Two-weighted norm inequality on weighted Morrey spaces

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Abstract: Let $u$ and $\omega$ be weight functions. We shall introduce the weighted Morrey spaces $L^{p,\omega}(\omega)$ and investigate the sufficient condition and necessary condition about the 2-weighted boundedness of the Hardy–Littlewood maximal operator.

Key words: Weighted Morrey spaces, Hardy–Littlewood maximal operator, $A_p$ weights

1. Introduction

Suppose $u(x)$ and $\omega(x)$ are weight functions on $\mathbb{R}^n$, and $T$ is an operator taking suitable functions on $\mathbb{R}^n$. In his survey article [10], Muckenhoupt raised the general question of characterization when the weighted norm inequality

$$
\left( \int_{\mathbb{R}^n} |Tf(x)|^q \omega(x) dx \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p u(x) dx \right)^{\frac{1}{p}}
$$

holds for any $1 \leq p, q \leq \infty$ and all appropriate $f$. In the case of one weight $u = \omega$, the inequality (1.1) can be characterized by remarkably simple conditions for many classical operators, e.g., the Hardy–Littlewood maximal operator, singular integral, and fractional operator (see [1, 9, 11]).

The case of different weights has been far less studied. Only for the Hardy–Littlewood maximal operator and other positive operators was this characterized in [13], while many classical operators are still open and only find sufficient conditions on weights for an operator to be bounded from $L^p(u)$ to $L^q(\omega)$. For the history of these results, we refer the reader to [2, 3, 5].

Weighted Morrey spaces $L^{p,\omega}(\omega)$ were first introduced recently by Komori and Shirai [7], where the boundedness of many classical operators was established. Later, many authors found that the weighted Morrey spaces were also used in harmonic analysis [14, 15]. However, this only gives sufficient conditions for the boundedness of classical operators. The necessary condition associated with Hilbert transform in Morrey spaces was discussed by Samko [12].

In this paper, we concentrate our attention on the 2-weighted norm inequality associated with the Hardy–Littlewood maximal operator in weighted Morrey spaces. The same as the above cases, we only give a sufficient condition and a necessary condition, respectively.
Throughout the paper cubes are assumed to have their sides parallel to the coordinate axes. Given cube $Q = Q(x, r)$ centered at $x$ with side length $r$, $\omega(Q)$ denotes $\int_Q \omega(x)dx$ and the measure $\omega(x)dx$ is often abbreviated to $\omega dx$. The Lebesgue measure of $Q$ is denoted by $|Q|$ and the characteristic function of $Q$ by $\chi_Q$.

2. Some notations and lemmas

In this section, we introduce some basic definitions and lemmas.

**Definition 2.1** Let $1 < p < \infty$, $0 < \kappa < 1$, and $w$ be a weight function. For any local integrable function $f$ in $\mathbb{R}^n$, if it satisfies

$$\|f\|_{L^p,\omega} := \sup_Q \left( \frac{1}{\omega(Q)^\kappa} \int_Q |f(x)|^p w(x)dx \right)^{1/p} < \infty.$$ 

Then $f$ belongs to weighted Morrey spaces and $\| \cdot \|_{L^p,\omega}$ denotes the norm.

Note that if $\omega = 1$, $L^{p,\kappa}(\omega) = L^{p,\kappa}(\mathbb{R}^n)$ is the classical Morrey spaces; if $\kappa = 0$, $L^{p,0}(\omega) = L^p(\omega)$ is the weighted Lebesgue spaces.

**Definition 2.2** The Hardy–Littlewood maximal operator $M$ is defined by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

and we define the maximal operator with respect to the measure $w(x)dx$ by

$$M_\omega f(x) = \sup_{x \in Q} \frac{1}{\omega(Q)} \int_Q |f(y)|\omega(y)dy.$$

Before the next definition, we recall that a dyadic cube is the product of the intervals that are divided by dyadic decomposition of the coordinate axis with side length $2^k$, $k \in \mathbb{Z}$.

