Seiberg–Witten-like equations on 5-dimensional contact metric manifolds**

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Abstract: In this paper, we write Seiberg–Witten-like equations on contact metric manifolds of dimension 5. Since any contact metric manifold has a $\text{Spin}^c$-structure, we use the generalized Tanaka–Webster connection on a $\text{Spin}^c$ spinor bundle of a contact metric manifold to define the Dirac-type operators and write the Dirac equation. The self-duality of 2-forms needed for the curvature equation is defined by using the contact structure. These equations admit a nontrivial solution on 5-dimensional strictly pseudoconvex CR manifolds whose contact distribution has a negative constant scalar curvature.

Key words: Seiberg–Witten equations, spinor, Dirac operator, contact metric manifold, self-duality

1. Introduction

Seiberg–Witten equations were defined on 4-dimensional Riemannian manifolds by Witten in [14]. The solution space of these equations gives differential topological invariants for 4-manifolds [1, 11]. Some generalizations were given later on higher dimensional manifolds [4, 7, 10].

Seiberg–Witten equations consist of 2 equations. The first is the Dirac equation, which is meaningful for the manifolds having $\text{Spin}^c$–structure. The second is the curvature equation, which couples the self-dual part of a connection 2-form with a spinor field. In order to be able to write down the curvature equation, the notion of the self-duality of a 2-form is needed. This notion is meaningful for 4-dimensional Riemannian manifolds. On the other hand, there are similar self-duality notions for some higher dimensional manifolds [5, 13]. In the present paper, we propose Seiberg–Witten-like equations for 5-dimensional contact metric manifolds by using the $\text{Spin}^c$-structure and the notion of self-duality given in [12] and [3], respectively.

The paper is organized as follows. We begin with a section introducing some basic facts concerning contact metric manifolds. In the following section, we study self-dual 2-forms on 5-dimensional contact metric manifolds. In Section 4, we discuss the $\text{Spin}^c$–structures and Dirac-type operators associated to the generalized Tanaka–Webster connection. In the final section we propose the Dirac and curvature equations and hence write Seiberg–Witten-like equations on contact metric manifolds of dimension 5. Finally, we obtain a special solution for these equations on the 5-dimensional strictly pseudoconvex CR manifolds whose contact distribution has a negative constant scalar curvature.

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2. Contact metric manifolds

A contact form on a smooth manifold $M$ of dimension $(2n+1)$ is a 1-form $\eta$ such that $\eta \wedge (d\eta)^{n} \neq 0$ everywhere on $M$. The contact form $\eta$ induces a hyperplane subbundle $H$ of the tangent bundle $TM$ given by $H = \text{Ker} \eta$. The Reeb vector field associated to $\eta$ is the vector field $\xi$ uniquely determined by $\eta(\xi) = 1$ and $d\eta(\xi,.) = 0$. Then $(M,\eta)$ is called a contact manifold.

Note that given $H = \text{Ker} \eta$ and $\xi$ such that $\eta(\xi) = 1$, we can split the tangent bundle into $TM = H \oplus \mathbb{R}\xi$. If $X$ is any vector field on $M$, then $X$ decomposes as $X = X_H + f\xi$ for any $f \in C^\infty(M, \mathbb{R})$. $X_H$ is called the horizontal part of $X$.

If $(M,\eta)$ is a contact manifold, then the pair $(H,d\eta|_H)$ is a symplectic vector bundle. We fix an almost complex structure $J_H$ on $H$ compatible with $d\eta|_H$, i.e. $d\eta|_H(J_H(X),J_H(Y)) = d\eta|_H(X,Y)$. We can extend $J_H$ to an endomorphism $J$ of the tangent bundle $TM$ by setting $J\xi = 0$. The relation $J^2 = -Id + \eta \otimes \xi$ then holds. With this in mind, $g_\eta$ given by

$$g_\eta(X,Y) = d\eta(X,JY) + \eta(X)\eta(Y),$$

defines a Riemannian metric on $TM$. The metric $g_\eta$ is called a Webster metric and is said to be associated to $\eta$. Moreover, the following relations hold:

$$g_\eta(\xi,X) = \eta(X), \quad g_\eta(JX,JY) = d\eta(X,Y), \quad g_\eta(JX,JY) = g_\eta(X,Y) - \eta(X)\eta(Y)$$

for any $X,Y \in \Gamma(TM)$. We call $(M,g_\eta,\eta,\xi,J)$ a contact metric manifold. For detailed information, see [2, 12].

