) of Theorem 1.3 we have $U = U'$; thus $G$ and $G'$ have the same $3$-element homogeneous subsets.

Case 2. $p = 3$ and $3 \mid k$. Then $k = [0, k_1, \ldots, k_{k(p)}]_p$. Let $t := 3 = [0, 1]_p$.

Case 2.1. $k_1 \in \{1, 2\}$; then from 1) of Theorem 1.3 we have $U = U'$; thus $G$ and $G'$ have the same $3$-element homogeneous subsets.

Case 2.2. $k_1 = 0$. By Ramsey’s theorem [33], every graph with at least 6 vertices contains a $3$-element homogeneous subset. Then $U$ and $U'$ are nonempty, and so from 2) of Theorem 1.3, $U = U'$; thus $G$ and $G'$ have the same $3$-element homogeneous subsets.

Let $G = (V, E)$ be a graph. From [34], every indecomposable graph of size 4 is isomorphic to $P_4 = \{(0, 1, 2, 3), \{0, 1\}, \{1, 2\}, \{2, 3\}\}$. Let $P^4(G)$ be the set of subsets $X$ of $V$ such that the induced subgraph $G_{1|X}$ is isomorphic to $P_4$. We set $p^4(G) := |P^4(G)|$. The following result concerns graphs $G$ and $G'$ such that $p^4(G_{1|K}) \equiv p^4(G'_{1|K})$ modulo a prime $p$, for all $k$-element subsets $K$ of $V$.
**Theorem 4.2** Let $G$ and $G'$ be 2 graphs on the same set $V$ of $v$ vertices. Let $p$ be a prime number and $k$ be an integer, $4 \leq k \leq v - 4$.

1) If $p^4(G|_K) = p^4(G'|_K)$ for all $k$-element subsets $K$ of $V$ then $G$ and $G'$ have the same indecomposable sets of size 4.

2) Assume $p^4(G|_K) \equiv p^4(G'|_K) \pmod{p}$ for all $k$-element subsets $K$ of $V$.

   a) If $p \geq 5$ and $k \not\equiv 1, 2, 3 \pmod{p}$, then $G$ and $G'$ have the same indecomposable sets of size 4.

   b) If $(p = 2, 4 \mid k$ and $8 \nmid k)$ or $(p = 3, 3 \mid k - 1$ and $9 \nmid k - 1)$, then $G$ and $G'$ have the same indecomposable sets of size 4.

   c) If $p = 2$ and $8 \nmid k$, then $G$ and $G'$ have the same indecomposable sets of size 4, or for all 4-element subsets $T$ of $V$, $G|_T$ is indecomposable if and only if $G'|_T$ is decomposable.

**Proof** Let $U := \{T \subseteq V : |T| = 4, G|_T \simeq P_4\} = \mathcal{P}^4(G)$, $U' := \{T \subseteq V : |T| = 4, G'|_T \simeq P_4\} = \mathcal{P}^4(G')$. For all $K \subseteq V$, we have $\{T \subseteq K : T \in U\} = \mathcal{P}_4(G|_K)$ and $\{T \subseteq K : T \in U'\} = \mathcal{P}_4(G'|_K)$. Set $t := |T| = 4$.

1) Since $p^4(G|_K) = p^4(G'|_K)$ then $|\{T \subseteq K : T \in U\}| = |\{T \subseteq K : T \in U'\}|$. From Lemma 1.2, $U = U'$; then $G$ and $G'$ have the same indecomposable sets of size 4.

2) We have $p^4(G|_K) \equiv p^4(G'|_K) \pmod{p}$ for all $k$-element subsets $K$ of $V$; then $|\{T \subseteq K : T \in U\}| \equiv |\{T \subseteq K : T \in U'\}| \pmod{p}$.

   a) Case 1. $k_0 \geq 4$. Then $p \geq 5$, $t = 4 = [4]_p$, and $t_0 = 4 \leq k_0$. From 1) of Theorem 1.3 we have $U = U'$; thus $G$ and $G'$ have the same indecomposable sets of size 4.

   Case 2. $k_0 = 0$. Let $t := 4 = [4]_p$.