**Definition 2.3** Supposing that $\mathcal{F}$ is the collection of the dyadic cubes, we define $M^*_\ell f(x)$ with translation operator $\tau_t$ as follows (see [4], p. 112, or [6], p. 431):

$$M^*_\ell f(x) = (\tau_{-t} \circ M^* \circ \tau_t) f(x) = M^*(\tau_t f)(x + t).$$

In the definition, $M^* f(x)$ is a dyadic maximal operator (see [4], p. 111), which is defended by

$$M^* f(x) = \sup_{x \in Q \in \mathcal{F}} \frac{1}{|Q|} \int_Q |f(y)|dy.$$ 

The following definition was considered by Fefferman and Stein (see [4], p. 112):

**Definition 2.4** Let $\ell(Q)$ be the side length of a cube $Q$. For a positive real number $N$, we define the locally maximal operator by

$$M_N f(x) = \sup_{x \in Q \subseteq N} \frac{1}{|Q|} \int_Q |f(y)|dy.$$
and the locally dyadic maximal operator by

$$
\tilde{M}_N^\star f(x) = \sup_{x \in Q \subseteq \mathbb{R}^n} \frac{1}{|Q|} \int_Q |f(y)| dy.
$$

**Definition 2.5** A weight function $\omega$ satisfies the $A_p$ condition with $1 < p < \infty$, if there exists a constant $C \geq 1$ such that for any cube $Q$,

$$
\left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q (w(x))^{1-p'} dx \right)^{p-1} \leq C,
$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

The definition of 2.5 can be found in [8] on page 21. In fact, the $A_p$ weights have the following important lemma (see [8], p. 22):

**Lemma 2.1** Given a weight function $w \in A_p$, $1 < p < \infty$, it also satisfies the doubling condition $\Delta_2$: for any cube $Q$, there exists a constant $C > 0$ such that $w(2Q) \leq Cw(Q)$.

The next 2 definitions have a relation with the 2-weighted inequality in weighted Morrey spaces.

**Definition 2.6** A weight $\omega$ is called a $(p, \kappa)$-permission weight if for every cube $Q$, the inequality

$$
\|\chi_Q\|_{L^p(\omega)} < \infty
$$

holds. Furthermore, a weight $\omega$ is called a $(p, \kappa)$-specific permission weight if it is a $(p, \kappa)$-permission weight and for every cube $Q$

$$
\|\chi_Q\|_{L^p(\omega)} < \infty,
$$

where $\sigma = \omega^{1-p'}$.

**Definition 2.7** We say $(u, \omega) \in S_{p, \kappa}$ if $u$ is a $(p, \kappa)$-permission weight and $\omega$ is a $(p, \kappa)$-specific permission weight, such that the following inequalities hold:

$$
\sup_Q \frac{\|\chi_Q\|_{L^p(\omega)}}{\|\chi_Q\|_{L^{p, \kappa}(\omega)}} < \infty \quad \text{and} \quad \sup_Q \frac{\sigma(3Q)}{|Q|} \times \frac{\|\chi_Q\|_{L^p(\omega)}}{\|\chi_Q\|_{L^{p, \kappa}(\omega)}} < \infty.
$$

The following lemmas play an important role in our proofs.

**Lemma 2.2** Let $1 < p < \infty$ and $\omega \in A_p$; then there exists an index $r$: $1 < r < p$, such that $\omega \in A_r$.

This lemma was first obtained by Muckenhoupt in [9], page 214. One can also find a clear statement in [8], page 26.

**Lemma 2.3** Let $1 < p < \infty$. $\sigma$ is a nonnegative locally integrable weight. Then $M_\sigma^\star$ is bounded in $L^p(\sigma)$.

Lemma 2.10 would be found in [6], page 426. In fact, $M_\sigma^\star$ is of weak type $(1,1)$ and bounded in $L^\infty(\sigma)$. By using the Marcinkiewicz interpolation theorem we can get this result.
Lemma 2.4 Suppose \( f \) is a locally integrable function in \( \mathbb{R}^n \); then for every integer \( k \) and \( x \in \mathbb{R}^n \) we have
\[
M_{2^k}f(x) \leq 2^{1-kn} \int_{Q(0,2^{k+3})} M^*_f(x) dt,
\]
where \( Q(0,2^{k+3}) \) means the cube centered at 0 with side length \( 2^{k+3} \).

