On Biharmonic Legendre curves in $S$-space forms

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Abstract: We study biharmonic Legendre curves in $S-$ space forms. We find curvature characterizations of these special curves in 4 cases.

Key words: $S-$ space form, Legendre curve, biharmonic curve, Frenet curve

1. Introduction

Let $(M, g)$ and $(N, h)$ be 2 Riemannian manifolds and $f : (M, g) \rightarrow (N, h)$ a smooth map. The energy functional of $f$ is defined by

$$E(f) = \frac{1}{2} \int_M |df|^2 g.$$ 

If $f$ is a critical point of the energy functional $E(f)$, then it is called harmonic [10]. $f$ is called a biharmonic map if it is a critical point of the bienergy functional

$$E_2(f) = \frac{1}{2} \int_M |\tau(f)|^2 g,$$ 

where $\tau(f)$ is the first tension field of $f$, which is defined by $\tau(f) = \text{trace} \nabla df$. The Euler-Lagrange equation of bienergy functional $E_2(f)$ gives the biharmonic map equation [16]

$$\tau_2(f) = -J^f(\tau(f)) = -\Delta \tau(f) - \text{trace} \mathcal{R}^N(df, \tau(f)) df = 0,$$

where $J^f$ is the Jacobi operator of $f$. It is trivial that any harmonic map is biharmonic. If the map is a nonharmonic biharmonic map, then we call it proper biharmonic. Biharmonic submanifolds have been studied by many geometers. For example, see [2], [3], [7], [8], [11], [12], [13], [14], [15], [18], [20], [21], [22], and the references therein. In a different setting, in [9], Chen defined a biharmonic submanifold $M \subset \mathbb{E}^n$ of the Euclidean space as its mean curvature vector field $H$ satisfies $\Delta H = 0$, where $\Delta$ is the Laplacian.

In [12] and [14], Fetcu and Oniciuc studied biharmonic Legendre curves in Sasakian space forms. As a generalization of their studies, in the present paper, we study biharmonic Legendre curves in $S-$ space forms. We obtain curvature characterizations of these kinds of curves.

The paper is organized as follows: In Section 2, we give a brief introduction about $S-$ space forms. In Section 3, we give the main results of the study.

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2. \( S \)-space forms and their submanifolds

Let \((M, g)\) be a \((2m + s)\)-dimensional framed metric manifold \([24]\) with a framed metric structure \((f, \xi_\alpha, \eta^\alpha, g)\), \(\alpha \in \{1, \ldots, s\}\), that is, \(f\) is a \((1, 1)\) tensor field defining an \( f \)-structure of rank \(2m\); \(\xi_1, \ldots, \xi_s\) are vector fields; \(\eta^1, \ldots, \eta^s\) are 1-forms; and \(g\) is a Riemannian metric on \(M\) such that for all \(X, Y, Z \in T_M\) and \(\alpha, \beta \in \{1, \ldots, s\}\),

\[
f^2 = -I + \sum_{\alpha=1}^s \eta^\alpha \otimes \xi_\alpha, \quad \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad f(\xi_\alpha) = 0, \quad \eta^\alpha \circ f = 0, \quad (2.1)
\]

\[
g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^s \eta^\alpha(X)\eta^\alpha(Y), \quad (2.2)
\]

\[
d\eta^\alpha(X, fY) = g(X, fY) = -d\eta^\alpha(Y, X), \quad \eta^\alpha(X) = g(X, \xi), \quad (2.3)
\]

\((M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)\) is also called a framed \( f \)-manifold \([19]\) or almost \( r \)-contact metric manifold \([23]\). If the Nijenhuis tensor of \(f\) equals \(-2d\eta^\alpha \otimes \xi_\alpha\) for all \(\alpha \in \{1, \ldots, s\}\), then \((f, \xi_\alpha, \eta^\alpha, g)\) is called \( S \)-structure \([4]\).