   A graph $H$ is $k$-monomorphic if $G|_X \simeq G|_Y$ for all $k$-element subsets $X$ and $Y$ of $V$. If a graph of order at least 6 is 4-monomorphic then it is 2-monomorphic and hence complete or empty. Since in every graph of order 6, there is a restriction of size 4 not isomorphic to $P_4$ then, from 2) of Theorem 1.3, $U = U'$; thus $G$ and $G'$ have the same indecomposable sets of size 4.

   b) Case 1. $p = 2, 4 \mid k$, and $8 \nmid k$. Then $t := 4 = [0, 0, 1]_p$ and $k = [0, 0, 1, k_3, \ldots, k_{k(p)}]_p$. From 1) of Theorem 1.3, we have $U = U'$; thus $G$ and $G'$ have the same indecomposable sets of size 4.

   Case 2. $p = 3, 3 \mid k - 1$, and $9 \nmid k - 1$. Then $t := 4 = [1, 1]_p$, $k = [1, k_1, \ldots, k_{k(p)}]_p$, and $t_1 = 1 \leq k_1$. From 1) of Theorem 1.3, $U = U'$, thus $G$ and $G'$ have the same indecomposable sets of size 4.

   c) We have $p = 2, t := 4 = [0, 0, 1]_p$, and $k = [0, 0, 0, k_3, \ldots, k_{k(p)}]_p$. Since in every graph of order 6, there is a restriction of size 4 not isomorphic to $P_4$, then from 2) of Theorem 1.3, $U = U'$, or for all 4-element subsets $T$ of $V$, $T \in U$ if and only if $T \not\in U'$. Thus $G$ and $G'$ have the same indecomposable sets of size 4, or for all 4-element subsets $T$ of $V$, $G|_T$ is indecomposable if and only if $G'|_T$ is decomposable. □

In a reconstruction problem of graphs up to complementation [13], Wilson’s theorem yielded the following result:

**Theorem 4.3** ([13]) Let $G$ and $G'$ be 2 graphs on the same set $V$ of $v$ vertices (possibly infinite). Let $k$ be an integer, $5 \leq k \leq v - 2$, $k \equiv 1 \pmod{4}$. Then the following properties are equivalent:

(i) $e(G|_K)$ has the same parity as $e(G'|_K)$ for all $k$-element subsets $K$ of $V$; and $G|_K$, $G'|_K$ have the same 3-homogeneous subsets;

(ii) $G' = G$ or $G' = \overline{G}$.
Here, we just want to point out that we can obtain a similar result for \( k \equiv 3 \pmod{4} \), namely Theorem 4.4, using the same proof as that of Theorem 4.3.

The boolean sum \( G + G' \) of 2 graphs \( G = (V, E) \) and \( G' = (V, E') \) is the graph \( U \) on \( V \) whose edges are pairs \( e \) of vertices such that \( e \in E \) if and only if \( e \notin E' \).

**Theorem 4.4** Let \( G \) and \( G' \) be 2 graphs on the same set \( V \) of \( v \) vertices (possibly infinite). Let \( k \) be an integer, \( 3 \leq k \leq v - 2 \), \( k \equiv 3 \pmod{4} \). Then the following properties are equivalent:

(i) \( e(G_{1|K}) \) has the same parity as \( e(G'_{1|K}) \) for all \( k \)-element subsets \( K \) of \( V \); and \( G_{1|K}, G'_{1|K} \) have the same \( 3 \)-homogeneous subsets;

(ii) \( G' = G \).

**Proof** It is exactly the same as that of Theorem 4.3 (see ([13])). The implication \( (ii) \Rightarrow (i) \) is trivial. We prove \( (i) \Rightarrow (ii) \). We may suppose \( V \) finite. We set \( U := G + G' \); let \( T_1, T_2, \ldots, T_{\binom{v}{2}} \) be an enumeration of the 2-element subsets of \( V \), and let \( K_1, K_2, \ldots, K_{\binom{v}{2}} \) be an enumeration of the \( k \)-element subsets of \( V \). Let \( w_U \) be the row matrix \( (u_1, u_2, \ldots, u_{\binom{v}{2}}) \) where \( u_i = 1 \) if \( T_i \) is an edge of \( U \), 0 otherwise. We have \( w_U W_{2|k} = (e(U_{1|K_1}), e(U_{1|K_2}), \ldots, e(U_{1|K_{\binom{v}{2}}})) \). From the fact that \( e(G_{1|K}) \) has the same parity as \( e(G'_{1|K}) \) and \( e(U_{1|K}) = e(G_{1|K}) + e(G'_{1|K}) - 2e(G_{1|K} \cap G'_{1|K}) \) for all \( k \)-element subsets \( K \), \( w_U \) belongs to \( \text{Ker}_2(\ell W_{2|k}) \). According to Theorem 2.2, \( \text{rank}_{2} W_{2|k} = \left( \binom{v}{2} \right) - v + 1 \). Hence \( \dim \text{Ker}_2(\ell W_{2|k}) = v - 1 \).