As to the proof of Lemma 2.11, we refer to [6], page 431. Note that the notation \( Q(0,2^{k+2}) \) in [6] means a cube centered at 0 with half side length \( 2^{k+2} \), which differs from our argument.

3. A sufficient condition of 2-weighted norm inequalities in weighted Morrey spaces

In this section we give a sufficient condition of 2-weighted boundedness of the Hardy–Littlewood maximal operator. The statement is the following theorem.

Theorem 3.1 Suppose \( 1 < p < \infty \), \( 0 < \kappa < 1 \), \((u, \omega)\) is a couple of weights, \( \sigma = \omega^{1-p'} \) and \( \omega \in A_p \). Then the Hardy–Littlewood maximal operator \( M \) is bounded from \( L^{p,\kappa}(\omega) \) to \( L^{p,\kappa}(u) \) if there exists a constant \( C > 0 \), such that for any cubes \( Q \) and \( Q' \)
\[
\frac{1}{u(Q)\kappa} \int_{Q'} M(\chi_{Q'} \sigma)(x)^p u dx \leq \frac{C}{\omega(Q)\kappa} \int_{Q'} \sigma dx < \infty.
\]

To prove Theorem 3.1, we need an auxiliary proposition as follows:

Proposition 3.1 Let \( 1 < p < \infty, 0 < \kappa < 1 \). If \((u, \omega)\) is a couple of weights and \( \sigma = \omega^{1-p'} \) is locally integrable, then the following statements are equivalent:

(a) There exists a constant \( C > 0 \), such that for any cube \( Q \)
\[
\frac{1}{u(Q)\kappa} \int_{\mathbb{R}^n} (Mf(x))^p u dx \leq \frac{C}{\omega(Q)\kappa} \int_{\mathbb{R}^n} |f(x)|^p \omega dx;
\]

(b) There exists a constant \( C > 0 \), such that for any cube \( Q \) and \( Q' \)
\[
\frac{1}{u(Q)\kappa} \int_{Q'} (M(\chi_{Q'} \sigma)(x))^p u dx \leq \frac{C}{\omega(Q)\kappa} \int_{Q'} \sigma dx < \infty.
\]

Proof The idea follows from [13] and [6]. Once having chosen \( f = \sigma(x)\chi_Q(x) \), we can easily draw the conclusion \((a) \Rightarrow (b)\). To verify the opposite, we partition it into 3 steps. First, it suffices to prove the result for the dyadic maximal operator \( M^* \); second, by using the first step, we show the result for the translation dyadic maximal operator \( M^*_t \); and, third, by using Lemma 2.11, we complete the proof for the maximal operator \( M \).

We first check the case of the dyadic maximal operator. Since \((b)\) is satisfied by \( M^* \), let us consider the locally dyadic maximal operator \( M^*_N \). Recall the definition of \( M^*_N f(x) \): it takes the supremum over all dyadic cubes \( Q \) that contain \( x \) with side length of less than \( N \); therefore, under the condition \( M^*_N f(x) > 2^k \), \( k \in \mathbb{Z}, x \in \mathbb{R}^n \), we get a family of countable such dyadic cubes \( \{Q^k_i\} \) satisfying
\[
2^k < \frac{1}{|Q^k_i|} \int_{Q^k_i} |f(y)| dy. \tag{3.1}
\]

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For any 2 dyadic cubes, either 1 is contained in the other or they are disjoint. Hence, we can choose the maximum ones in the family \( \{ Q^k_l \}_l \). The collection of these maximum dyadic cubes is denoted by \( \{ Q^k_j \}_j \). They satisfy the same inequality as in (3.1). Moreover, for any dyadic cube \( Q \supsetneq Q^k_j \) with side length \( \ell(Q) \leq N \), we have

\[
\frac{1}{|Q|} \int_Q |f(y)| dy \leq 2^k.
\]