The generalized Tanaka–Webster connection $\nabla$ is a well-known connection on the contact metric manifold $(M,g_\eta,\eta,\xi,J)$. This connection satisfies the conditions $\nabla \eta = 0$ and $\nabla g_\eta = 0$. Moreover, if $J$ is integrable, i.e. $\nabla J = 0$, then the contact metric manifold $(M,g_\eta,\eta,\xi,J)$ is called a strictly pseudoconvex CR manifold [12].

3. Self-dual 2-forms on 5-dimensional contact metric manifolds

Let $(M,g_\eta,\eta,\xi,J)$ be a 5-dimensional contact metric manifold. The $p$-form $\alpha$ is called a horizontal $p$-form if $i(\xi)\alpha = 0$ where $i$ is contraction operator. For any 2-form $\alpha \in \Omega^2(M)$ we have the splitting $\alpha = \alpha_H + \alpha_\xi$ where $\alpha_H = \alpha \circ \Pi$, $\Pi : TM \to H$ is the canonical projection and $\alpha_\xi = \eta \wedge i(\xi)\alpha$. The decomposition of $\Omega^2(M)$ is then given by

$$\Omega^2(M) = \Omega^2_H(M) \oplus \eta \wedge \Omega^1_H(M),$$

where $\Omega^2_H(M)$ and $\Omega^1_H(M)$ are the bundles of horizontal forms. Moreover, any horizontal 2-form can be split into its self-dual and anti-self dual parts as follows.

Let $\ast$ be the Hodge-star operator acting on the cotangent bundle $T^*M$. We can define the operator

$$\ast : \Omega^2(M) \to \Omega^2(M), \quad \ast(\beta) := \ast(\eta \wedge \beta).$$

We can restrict the operator $\ast$ to the space of horizontal 2-forms $\Omega^2_H(M)$:

$$\ast_H : \Omega^2_H(M) \to \Omega^2_H(M), \quad \ast_H(\beta) := \ast(\eta \wedge \beta).$$

This operator satisfies $\ast_H^2 = id$. Then we have the following orthogonal decomposition:

$$\Omega^2_H(M) = \Omega^2_H(M)^+ \oplus \Omega^2_H(M)^-,$$
where $\Omega^2_H(M)^\pm$ is the eigenspace associated to eigenvalue $\pm 1$ of the operator $\star_H$. The eigenspace $\Omega^2_H(M)^+$ is called as the space of self-dual 2-forms. In a similar way, the eigenspace $\Omega^2_H(M)^-$ is called the space of anti-self-dual 2-forms (see [3, 8]). From equalities (1) and (2), we have
\[ \Omega^2(M) = \Omega^2_H(M)^+ \oplus \Omega^2_H(M)^- \oplus \eta \wedge \Omega^1_H(M). \]

Hence, any 2-form $\alpha$ can be written as $\alpha = \alpha_H^+ + \alpha_H^- + \eta \wedge \beta$ where $\beta$ is a 1-form on $H$. The self-dual part of $\alpha$ is defined as the self-dual part of $\alpha_H$, i.e. $\alpha^+ := \alpha_H^+$.