If \(s = 1\), a framed metric structure is an almost contact metric structure and an \( S \)-structure is a Sasakian structure. If a framed metric structure on \(M\) is an \( S \)-structure, then the following equations hold \([4]\):

\[
(\nabla_X f)Y = \sum_{\alpha=1}^s \left\{ g(fX, fY)\xi_\alpha - \eta^\alpha(Y)f^2X \right\}, \quad (2.4)
\]

\[
\nabla\xi_\alpha = -f, \quad \alpha \in \{1, \ldots, s\}. \quad (2.5)
\]

In the case of Sasakian structure \((s = 1)\), \((2.5)\) can be calculated using \((2.4)\).

A plane section in \(T_pM\) is an \( f \)-section if there exists a vector \(X \in T_pM\) orthogonal to \(\xi_1, \ldots, \xi_s\) such that \(\{X, fX\}\) span the section. The sectional curvature of an \( f \)-section is called an \( f \)-sectional curvature. In an \( S \)-manifold of constant \( f \)-sectional curvature, the curvature tensor \(R\) of \(M\) is of the form

\[
R(X, Y)Z = \sum_{\alpha, \beta} \left\{ \eta^\alpha(X)\eta^\beta(Z)f^2Y - \eta^\beta(Y)\eta^\alpha(Z)f^2X \right. \\
- g(fX, fZ)\eta^\alpha(Y)\xi_\beta + g(fY, fZ)\eta^\alpha(X)\xi_\beta \right. \\
+ \frac{c-1}{c} \left. \left\{ -g(fY, fZ)f^2X + g(fX, fZ)f^2Y \right\} \right. \\
\left. + \frac{c}{c-s} \{g(X, fZ)fY - g(Y, fZ)fX + 2g(X, fY)fZ\}, \quad (2.6)\right.
\]

for all \(X, Y, Z \in TM\) \([6]\). An \( S \)-manifold of constant \( f \)-sectional curvature \(c\) is called an \( S \)-space form, which is denoted by \(M(c)\). When \(s = 1\), an \( S \)-space form becomes a Sasakian space form \([5]\).

A submanifold of an \( S \)-manifold is called an integral submanifold if \(\eta^\alpha(X) = 0, \ \alpha = 1, \ldots, s\), for every tangent vector \(X\) \([17]\). We call a 1-dimensional integral submanifold of an \( S \)-space form \((M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)\) a Legendre curve of \(M\). In other words, a curve \(\gamma : I \rightarrow M = (M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)\) is called a Legendre curve if \(\eta^\alpha(T) = 0, \ \text{for every} \ \alpha = 1, \ldots s\), where \(T\) is the tangent vector field of \(\gamma\).
3. Biharmonic Legendre curves in $S$-space forms

Let $\gamma : I \to M$ be a curve parametrized by arc length in an $n$-dimensional Riemannian manifold $(M, g)$. If there exists orthonormal vector fields $E_1, E_2, \ldots, E_r$ along $\gamma$ such that

\begin{align*}
E_1 &= \gamma' = T, \\
\nabla_T E_1 &= \kappa_1 E_2, \\
\nabla_T E_2 &= -\kappa_1 E_1 + \kappa_2 E_3, \\
\quad \vdots \\
\nabla_T E_r &= -\kappa_{r-1} E_{r-1},
\end{align*}

then $\gamma$ is called a Frenet curve of osculating order $r$, where $\kappa_1, \ldots, \kappa_{r-1}$ are positive functions on $I$ and $1 \leq r \leq n$.

A Frenet curve of osculating order 1 is a geodesic; a Frenet curve of osculating order 2 is called a circle if $\kappa_1$ is a nonzero positive constant; a Frenet curve of osculating order $r \geq 3$ is called a helix of order $r$ if $\kappa_1, \ldots, \kappa_{r-1}$ are nonzero positive constants; a helix of order 3 is shortly called a helix.