Obviously

\[
\{ x \in \mathbb{R}^n | M_N^\ast f(x) > 2^k \} = \bigcup_j Q^k_j.
\]

Let \( E^k_j = Q^k_j \setminus \{ x \in \mathbb{R}^n | M_N^\ast f(x) > 2^{k+1} \} \). For \( k_1 \neq k_2 \) or \( j_1 \neq j_2 \), it is easy to check that \( E^k_{j_1} \) and \( E^k_{j_2} \) are disjoint and

\[
\bigcup_{j,k} E^k_j = \bigcup_{j,k} Q^k_j.
\]

Hence, for any cube \( Q \), we have

\[
\frac{1}{u(Q)^\kappa} \int_{\mathbb{R}^n} (M_N^\ast f(x))^p u dx = \frac{1}{u(Q)^\kappa} \sum_{j,k} \int_{E^k_j} (M_N^\ast f(x))^p u dx
\]

\[
\leq \frac{2^p}{u(Q)^\kappa} \sum_{j,k} u(E^k_j) \left( \frac{1}{|Q^k_j|} \int_{Q^k_j} \sigma dx \right)^p \left( \frac{1}{\sigma(Q^k_j)} \int_{Q^k_j} |f(x)| \omega^\sigma \sigma dx \right)^p
\]

\[
= \frac{C}{u(Q)^\kappa} \sum_{j,k} \gamma^k_j \left( \frac{1}{\sigma(Q^k_j)} \int_{Q^k_j} g(x) \sigma dx \right)^p,
\]

where \( g = |f| \omega^\sigma \) and

\[
\gamma^k_j = u(E^k_j) \left( \frac{1}{|Q^k_j|} \int_{Q^k_j} \sigma dx \right)^p.
\]

Next we define the measure \( \gamma \) on the measure space \( \mathcal{M} \) where \( \mathcal{M} = \mathbb{Z} \times \mathbb{Z}_+ \). Let \( \mathcal{M}_0 = \{(k,j) \in \mathcal{M} | \ k, j \ \text{is} \ \text{the} \ \text{index} \ \text{of} \ Q^k_j \} \), and

\[
\bar{g}(k,j) = \begin{cases} 
\left( \frac{1}{\sigma(Q^k_j)} \int_{Q^k_j} g(x) \sigma dx \right)^p, & (k,j) \in \mathcal{M}_0 \\
0, & \text{otherwise}.
\end{cases}
\]

Then we have

\[
\frac{C}{u(Q)^\kappa} \sum_{j,k} \gamma^k_j \left( \frac{1}{\sigma(Q^k_j)} \int_{Q^k_j} g(x) \sigma dx \right)^p
\]

\[
= \frac{C}{u(Q)^\kappa} \int_{\mathcal{M}} \bar{g}(k,j) d\gamma
\]

\[
= C \int_0^\infty \frac{\gamma(S_\lambda)}{u(Q)^\kappa} d\lambda,
\]

(3.3)
where

\[ S_\lambda = \left\{ (k, j) \in \mathcal{M}_0 \mid \left( \frac{1}{\sigma(Q^k_j)} \int_{Q^k_j} g(x)\sigma dx \right)^p > \lambda \right\}. \]

Note that all the cubes in \( \{ Q^k_j \}_{j,k} \) have side length of at most \( N \). For the same reason, we can choose maximum dyadic cubes in \( \{ Q^k_j : (k, j) \in S_\lambda \} \). These maximum dyadic cubes are relabeled by \( \{ Q_i^\lambda \} \). Thus:

\[ \bigcup_i Q_i^\lambda \subseteq \{ x \in \mathbb{R}^n | (M^*_\sigma g(x))^p > \lambda \}. \]