We give a similar claim as Claim 2.8 of [13]; the proof is identical.

**Claim 4.5** Let \( k \) be an integer such that \( 3 \leq k \leq v - 2 \), \( k \equiv 3 \pmod{4} \); then \( \text{Ker}_2(\ell W_{2|k}) \) consists of complete bipartite graphs (including the empty graph).

**Proof** Let us recall that a star-graph of \( v \) vertices consists of a vertex linked to all other vertices, those \( v - 1 \) vertices forming an independent set. First we prove that each star-graph \( S \) belongs to \( \mathbb{K} := \text{Ker}_2(\ell W_{2|k}) \). Let \( w_S \) be the row matrix \( (s_1, s_2, \ldots, s_{\binom{v}{2}}) \) where \( s_i = 1 \) if \( T_i \) is an edge of \( S \), 0 otherwise. We have \( w_S W_{2|k} = (e(S_{1|K_1}), e(S_{1|K_2}), \ldots, e(S_{1|K_{\binom{v}{2}}})) \). For all \( i \in \{1, \ldots, \binom{v}{k}\} \), \( e(S_{1|K_i}) = k - 1 \) if the center of the star-graph belongs to \( K_i \), 0 otherwise. Since \( k \) is odd, each star-graph \( S \) belongs to \( \mathbb{K} \). The vector space (over the 2-element field) generated by the star-graphs on \( V \) consists of all complete bipartite graphs; since \( v \geq 3 \), these are distinct from the complete graph (but include the empty graph). Moreover, its dimension is \( v - 1 \) (a basis being made of star-graphs). Since \( \dim \text{Ker}_2(\ell W_{2|k}) = v - 1 \), then \( \mathbb{K} \) consists of complete bipartite graphs as claimed.

A claw is a star-graph on 4 vertices, that is a graph made of a vertex joined to 3 other vertices, with no edges between these 3 vertices. A graph is claw-free if no induced subgraph is a claw.

**Claim 4.6** ([13]) Let \( G \) and \( G' \) be 2 graphs on the same set and having the same \( 3 \)-homogeneous subsets; then the boolean sum \( U := G + G' \) is claw-free.

From Claim 4.5, \( U \) is a complete bipartite graph and, from Claim 4.6, \( U \) is claw-free. Since \( v \geq 5 \), it follows that \( U \) is the empty graph. Hence \( G' = G \) as claimed.
5. Illustrations to tournaments

Let $T = (V, E)$ be a tournament. For 2 distinct vertices $x$ and $y$ of $T$, $x \rightarrow_T y$ (or simply $x \rightarrow y$) means that $(x, y) \in E$. For $A \subseteq V$ and $y \in V$, $A \rightarrow y$ means $x \rightarrow y$ for all $x \in A$. The degree of a vertex $x$ of $T$ is $d_T(x) := |\{y \in V : x \rightarrow y\}|$. We denote by $T^*$ the dual of $T$ that is $T^* = (V, E^*)$ with $(x, y) \in E^*$ if and only if $(y, x) \in E$. A transitive tournament or a total order or $k$-chain (denoted $O_k$) is a tournament of cardinality $k$, such that for $x, y, z \in V$, if $x \rightarrow y$ and $y \rightarrow z$, then $x \rightarrow z$. If $x$ and $y$ are 2 distinct vertices of a total order, the notation $x < y$ means that $x \rightarrow y$. The tournament $C_3 := \{(0, 1, 2), (0, 1), (1, 2), (2, 0)\}$ (resp. $C_4 := \{(0, 1, 2, 3), ((0, 3), (0, 1), (3, 1), (1, 2), (2, 0), (2, 3))\}$) is a 3-cycle (resp. 4-cycle) (see Figure 1). A diamond is a tournament on 4 vertices admitting only 1 interval of cardinality 3, which is a 3-cycle. Up to isomorphism, there are exactly 2 diamonds $\delta^+$ and $\delta^- = (\delta^+)^*$, where $\delta^+$ is the tournament defined on $\{0, 1, 2, 3\}$ by $\delta^+_{\{0, 1, 2\}} = C_3$ and $\{0, 1, 2\} \rightarrow 3$. A tournament isomorphic to $\delta^+$ (resp. isomorphic to $\delta^-$) is said to be a positive diamond (resp. negative diamond) (see Figure 1). The boolean sum $U := T_1 + T_2$ of 2 tournaments, $T = (V, E)$ and $T' = (V, E')$, is the graph $U$ on $V$ whose edges are pairs $\{x, y\}$ of vertices such that $(x, y) \in E$ if and only if $(x, y) \notin E'$.