Now let $(M^{2m+s}, f, \xi, \eta^\alpha, g)$ be an $S$-space form and $\gamma : I \to M$ a Legendre Frenet curve of osculating order $r$. Differentiating

\[ \eta^\alpha(T) = 0 \]

and using (3.7), we find

\[ \eta^\alpha(E_2) = 0, \quad \alpha \in \{1, \ldots, s\}. \]  

By the use of (2.1), (2.2), (2.3), (2.6), (3.7), and (3.9), it can be seen that

\[ \nabla_T \nabla_T T = -\kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3, \]

\[ \nabla_T \nabla_T \nabla_T T = -3\kappa_1\kappa_1' E_1 + (\kappa''_1 - \kappa^3_1 - \kappa_1 \kappa_2^2) E_2 \]

\[ + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 + \kappa_1 \kappa_2 \kappa_3 E_4, \]

\[ R(T, \nabla_T T)T = -\kappa_1 \frac{(c + 3s)}{4} E_2 - 3\kappa_1 \frac{(c - s)}{4} g(fT, E_2)fT. \]

Thus, we have

\[ \tau_2(\gamma) = \nabla_T \nabla_T \nabla_T T - R(T, \nabla_T T)T \]

\[ = -3\kappa_1 \kappa_1' E_1 \]

\[ + \left( \kappa''_1 - \kappa^3_1 - \kappa_1 \kappa_2^2 + \kappa_1 \frac{(c + 3s)}{4} \right) E_2 \]

\[ + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 + \kappa_1 \kappa_2 \kappa_3 E_4 \]

\[ + 3\kappa_1 \frac{(c - s)}{4} g(fT, E_2)fT. \]  

Let $k = \min \{r, 4\}$. From (3.10), the curve $\gamma$ is proper biharmonic if and only if $\kappa_1 > 0$ and

1. $c = s$ or $fT \perp E_2$ or $fT \in \text{span} \{E_2, \ldots, E_k\}$; and
2. $g(\tau(\gamma), E_i) = 0$, for any $i = 1, k$.

We can therefore state the following theorem:
Theorem 3.1 Let $\gamma$ be a Legendre Frenet curve of osculating order $r$ in an $S$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$\), $\alpha \in \{1, \ldots, s\}$, and $k = \min \{r, 4\}$. Then $\gamma$ is proper biharmonic if and only if

1. $c = s$ or $fT \perp E_2$ or $fT \in \text{span}\{E_2, \ldots, E_k\}$; and
2. the first $k$ of the following equations are satisfied (replacing $\kappa_k = 0$):

$$\kappa_1 = \text{constant} > 0,$$
$$\kappa_1^2 + \kappa_2^2 = \frac{c + 3s}{4} + \frac{3(c - s)}{4} [g(fT, E_2)]^2,$$
$$\kappa_2 + \frac{3(c - s)}{4} g(fT, E_2) g(fT, E_3) = 0,$$
$$\kappa_2 \kappa_3 + \frac{3(c - s)}{4} g(fT, E_2) g(fT, E_4) = 0.$$

Now we give the interpretations of Theorem 3.1.

Case I. $c = s$.
In this case $\gamma$ is proper biharmonic if and only if

$$\kappa_1 = \text{constant} > 0,$$
$$\kappa_1^2 + \kappa_2^2 = s,$$
$$\kappa_2 = \text{constant},$$
$$\kappa_2 \kappa_3 = 0.$$

Theorem 3.2 Let $\gamma$ be a Legendre Frenet curve in an $S$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \ldots, s\}$, $c = s$, and $(2m + s) > 3$. Then $\gamma$ is proper biharmonic if and only if either $\gamma$ is a circle with $\kappa_1 = \sqrt{s}$ or a helix with $\kappa_1^2 + \kappa_2^2 = s$.

Remark 3.1 If $2m + s = 3$, then $m = s = 1$. So $M$ is a 3-dimensional Sasakian space form. Since a Legendre curve in a Sasakian 3-manifold has torsion 1 (see [1]), we can write $\kappa_1 > 0$ and $\kappa_2 = 1$, which contradicts $\kappa_1^2 + \kappa_2^2 = s = 1$. Hence, $\gamma$ cannot be proper biharmonic.