Joining (3.2) and (3.3) and by using condition (b) and Lemma 2.10, we have

\[
\frac{1}{u(Q)^\kappa} \int_{\mathbb{R}^n} (M^*_\sigma f(x))^p udx \leq C \int_0^\infty \frac{\gamma(S_\lambda)}{u(Q)^\kappa} d\lambda \\
= C \int_0^\infty \frac{1}{u(Q)^\kappa} \sum_i \sum_{(k,j) \in \mathcal{M}_0} \sigma(E^\lambda_i) \left( \frac{1}{|Q^\lambda_i|} \int_{Q^\lambda_i} \sigma \right)^p d\lambda \\
\leq C \int_0^\infty \left( \sum_i \frac{1}{u(Q)^\kappa} \int_{Q^\lambda_i} (M^*(\chi_{Q^\lambda_i}\sigma)(x))^p udx \right) d\lambda \\
\leq C \int_0^\infty \frac{1}{\omega(Q)^\kappa} \sum_i \sigma(Q_i^\lambda) d\lambda \\
\leq \frac{C}{\omega(Q)^\kappa} \int_0^\infty \sigma(\{ x \in \mathbb{R}^n | (M^*_\sigma g(x))^p > \lambda \}) d\lambda \\
= \frac{C}{\omega(Q)^\kappa} \int_{\mathbb{R}^n} \left( M^*_\sigma \left( \frac{f}{\sigma} \right)(x) \right)^p \sigma dx \\
\leq \frac{C}{\omega(Q)^\kappa} \int_{\mathbb{R}^n} |f(x)|^p \omega dx.
\]

Letting \( N \) tend to \( \infty \), we get

\[
\frac{1}{u(Q)^\kappa} \int_{\mathbb{R}^n} (M^* f(x))^p udx \leq \frac{C}{\omega(Q)^\kappa} \int_{\mathbb{R}^n} |f(x)|^p \omega dx.
\]

Now we prove the case of maximal operator \( M \). Note that \( (\tau_t u, \tau_t \omega) \) is also a couple of weights and \( \tau_t u(Q) = u(Q - t) \). Then for 2 arbitrary cubes \( Q \) and \( Q' \), we have
\[
\frac{1}{\tau_t u(Q)} \int_{Q^t} (M^*((\tau_t \sigma)\chi_{Q^t}))(x)^p \tau_t u(x)dx 
\]
\[
= \frac{1}{\tau_t u(Q)} \int_{Q^t} (M^*((\tau_t \sigma \chi_{Q^t}^{-1}))(x)^p u(x-t)dx 
\]
\[
= \frac{1}{\tau_t u(Q)} \int_{Q^t} (M^*\sigma(\chi_{Q^t}^{-1}))(y)^p u(y)dy 
\]
\[
\leq \frac{1}{\tau_t u(Q)} \int_{Q^t} (M\sigma(\chi_{Q^t}^{-1}))(x)^p u(x)dx 
\]
\[
= \frac{1}{\tau_t u(Q)} \int_{Q^t} (M(\tau_t \sigma\chi_{Q^t}))(x)^p \tau_t u(x)dx 
\]
\[
\leq \frac{C}{\tau_t \omega(Q)} \int_{Q^t} \tau_t dx.
\]

Hence:

\[
\frac{1}{u(Q)} \int_{\mathbb{R}^n} (M^*_t f(x))^p u(x)dx 
\]
\[
= \frac{1}{\tau_t u(Q + t)} \int_{\mathbb{R}^n} (M^*(\tau_t f)(x))^p (\tau_t u)dx 
\]
\[
\leq \frac{C}{\tau_t \omega(Q + t)} \int_{\mathbb{R}^n} |\tau_t f(x)|^p \tau_t \omega dx 
\]
\[
= \frac{C}{\omega(Q)} \int_{\mathbb{R}^n} |f(x)|^p \omega dx.
\]

Using Lemma 2.11, for each \( k \in \mathbb{Z} \), we have

\[
\left( \frac{1}{u(Q)^c} \int_{\mathbb{R}^n} (M_{2k} f(x))^p u dx \right)^{\frac{1}{p}} 
\]
\[
\leq 2^{1-\frac{kn}{p}} \left( \int_{\mathbb{R}^n} \left( \int_{Q_{0,2k+1}} M^* f(x) dt \right)^p u dx \right)^{\frac{1}{p}} 
\]
\[
\leq 2^{1-\frac{kn}{p}} \int_{Q_{0,2k+1}} \left( \frac{1}{u(Q)^c} \int_{\mathbb{R}^n} (M^* f(x))^p u dx \right)^{\frac{1}{p}} dt 
\]
\[
\leq C \left( \frac{1}{\omega(Q)} \int_{\mathbb{R}^n} |f(x)|^p \omega dx \right)^{\frac{1}{p}}.
\]