**Theorem 5.1** Let $T = (V, E)$ and $T' = (V, E')$ be 2 tournaments on the same set $V$ of $v$ vertices (possibly infinite). Let $p$ be a prime number and $k$ be an integer, $2 \leq k \leq v - 2$. Let $G := T_1 + T_2$. We assume that for all $k$-element subsets $K$ of $V$, $e(G_{|K}) \equiv 0$ (mod $p$). Then

1) $T_1 = T_2$ if $(p > 3, k \neq 0, 1$ (mod $p$)) or $(p = 2, k \equiv 2$ (mod $4$)).

2) $T = T_1$ or $T = T_2$ if $(p > 3, k \equiv 0$ (mod $p$)) or $(p = 2, k \equiv 0$ (mod $4$)).

**Proof** We may suppose $V$ finite. The proof reduces to say when $G$ is the empty graph or when $G$ is either empty or full. We set $G' :=$ The empty graph. Then $e(G_{|K}) \equiv e(G'_{|K})$ (mod $p$).

1) Use respectively 1) of Theorem 1.5 and 3) of Theorem 1.5.

2) Use respectively 2) of Theorem 1.5 and Theorem 1.4. \qed

Let $T$ be a tournament; we set $C^{(3)}(T) := \{a, b, c : T_{|\{a, b, c\}}$ is a 3-cycle$\}$, and $e^{(3)}(T) := |C^{(3)}(T)|$. Let $T = (V, E)$ and $T' = (V, E')$ be 2 tournaments and let $k$ be a nonnegative integer; $T$ and $T'$ are $k$-hypomorphic [8, 27] (resp. $k$-hypomorphic up to duality) if for every $k$-element subset $K$ of $V$, the induced subtournaments $T_{|K}$ and $T'_{|K}$ are isomorphic (resp. $T_{|K}$ is isomorphic to $T'_{|K}$ or to $T_{|K}^+$). We say that $T$ and $T'$ are $(\leq k)$-hypomorphic if $T$ and $T'$ are $h$-hypomorphic for every $h \leq k$. Similarly, we say that $T$ and $T'$ are $(\leq k)$-hypomorphic up to duality if $T$ and $T'$ are $h$-hypomorphic up to duality for every $h \leq k$. Clearly, 2 $(\leq 3)$-hypomorphic tournaments have the same diamonds. Furthermore, note that 2 $(\leq 3)$-hypomorphic tournaments have the same indecomposable structures and if a component in the tree decomposition is indecomposable, then the corresponding one is equal or dual [9].

![Figure 1](image-url)  
Figure 1. Cycle $C_3$, Cycle $C_4$, Positive Diamond, Negative Diamond.
Theorem 5.2 Let $T$ and $T'$ be 2 tournaments on the same set $V$ of $v$ vertices. Let $p$ be a prime number and $k$ be an integer, $3 \leq k \leq v - 3$.

1) If $c^3(T↾|K) = c^3(T'↾|K)$ for all $k$-element subsets $K$ of $V$ then $T$ and $T'$ are $(\leq 3)$-hypomorphic.

2) Assume $p \geq 5$. If $k \not\equiv 1, 2 \pmod{p}$, and $c^3(T↾|K) \equiv c^3(T'↾|K) \pmod{p}$ for all $k$-element subsets $K$ of $V$, then $T$ and $T'$ are $(\leq 3)$-hypomorphic.

3) If $(p = 2$ and $k \equiv 3 \pmod{4})$ or $(p = 3$ and $3 | k$), and $c^3(G↾|K) \equiv c^3(G'↾|K) \pmod{p}$ for all $k$-element subsets $K$ of $V$, then $T$ and $T'$ are $(\leq 3)$-hypomorphic.

Proof Since every tournament of cardinality $\geq 4$ has at least a restriction of cardinality 3 that is not a 3-cycle, then the proof is similar to that of Theorem 4.1. □

Let $T$ be a tournament, we set $D^+_k(T) := \{a, b, c, d \mid T↾|\{a, b, c, d\} \simeq \delta^+\}$, $D^-_k(T) := \{a, b, c, d \mid T↾|\{a, b, c, d\} \nsimeq \delta^+\}$, $d^+_k(T) := |D^+_k(T)|$, and $d^-_k(T) := |D^-_k(T)|$.