Case II. $c \neq s$, $fT \perp E_2$.
In this case, $g(fT, E_2) = 0$. From Theorem 3.1, we obtain

$$\kappa_1 = \text{constant} > 0,$$
$$\kappa_1^2 + \kappa_2^2 = \frac{c + 3s}{4},$$
$$\kappa_2 = \text{constant},$$
$$\kappa_2 \kappa_3 = 0. \quad (3.11)$$

First, we give the following proposition:

Proposition 3.1 Let $\gamma$ be a Legendre Frenet curve of osculating order 3 in an $S$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \ldots, s\}$, and $fT \perp E_2$. Then $\{T = E_1, E_2, E_3, fT, \nabla_T fT, \xi_1, \ldots, \xi_s\}$ is linearly independent at any point of $\gamma$. Therefore, $m \geq 3$.

Proof Since $\gamma$ is a Frenet curve of osculating order 3, we can write

$$E_1 = \gamma' = T,$$
$$\nabla_T E_1 = \kappa_1 E_2,$$
$$\nabla_T E_2 = -\kappa_1 E_1 + \kappa_2 E_3,$$
$$\nabla_T E_3 = -\kappa_2 E_2. \quad (3.12)$$
The system
\[ S_1 = \{T = E_1, E_2, E_3, fT, \nabla_T fT, \xi_1, \ldots, \xi_s\} \]
has only nonzero vectors. Using (2.1), (2.2), (2.3), and (2.4), we find
\[ \nabla_T fT = \sum_{\alpha=1}^{s} \xi_\alpha + \kappa_1 fE_2. \]  
(3.13)

So by the use of (3.8), (3.9), (3.12), and (3.13), we have
\[ T \perp E_2, T \perp E_3, T \perp E_4, T \perp fT, \]
\[ T \perp \nabla_T fT, T \perp \xi_\alpha \text{ for all } \alpha \in \{1, \ldots, s\}. \]

Hence, \( S_1 \) is linearly independent if and only if \( S_2 = \{E_2, E_3, fT, \nabla_T fT, \xi_1, \ldots, \xi_s\} \) is linearly independent. From the assumption we have \( E_2 \perp fT \). From (3.9), \( E_2 \perp \xi_\alpha \) for all \( \alpha \in \{1, \ldots, s\} \). Using (2.3), (3.12), and (3.13), we have \( E_2 \perp E_3 \) and \( E_2 \perp \nabla_T fT \). So \( S_2 \) is linearly independent if and only if \( S_3 = \{E_3, fT, \nabla_T fT, \xi_1, \ldots, \xi_s\} \) is linearly independent. Differentiating \( g(fT, E_2) = 0 \) and using (3.12) and (3.13), we find \( g(fT, E_3) = 0 \). Hence, \( fT \perp E_3 \). Using (2.1) and (2.3), we find \( g(fT, \xi_\alpha) = 0, \) that is, \( fT \perp \xi_\alpha \) for all \( \alpha \in \{1, \ldots, s\} \). Using (2.2) and (3.13), we obtain \( g(fT, \nabla_T fT) = 0 \). So \( S_3 \) is linearly independent if and only if \( S_4 = \{E_3, \nabla_T fT, \xi_1, \ldots, \xi_s\} \) is linearly independent. Differentiating \( \eta^\alpha(E_2) = 0 \), we have \( \eta^\alpha(E_3) = 0, \) \( \alpha \in \{1, \ldots, s\} \). Thus \( E_3 \perp \xi_\alpha \) for all \( \alpha \in \{1, \ldots, s\} \). If we differentiate \( g(fT, E_3) = 0 \), we get \( g(\nabla_T fT, E_3) = 0 \), that is, \( E_3 \perp \nabla_T fT \). So \( S_4 \) is linearly independent if and only if \( S_5 = \{\nabla_T fT, \xi_1, \ldots, \xi_s\} \) is linearly independent. Since \( \kappa_1 \neq 0 \) and \( fE_2 \perp \xi_\alpha \) for all \( \alpha \in \{1, \ldots, s\} \), equation (3.13) gives us \( \nabla_T fT \notin \text{span} \{\xi_1, \ldots, \xi_s\} \). So \( S_5 \) is linearly independent.