Locally, we can specify the self-dual and anti-self-dual 2-forms. For this, choose a local orthonormal frame field $\{e_1, e_2 = J(e_1), e_3, e_4 = J(e_3), \xi\}$ and denote by $\{e^1, e^2, e^3, e^4, \eta\}$ the dual basis. From (2), the 2-form $d\eta$ has the form $d\eta = e^1 \wedge e^2 + e^3 \wedge e^4$. The forms $e^1 \wedge e^2 + e^3 \wedge e^4$, $e^1 \wedge e^3 - e^2 \wedge e^4$ and $e^1 \wedge e^4 + e^2 \wedge e^3$ are an orthonormal basis for $\Omega^2_H(M)^+$. An orthonormal basis for $\Omega^2_H(M)^-$ is given by the forms $e^1 \wedge e^2 - e^3 \wedge e^4$, $e^1 \wedge e^3 + e^2 \wedge e^4$, and $e^1 \wedge e^4 - e^2 \wedge e^3$.

4. Dirac operators on contact metric manifolds

In this section we will describe Dirac operators on contact metric manifolds. For this, we need a Spin$^c$-structure. Any contact metric manifold admits a canonical Spin$^c$-structure. Then we have a Spin$^c$-bundle $P_{Spin^c(2n)}$, an $S^1$-bundle $P_{S^1}$, and the canonical line bundle $\mathcal{L}$. The spinor bundle $S$ can be identified with the bundle $\wedge^* \mathcal{L}$ of the $(0,*)$ forms. For the definitions and more details about these notions, we refer to [12]. For our purpose, we use the following representation of the complex Clifford algebra $\mathbb{C}l_5$:

\[
\kappa(e_1) = \begin{pmatrix}
0 & i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & i
\end{pmatrix}, \quad \kappa(e_2) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]
\[
\kappa(e_3) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}, \quad \kappa(e_4) = \begin{pmatrix}
0 & 0 & 0 & i \\
0 & 0 & -i & 0 \\
0 & i & 0 & 0 \\
i & 0 & 0 & 0
\end{pmatrix},
\]
\[
\kappa(e_5) = \begin{pmatrix}
i & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & -i
\end{pmatrix}, \quad \kappa(d\eta) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 2i & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2i
\end{pmatrix}.
\]

Let $(M, g, \eta, \xi, J)$ be a contact metric manifold equipped with a Spin$^c$-structure. Each unitary connection $\nabla$ on $\mathcal{L}$ induces a spinorial connection $\nabla^A$ on $S$ with the generalized Tanaka–Webster connection $\nabla$. The Kohn–Dirac operator $D^A_H$ is defined as follows:
\[
D^A_H = \sum_{i=1}^{2n} \kappa(e_i)(\nabla^A_{e_i}),
\]
where $\{e_i\}$ is a local orthonormal frame of $H$. The Dirac operator $D_A$ is defined by
\[
D_A = D^A_H + \xi \cdot \nabla^A
\]
(see also [12]).
5. Seiberg–Witten-like equations on 5-dimensional contact metric manifolds

In [6], Seiberg–Witten-like equations on 5-dimensional Euclidean space $\mathbb{R}^5$ were written. In this section, we will write Seiberg–Witten-like equations on 5-dimensional contact metric manifolds and give a solution to these equations on strictly pseudoconvex CR manifolds.

For a spinor $\psi$ we define a 2-form $\sigma(\psi)$ by the following formula:

$$\sigma(\psi)(X,Y) = \langle X \cdot Y \cdot \psi, \psi \rangle + g_\eta(X,Y)|\psi|^2,$$

where $X, Y \in \Gamma(TM)$ and $\langle , , \rangle$ is the Hermitian inner product on the spinor space $S$. Note that $\sigma(\psi)$ is an imaginary valued 2-form. The restriction of $\sigma(\psi)$ to $H$ is denoted by $\sigma_H(\psi)$.

**Definition 1** Let $(M,g_\eta,\eta,\xi,J)$ be a contact metric 5-manifold. Fix a Spin$^c$-structure and a connection $A$ in the $U(1)$-principal bundle associated with the Spin$^c$-structure. For any $\psi \in \Gamma(S)$ Seiberg–Witten equations are defined by

$$D_A(\psi) = 0,$$

$$F_A^+ = -\frac{1}{4}\sigma(\psi)^+, \quad (3)$$

where $F_A^+$ is the self-dual part of the curvature $F_A$ and $\sigma(\psi)^+$ is the self-dual part of the 2-form $\sigma(\psi)$.