It is well known that every subtournament of order 4 of a tournament is a diamond, a 4-chain, or a 4-cycle subtournament. We have $c^3(O_4) = 0$, $c^3(\delta^+) = c^3(\delta^-) = 1$, $c^3(C_4) = 2$, and $C_4 \simeq C_4$. The $(\leq 4)$-hypomorphy has been studied by G. Lopez and C. Rauzy [27, 28].

Theorem 5.3 Let $T$ and $T'$ be 2 $(\leq 3)$-hypomorphic tournaments on the same set $V$ of $v$ vertices. Let $p$ be a prime number and $k$ be an integer, $4 \leq k \leq v - 4$.

1) If $d^+_k(T↾|K) = d^+_k(T'↾|K)$ for all $k$-element subsets $K$ of $V$ then $T'$ and $T$ are $(\leq 4)$-hypomorphic.

2) Assume $d^+_k(T↾|K) \equiv d^+_k(T'↾|K) \pmod{p}$ for all $k$-element subsets $K$ of $V$.

a) If $p \geq 5$ and $k \not\equiv 1, 2, 3 \pmod{p}$, then $T'$ and $T$ are $(\leq 4)$-hypomorphic.

b) If $(p = 3, 3 | k - 1$ and $9 \not| k - 1)$ or $(p = 2, 4 | k$ and $8 \not| k)$, then $T'$ and $T$ are $(\leq 4)$-hypomorphic.

c) If $p = 2$ and $8 | k$, then $T'$ and $T$ are $(\leq 4)$-hypomorphic.

Proof Let $U^+ := \{S \subseteq V, T↾|S \simeq \delta^+\} = D^+_k(T)$, $U'^+ := D^+_k(T')$, $U^- := D^-_k(T)$, and $U'^- := D^-_k(T')$.

Claim 5.4 If $T$ and $T'$ are $(\leq 3)$-hypomorphic and $U^+ = U'^+$, then $U^- = U'^-; T$ and $T'$ are $(\leq 4)$-hypomorphic.

Proof Let $S \subseteq U^-$, $T↾|S \simeq \delta^-$. Since $T$ and $T'$ are $(\leq 3)$-hypomorphic, then $T↾|S \simeq \delta^+$ or $T'↾|S \simeq \delta^-$. We have $\{S \subseteq V, T↾|S \simeq \delta^+\} = \{S \subseteq V, T↾|S \simeq \delta^-\}$; then $T↾|S \simeq \delta^-$, $S \subseteq U'^-$ and $U^- = U'^-$. Therefore, for $X \subseteq V$, if $T↾|X$ is a diamond then $T'↾|X \simeq T↾|X$.

Now we prove that $T$ and $T'$ are 4-hypomorphic. Let $X \subseteq V$ such that $|X| = 4$. If $T↾|X \simeq C_4$, then $c^3(T↾|X) = 2$. Since $T$ and $T'$ are $(\leq 3)$-hypomorphic then $c^3(T↾|X) = 2$; thus $T'↾|X \simeq T↾|X \simeq C_4$. The same, if $T↾|X \simeq O_4$ then $T'↾|X \simeq T↾|X \simeq O_4$. Therefore, $T'$ and $T$ are $(\leq 4)$-hypomorphic. □

From Claim 5.4, it is sufficient to prove that $U^+ = U'^+.$

For all $K \subseteq V$ with $|K| = k$, we have $\{S \subseteq K : S \subseteq U^+\} = D^+_k(T↾|K)$ and $\{S \subseteq K : S \subseteq U'^+\} = D^+_k(T'↾|K)$.

1) Since $d^+_k(T↾|K) = d^+_k(T'↾|K)$ then $\{|S \subseteq K : S \subseteq U^+\} = \{|S \subseteq K : S \subseteq U'^+\}$. From Lemma 1.2, we have $U^+ = U'^+$. 959
Figure 3). We set \( \sum S \subseteq K : S \in U^+ \) (mod \( p \)) for all \( k \)-element subsets \( K \) of \( V \); then \(|\{S \subseteq K : S \in U^+\}| \equiv |\{S \subseteq K : S \in U^+\}| \pmod{p} \).