Since \( \{T = E_1, E_2, E_3, fT, \nabla_T fT, \xi_1, \ldots, \xi_s\} \) is linearly independent, \( \dim M = 2m + s \geq s + 5 \). Hence, \( m \geq 3 \).

Now we can state the following Theorem:

**Theorem 3.3** Let \( \gamma \) be a Legendre Frenet curve in an \( S \)-space form \( (M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g) \), \( \alpha \in \{1, \ldots, s\} \), \( c \neq s \), and \( fT \perp E_2 \). Then \( \gamma \) is proper biharmonic if and only if either

1. \( m \geq 2 \) and \( \gamma \) is a circle with \( \kappa_1 = \frac{1}{2} \sqrt{c + 3s} \), where \( c > -3s \) and \( \{T = E_1, E_2, fT, \nabla_T fT, \xi_1, \ldots, \xi_s\} \) is linearly independent; or

2. \( m \geq 3 \) and \( \gamma \) is a helix with \( \kappa_1^2 + \kappa_2^2 = \frac{c + 3s}{1} \), where \( c > -3s \) and \( \{T = E_1, E_2, E_3, fT, \nabla_T fT, \xi_1, \ldots, \xi_s\} \) is linearly independent.

If \( c \leq -3s \), then \( \gamma \) is biharmonic if and only if it is a geodesic.

**Case III.** \( c \neq s \), \( fT \parallel E_2 \).

In this case, \( fT = \pm E_2 \), \( g(fT, E_2) = \pm 1 \), \( g(fT, E_3) = g(\pm E_2, E_3) = 0 \), and \( g(fT, E_4) = g(\pm E_2, E_4) = 0 \). From Theorem 3.1, \( \gamma \) is biharmonic if and only if

\[ \kappa_1 = \text{constant} > 0, \]
\[ \kappa_1^2 + \kappa_2^2 = c, \]
\[ \kappa_2 = \text{constant}, \]
\[ \kappa_2 \kappa_3 = 0. \]
We can assume that $fT = E_2$. From equation (2.1), we get

$$fE_2 = f^2T = -T + \sum_{\alpha=1}^{s} \eta^\alpha(T) \xi_\alpha = -T.$$  \hspace{1cm} (3.14)

From (3.13) and (3.14), we find

$$\nabla_T fT = \sum_{\alpha=1}^{s} \xi_\alpha - \kappa_1 T.$$  \hspace{1cm} (3.15)

Using (3.7) and (3.15), we can write

$$\kappa_2 E_3 = \sum_{\alpha=1}^{s} \xi_\alpha,$$

which gives us

$$\kappa_2 = \sqrt{\sum_{\alpha=1}^{s} \xi_\alpha} = \sqrt{s},$$

$$E_3 = \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha,$$

$$\eta^\alpha(E_3) = \frac{1}{\sqrt{s}}, \quad \alpha \in \{1, \ldots, s\}.$$

Thus by the use of Theorem 3.1, we have the following Theorem:

**Theorem 3.4** Let $\gamma$ be a Legendre Frenet curve in an $\mathcal{S}$-space form $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \ldots, s\}$, $c \neq s$, and $fT \parallel E_2$. Then

$$\left\{ T, fT, \frac{1}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha \right\}$$

is the Frenet frame field of $\gamma$ and $\gamma$ is proper biharmonic if and only if it is a helix with $\kappa_1 = \sqrt{c-s}$ and $\kappa_2 = \sqrt{s}$, where $c > s$. If $c \leq s$, then $\gamma$ is biharmonic if and only if it is a geodesic.