Now we give a solution for Seiberg–Witten equations in dimension 5. To do this, we follow the method given in [9]. From now on we suppose that $(M,g_\eta,\eta,\xi,J)$ is a strictly pseudoconvex CR manifold.

Let $(M,g_\eta)$ be a contact metric manifold endowed with Spin$^c$-structure. The spinor bundle is then $S = \Lambda^{0,0}_H(M)$. Namely,

$$S = \Lambda^{0,2}_H(M) \oplus \Lambda^{0,1}_H(M) \oplus \Lambda^{0,0}_H(M),$$

where $\Lambda^{0,2}_H(M)$ is the eigenspace corresponding to the eigenvalue $2i$ of the mapping $\kappa(d\eta) : S \to S$ and has dimension 1, $\Lambda^{0,1}_H(M)$ is the eigenspace corresponding to the eigenvalue 0 of the mapping $\kappa(d\eta) : S \to S$ and has dimension 2, and $\Lambda^{0,0}_H(M)$ is the eigenspace corresponding to the eigenvalue $-2i$ of the mapping $\kappa(d\eta) : S \to S$ and has dimension 1.

If $\psi_0 \in \Lambda^{0,0}_H(M)$, then $\psi_0$ denotes the spinor corresponding to the constant function 1 in the chosen coordinates

$$\psi_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$ 

Moreover, we have $d\eta \cdot \psi_0 = -2i\psi_0$. By using the expression of $\sigma_H(\psi)$ in the local coordinates, we obtain the following identity:

$$\sigma_H(\psi_0) = -i d\eta. \quad (4)$$

5.1. Some identities

In this part, we collect some identities needed for the special solution of Seiberg–Witten equations.
When $M$ is a strictly pseudoconvex CR manifold, $M$ also has a complex CR structure [2]. Let \( \{Z_1, \ldots, Z_n\} \) be a local unitary frame of $T^{1,0}$ over $U \subset M$ where $Z_\alpha = \frac{1}{\sqrt{2}}(e_\alpha - \sqrt{-1}J_e_\alpha)$, $1 \leq \alpha \leq n$. Let us denote by $\omega := (\omega_{\alpha\beta})$ the matrix of the connection form of $\nabla$ with respect to the frame. Then we can write the following:

$$\nabla Z_\alpha = \sum_\beta \omega_{\alpha\beta} Z_\beta.$$

$\{Z_1, \ldots, Z_n, \zeta_1, \ldots, \zeta_n, \xi\}$ is a local frame of the complexified tangent bundle $TM^C$ over $U$.

Let $\{\theta^1, \ldots, \theta^n, \bar{\theta}^1, \ldots, \bar{\theta}^n, \eta\}$ be the corresponding dual basis. Thus,

$$\zeta = \bar{\theta}^1 \wedge \ldots \wedge \bar{\theta}^n : U \to \Lambda^{0,n}_H(M)$$

is a local section in determinant line bundle $\Lambda^{0,n}_H(M)$. The Webster connection $\nabla$ defines a covariant derivative in the canonical line bundle $\Lambda^{0,n}_H(M)$ such that

$$\nabla(\bar{\theta}^1 \wedge \ldots \wedge \bar{\theta}^n) = -Tr(\omega)\bar{\theta}^1 \wedge \ldots \wedge \bar{\theta}^n.$$

Since $\nabla$ is a metric with respect to $g_\eta$, the trace $Tr(\omega)$ is purely imaginary. Therefore, this connection $\nabla$ in $\Lambda^{0,n}_H(M)$ induces a connection on the associated $S^1$-principal bundle $P_{S^1}$. Let us denote this connection by $A$. Then,

$$\zeta^* A = -Tr \omega = Tr \omega$$

is a local connection form on $S^1$-bundle $P_{S^1}$. Let $F_A$ be the curvature form of the connection $A$. The curvature form $F_A$ is a 2-form on $M$ with values in $i\mathbb{R}$. Over $U \subset M$ we have

$$F_A = dA = Trd\omega. \quad (5)$$

Moreover,

$$Ric(X,Y) = Tr(d\omega) - Tr(\omega \wedge \omega) = Trd\omega. \quad (6)$$

From (5) and (6) it follows that

$$F_A = Ric. \quad (7)$$

Here we follow the similar procedures given in [2].