2) We have \( d^+_4(T_{|K}) \equiv d^+_4(T'_{|K}) \pmod{p} \) for all \( k \)-element subsets \( K \) of \( V \); then \(|\{S \subseteq K : S \in U^+\}| \equiv |\{S \subseteq K : S \in U^+\}| \pmod{p} \).

   a) Case 1. \( k_0 \geq 4 \). Then \( p \geq 5 \), \( t := 4 \equiv [4]_p \), \( k = [k_0, \ldots]_p \), and \( t_0 = 4 \leq k_0 \). From 1) of Theorem 1.3 we have \( U^+ = U^+ \).

   Case 2. \( k_0 = 0 \). Then \( p \geq 5 \), \( t := 4 \equiv [4]_p \), and \( k = [0, k_1, \ldots]_p \). Since every tournament of cardinality \( \geq 5 \) has at least a restriction of cardinality 4 that is not a diamond, then from 2) of Theorem 1.3, \( U^+ = U^+ \).

   b) Case 1. \( p = 3 \), \( 3 \mid k - 1 \) and \( 9 \mid k - 1 \). Then \( t := 4 \equiv [1, 1]_p \), \( k = [1, k_1, \ldots, k_{k(p)}]_p \) and \( t_1 = 1 \leq k_1 \). From 1) of Theorem 1.3 we have \( U^+ = U^+ \).

   Case 2. \( p = 2 \), \( 4 \mid k \) and \( 8 \mid k \). Then \( t := 4 \equiv [0, 0, 1]_p \) and \( k = [0, 0, 1, k_3, \ldots, k_{k(p)}]_p \).

   From 1) of Theorem 1.3 we have \( U^+ = U^+ \).

   c) We have \( p = 2 \), \( t := 4 \equiv [0, 0, 1]_p \), \( k = [0, 0, 0, k_3, \ldots, k_{k(p)}]_p \). Since every tournament of cardinality \( \geq 5 \) has at least a restriction of cardinality 4 that is not a diamond, and the fact that \( T \) and \( T' \) are 3-hypomorphic, then from 2) of Theorem 1.3, \( U^+ = U^+ \); thus \( T' \) and \( T \) are \((\leq 5)-\)hypomorphic, or for all 4-element subsets \( S \) of \( V \), \( T_{|S} \) is isomorphic to \( \delta^+ \) if and only if \( T'_{|S} \) is isomorphic to \( \delta^- \).

In fact, in Theorem 5.3, the conclusion is that \( T' \) and \( T \) are \((\leq 5)-\)hypomorphic; this follows from Lemma 5.5 below.

**Lemma 5.5** ([5]) Let \( T \) and \( T' \) be \((\leq 4)-\)hypomorphic tournaments on at least 5 vertices. Then, \( T \) and \( T' \) are \((\leq 5)-\)hypomorphic.

**Comment.** Let \( T \) and \( T' \) be \((\leq 3)-\)hypomorphic tournaments on the same set \( V \) of \( v \) vertices. Let \( U \) (respectively \( U' \)) be the set of positive diamonds of \( T \) (respectively of \( T' \)). Then 2) of Theorem 1.3 with \( U \neq U' \) cannot occur. Indeed, from 2) of Theorem 1.3, it follows that if \( U \neq U' \) then for every 4-element subset \( X \) of \( V \), \( T_{|X} \) is a positive diamond if and only if \( T'_{|X} \) is not a positive diamond. This implies that for every 4-element subset \( Y \) of \( V \) such that \( T'_{|Y} \) is not a diamond, \( T_{|Y} \) is a positive diamond. Since there are such \( Y \) (a 5-element tournament has 0 or 2 diamonds, see H. Bouchaala [4]), this contradicts the 3-hypomorphy.

Let \( m \) be an integer, \( m \geq 1 \), \( S = \{\{0, 1, \ldots, m - 1\}, A\} \) be a digraph and for \( i < m \) a digraph \( G_i = (V_i, A_i) \) such that the \( V_i \)’s are nonempty and pairwise disjoint. The lexicographic sum over \( S \) of the \( G_i \)’s or simply the \( S \)-sum of the \( G_i \)’s is the digraph denoted by \( S(G_0, G_1, \ldots, G_{m-1}) \) and defined on the union of the \( V_i \)’s as follows: given \( x \in V_i \) and \( y \in V_j \), where \( i, j \in \{0, 1, \ldots, m - 1\} \), \((x, y)\) is an arc of \( S(G_0, G_1, \ldots, G_{m-1}) \) if either \( i = j \) and \((x, y)\) is in \( A_i \) or \( i \neq j \) and \((i, j)\) is in \( A \): this digraph replaces each vertex \( i \) of \( S \) by \( G_i \). We say that the vertex \( i \) of \( S \) is dilated by \( G_i \).