**Case IV.** $c \neq s$ and $g(fT, E_2)$ is not constant 0, 1, or $-1$.

Now, let $(M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g)$ be an $\mathcal{S}$-space form, $\alpha \in \{1, \ldots, s\}$, and $\gamma: I \rightarrow M$ a Legendre curve of osculating order $r$, where $4 \leq r \leq 2m + s$ and $m \geq 2$. If $\gamma$ is biharmonic, then $fT \in \text{span} \{E_2, E_3, E_4\}$. Let $\theta(t)$ denote the angle function between $fT$ and $E_2$, that is, $g(fT, E_2) = \cos \theta(t)$. Differentiating $g(fT, E_2)$ along $\gamma$ and using (2.1), (2.3), (3.7), and (3.13), we find

$$- \theta'(t) \sin \theta(t) = \nabla_T g(fT, E_2) = g(\nabla_T fT, E_2) + g(fT, \nabla_T E_2)$$

$$= g(\sum_{\alpha=1}^{s} \xi_\alpha + \kappa_1 fE_2, E_2) + g(fT, -\kappa_1 T + \kappa_2 E_3)$$

$$= \kappa_2 g(fT, E_3).$$
If we write \( fT = g(fT, E_2)E_2 + g(fT, E_3)E_3 + g(fT, E_4)E_4 \), Theorem 3.1 gives us

\[
\begin{align*}
\kappa_1 &= \text{constant} > 0, \\
\kappa_1^2 + \kappa_3^2 &= \frac{c+3s}{4} + \frac{3(c-s)}{4} \cos^2 \theta, \\
\kappa_2^2 + \frac{3(c-s)}{4} \cos \theta g(fT, E_3) &= 0, \\
\kappa_2 \kappa_3 + \frac{3(c-s)}{4} \cos \theta g(fT, E_4) &= 0.
\end{align*}
\]

If we multiply the third equation of the above system with \( 2\kappa_2 \), using (3.16), we obtain

\[
2\kappa_2 \kappa_2' + \frac{3(c-s)}{4} \left(-2\theta' \cos \theta \sin \theta\right) = 0,
\]

which is equivalent to

\[
\kappa_2^2 = -\frac{3(c-s)}{4} \cos^2 \theta + \omega_0, \tag{3.17}
\]

where \( \omega_0 \) is a constant. If we write (3.17) in the second equation, we have

\[
\kappa_1^2 = \frac{c+3s}{4} + \frac{3(c-s)}{2} \cos^2 \theta + \omega_0.
\]

Thus, \( \theta \) is a constant. From (3.16) and (3.17), we find \( g(fT, E_4) = 0 \) and \( \kappa_2 = \text{constant} > 0 \). Since \( \|fT\| = 1 \) and \( fT = \cos \theta E_2 + g(fT, E_4)E_4 \), we get \( g(fT, E_4) = \sin \theta \). From the assumption \( g(fT, E_2) \) is not constant 0, 1, or \(-1\), it is clear that \( \theta \in (0, 2\pi) \setminus \left\{ \frac{\pi}{2}, \pi, \frac{3\pi}{2} \right\} \). Now we can state the following Theorem:

**Theorem 3.5** Let \( \gamma : I \to M \) be a Legendre curve of osculating order \( r \) in an \( S \)-space form \( (M^{2m+s}, f, \xi_\alpha, \eta^\alpha, g) \), \( \alpha \in \{1, \ldots, s\} \), where \( r \geq 4 \), \( m \geq 2 \), \( c \neq s \), \( g(fT, E_2) \) is not constant 0, 1, or \(-1\). Then \( \gamma \) is proper biharmonic if and only if

\[
\begin{align*}
\kappa_i &= \text{constant} > 0, \quad i \in \{1, 2, 3\}, \\
\kappa_1^2 + \kappa_2^2 &= \frac{1}{4} \left[ c+3s+3(c-s) \cos^2 \theta \right], \\
\kappa_2 \kappa_3 &= \frac{3(s-c) \sin 2\theta}{8},
\end{align*}
\]

where \( c > -3s \), \( fT = \cos \theta E_2 + \sin \theta E_4 \), \( \theta \in (0, 2\pi) \setminus \left\{ \frac{\pi}{2}, \pi, \frac{3\pi}{2} \right\} \) is a constant such that \( c+3s+3(c-s) \cos^2 \theta > 0 \), and \( 3(s-c) \sin 2\theta > 0 \). If \( c \leq -3s \), then \( \gamma \) is biharmonic if and only if it is a geodesic.

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