In the following, the Ricci form $\rho_H$ is defined by

$$\rho_H(X,Y) = Ric(X, J_H Y) = g_\eta(X, J_H Ric Y)$$

for any $X,Y \in \Gamma(H)$. In the case of a strictly pseudoconvex CR manifold, the almost complex structure $J_H$ is complex. Therefore, we have the equation

$$Ric(X,Y) = i\rho_H(X,Y) \quad (8)$$

for any $X,Y \in \Gamma(H)$. 

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Proposition 2 Let $\rho_H$ be a Ricci form on $H$ and $s_H$ be a scalar curvature of the subbundle $H$. Then the following identity holds:

$$\rho_H^+ = -\frac{s_H}{4} \, d\eta,$$

(9)

where $\rho_H^+$ is a the self-dual part of the Ricci form $\rho_H$.

Proof In local coordinates the almost complex structure $J$ is given as follows.

$$J = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Since $J \circ \text{Ric} = \text{Ric} \circ J$, we obtain the reduced form of the Ric as follows.

$$\text{Ric} = \begin{pmatrix}
R_{11} & 0 & R_{13} & R_{14} & 0 \\
0 & R_{11} & -R_{14} & R_{13} & 0 \\
R_{13} & -R_{14} & R_{33} & 0 & 0 \\
R_{14} & R_{13} & 0 & R_{33} & 0 \\
0 & 0 & 0 & 0 & R_{55}
\end{pmatrix}$$

The Ricci form $\rho_H$ can be written in the following way:

$$\rho_H = -R_{11} e^1 \wedge e^2 - R_{33} e^3 \wedge e^4 - R_{13} (e^1 \wedge e^4 - e^2 \wedge e^3) + R_{14} (e^1 \wedge e^3 + e^2 \wedge e^4).$$

Since the 2-forms $e^1 \wedge e^4 - e^2 \wedge e^3$ and $e^1 \wedge e^3 + e^2 \wedge e^4$ are anti-self-dual 2-forms, the self-dual part of $\rho_H$ is given by

$$\rho_H^+ = \frac{R_{11} - R_{33}}{2} \, d\eta = \frac{R_{11} + R_{22} + R_{33} + R_{44} - s_H}{4} \, d\eta = -\frac{s_H}{4} \, d\eta,$$

where $s_H$ is the restricted scalar curvature to $H$. 

5.2. A special solution to 5-dimensional Seiberg–Witten equations

Let $(M, g, \eta, \xi, J)$ be a strictly pseudoconvex contact manifold of dimension 5. Suppose that the scalar curvature $s_H$ of the subbundle $H$ is negative and constant. Then let $\psi = \sqrt{-s_H} \psi_0$. In this case, $\psi \in \Lambda^{0,0}_H(M)$.

From (4) we have

$$\sigma_H(\psi) = i s_H d\eta.$$  \hfill (10)

By using (7),(8), (9), and (10) we obtain

$$F_A^+ = \text{Ric}^+ = i \rho_H^+ = -\frac{i s_H}{4} \, d\eta = -\frac{1}{4} \sigma_H(\psi).$$

Note that since $d\eta$ is a self-dual 2-form, $\sigma_H(\psi)$ is also i.e., $\sigma_H(\psi)^+ = \sigma_H(\psi)$. Because of $\sigma(\psi)^+ = \sigma_H(\psi)^+$ and with identity (11), we get

$$F_A^+ = -\frac{1}{4} \sigma(\psi)^+.$$
One can show that $\nabla e_i \psi_0 = 0$. Therefore, we deduce that

$$D^A_H \psi = 0.$$ 

Moreover,

$$D_A \psi = 0.$$ 

The pair $(A, \psi = \sqrt{-1} \psi_0)$ is a solution of Seiberg–Witten-like equations in (3).

References