We define, for each integer \( h \geq 0 \), the tournament \( T_{2h+1} \) (see Figure 2) on \( \{0, \ldots, 2h\} \) as follows. For \( i, j \in \{0, \ldots, 2h\} \), \( i \rightarrow j \) if there exists \( k \in \{1, \ldots, h\} \) such that \( j = i + k \) modulo \( 2h + 1 \). A tournament \( T \) is said to be an element of \( D(T_{2h+1}) \) if \( T \) is obtained by dilating each vertex of \( T_{2h+1} \) by a finite chain \( p_i \), and then \( T = T_{2h+1}(p_0, p_1, \ldots, p_{2h}) \). We recall that \( T_{2h+1} \) is indecomposable and \( D(T_{2h+1}) \) is the class of finite tournaments without a diamond [27]; this class was obtained previously by Moon [30].

We define the tournament \( T_{6} = T_2(p_0, p_1, p_2) \) with \( p_0 = (0 < 1 < 2) \), \( p_1 = (3 < 4) \), and \( |p_2| = 1 \) (see Figure 3). We set \( \beta_{6}^+ := (\beta_{6}^-)^* \).
Theorem 5.8

Lemma 5.7

Corollary 5.6

A prime number and

then:

Two tournaments $T$ and $T'$ on the same vertex set $V$ are hereditarily isomorphic if for all $X \subseteq V$, $T|_X$ and $T'|_X$ are isomorphic.

Let $G = (V, E)$ and $G' = (V, E')$ be $(\leq 2)$-hypomorphic digraphs. Denote $D_{G,G'}$ the binary relation on $V$ such that: for $x \in V$, $xD_{G,G'}x$; and for $x \neq y \in V$, $xD_{G,G'}y$ if there exists a sequence $x_0 = x, \ldots, x_n = y$ of elements of $V$ satisfying $(x_i, x_{i+1}) \in E$ if and only if $(x_i, x_{i+1}) \notin E'$, for all $i$, $0 \leq i \leq n - 1$. The relation $D_{G,G'}$ is an equivalence relation called the difference relation; its classes are called difference classes.

Using difference classes, G. Lopez [25, 26] showed that if $T$ and $T'$ are $(\leq 6)$-hypomorphic then $T$ and $T'$ are isomorphic. One may deduce the next corollary.

Corollary 5.6 ([25, 26]) Let $T$ and $T'$ be $2$ tournaments. We have the following properties:

1) If $T$ and $T'$ are $(\leq 6)$-hypomorphic then $T$ and $T'$ are hereditarily isomorphic.

2) If for each equivalence class $C$ of $D_{T,T'}$, $C$ is an interval of $T$ and $T'$, and $T'_C$, $T|_C$ are $(\leq 6)$-hypomorphic, then $T$ and $T'$ are hereditarily isomorphic.

Lemma 5.7 [27] Given $(\leq 4)$-hypomorphic tournaments $T$ and $T'$, and $C$ an equivalence class of $D_{T,T'}$, then:

1) $C$ is an interval of $T'$ and $T$.

2) Every $3$-cycle in $T|_C$ is reversed in $T'_C$.

3) There exists an integer $h \geq 0$ such that $T|_C = T_{2h+1}(p_0, p_1, \ldots, p_{2h})$ and $T'_C = T_{2h+1}(p'_0, p'_1, \ldots, p'_{2h})$ with $p_i$, $p'_i$ as chains on the same basis, for all $i \in \{0, 1, \ldots, 2h\}$.

Theorem 5.8 Let $T$ and $T'$ be $(\leq 4)$-hypomorphic tournaments on the same set $V$ of $v$ vertices. Let $p$ be a prime number and $k = [k_0, k_1, \ldots, k_{k(p)}]_p$ be an integer, $6 \leq k \leq v - 6$. 

961
1) If $b_6^+(T_{1|K}) = b_6^+(T'_{1|K})$ for all $k$-element subsets $K$ of $V$ then $T'$ and $T$ are $(\leq 6)$-hypomorphic and thus hereditarily isomorphic.

2) Assume $b_6^+(T_{1|K}) \equiv b_6^+(T'_{1|K})$ (mod $p$) for all $k$-element subsets $K$ of $V$.

a) If $p \geq 7$, and $k_0 \geq 6$ or $k_0 = 0$, then $T'$ and $T$ are $(\leq 6)$-hypomorphic and thus hereditarily isomorphic.

b) If $(p = 5, k_0 = 1, \text{ and } k_1 \neq 0)$ or $(p = 3, k_0 = 0, \text{ and } k_1 = 2)$ or $(p = 3 \text{ and } k_0 = k_1 = 0) \text{ or } (p = 2, k_0 = 0, \text{ and } k_1 = k_2 = 1)$, then $T'$ and $T$ are $(\leq 6)$-hypomorphic and thus hereditarily isomorphic.

**Proof** From Lemma 5.5, $T$ and $T'$ are $(\leq 5)$-hypomorphic. Let $U^+ := \{S \subseteq V, T_{1|S} \simeq \beta_6^+\} = B_6^+(T)$, $U'^+ := B_6^+(T')$, $U^- := \{S \subseteq V, T_{1|S} \simeq \beta_6^-\} = B_6^-(T)$, $U'^- := B_6^-(T')$.

Every tournament of cardinality $\geq 7$ has at least a restriction of cardinality 6 that is neither isomorphic to $\beta_6^+$ nor to $\beta_6^-$. Then, for all cases, similarly to the proof of Theorem 5.3, we have $U^+ = U'^+$.

Let $C$ be an equivalence class of $D_{T,T'}$, $S \in U^-$, $T_{1|S} \simeq \beta_6^-$. Since $T$ and $T'$ are $(\leq 3)$-hypomorphic, then $T'_{1|S} \simeq \beta_6^+$ or $T'_{1|S} \simeq \beta_6^-$. We have $S \subseteq V$, $T'_{1|S} \simeq \beta_6^+\}$ $\cup$ $\{S \subseteq V, T'_{1|S} \simeq \beta_6^-\}$. Then, $T'_{1|S} \simeq \beta_6^-$, $S \in U'^-$, and $U^- = U'^-$. Let $X \subseteq C$ such that $|X| = 6$; if $T_X \simeq \beta_6^+$ then, from 2) of Lemma 5.7, $T_X \simeq \beta_6^-$, which is impossible, and so $T_C$ and $T'_C$ do not have a restriction of cardinality 6 isomorphic to $\beta_6^+$ and $\beta_6^-$. From Lemma 5.9 below, $T_C$ and $T'_C$ are $(\leq 6)$-hypomorphic.

**Lemma 5.9** ([?]) Let $T$ and $T'$ be 2 $(\leq 5)$-hypomorphic tournaments defined on a vertex set $V$ such that for all $X \subseteq V$, if $T_{1|X}$ is isomorphic to $\beta_6^+$ or to $\beta_6^-$, then $T'_{1|X}$ is isomorphic to $T_{1|X}$. Then $T$ and $T'$ are $(\leq 6)$-hypomorphic.

From 1) of Lemma 5.7, $C$ is an interval of $T'$ and $T$. Then, from 2) of Corollary 5.6, $T$ and $T'$ are hereditarily isomorphic.

From Theorem 5.2, Theorem 5.3, and Theorem 5.8, we deduce the following result.

**Corollary 5.10** Let $T$ and $T'$ be 2 tournaments on the same set $V$ of $v$ vertices. Let $p$ be a prime number and $k = [k_0, k_1, \ldots, k_{k(p)}]_p$ be an integer, $6 \leq k \leq v - 6$.

1) If $c^{(3)}(T_{1|K}) = c^{(3)}(T'_{1|K})$, $d_1^+(T_{1|K}) = d_1^+(T'_{1|K})$, and $b_6^+(T_{1|K}) = b_6^+(T'_{1|K})$ for all $k$-element subsets $K$ of $V$ then $T'$ and $T$ are hereditarily isomorphic.

2) Assume $c^{(3)}(T_{1|K}) \equiv c^{(3)}(T'_{1|K})$, $d_1^+(T_{1|K}) \equiv d_1^+(T'_{1|K})$, and $b_6^+(T_{1|K}) \equiv b_6^+(T'_{1|K})$ (mod $p$) for all $k$-element subsets $K$ of $V$.

If $p \geq 7$, and $k_0 \geq 6$ or $k_0 = 0$, then $T'$ and $T$ are hereditarily isomorphic.

**References**