Letting \( k \) tend to \( \infty \), we get

\[
\frac{1}{u(Q)^c} \int_{\mathbb{R}^n} (M f(x))^p u dx \leq \frac{C}{\omega(Q)} \int_{\mathbb{R}^n} |f(x)|^p \omega dx.
\]

This completes the proof. \( \square \)
Next we shall prove Theorem 3.1.

**Proof** Suppose \( f = f \chi_{3Q} + f \chi_{(3Q)^c} \equiv f_1 + f_2 \). Since \( \omega \in A_p \), \( \sigma \) is locally integrable, by Proposition 3.2:

\[
\left( \frac{1}{\omega(Q)^\alpha} \int_Q (Mf_1(x))^p \omega dx \right)^{\frac{1}{p}} \leq \frac{C}{\omega(Q)^\alpha} \int_{3Q} |f(x)|^p \omega dx \right)^{\frac{1}{p}} \\
\leq C\|f\|_{L^p,\omega}.
\]

On the other hand, from [7] we know that, for every \( x \in Q \),

\[
M_\omega f_2(x) \leq \sup_{R:Q \subseteq 3R} \left( \frac{1}{\omega(R)} \int_R |f(x)|^p \omega dx \right)^{\frac{1}{p}}.
\]

Noting that \( \omega \in A_p \), by Lemma 2.9, there exists an index \( r : 1 < r < p \), such that \( \omega \in A_r \), and then \( Mf_2(x) \leq C(M_\omega |f_2|^r(x))^{\frac{1}{r}} \). By inequality (3.4), for every \( x \in Q \), we have

\[
Mf_2(x) \leq C(M_\omega |f_2|^r(x))^{\frac{1}{r}} \\
\leq C \sup_{R:Q \subseteq 3R} \left( \frac{1}{\omega(R)} \right)^{\frac{1}{r}} \int_R |f(x)|^p \omega dx \right)^{\frac{1}{p}} \\
\leq C \sup_{R:Q \subseteq 3R} \left( \frac{1}{\omega(R)} \right)^{\frac{1}{r}} \int_R |f(y)|^p \omega dy \right)^{\frac{1}{p}} \omega(R)^{\frac{r-1}{p}} \\
\leq C\|f\|_{L^p,\omega} \omega(Q)^{\frac{r-1}{p}}.
\]

Hence:

\[
\left( \frac{1}{u(Q)^\alpha} \int_Q (Mf_2(x))^p \omega dx \right)^{\frac{1}{p}} \leq C\omega(Q)^{\frac{r-1}{p}} \|f\|_{L^p,\omega}.
\]

Using Proposition 3.2 again, let \( f = \chi_Q \); then for every \( x \in Q^c \), \( M(\chi_Q)(x) \equiv 1 \). We have

\[
u(Q)^{\frac{1}{p}} = \left( \frac{1}{\nu(Q)^\alpha} \int_Q (M(\chi_Q)(x))^p \omega dx \right)^{\frac{1}{p}} \leq C\omega(Q)^{\frac{1}{p}},
\]

and then

\[
\left( \frac{1}{\nu(Q)^\alpha} \int_Q (Mf_2(x))^p \omega dx \right)^{\frac{1}{p}} \leq C\|f\|_{L^p,\omega}.
\]

Therefore, we complete the proof of Theorem 3.1.

\[\square\]

4. A necessary condition of 2-weighted norm inequalities in weighted Morrey spaces

In this section we give a necessary condition of 2-weighted boundedness of the Hardy–Littlewood maximal operator. The idea goes back to Samko [12].
Theorem 4.1 If \( u, \omega, \) and \( \sigma = \omega^{1-p'} \) are respectively \((p, \kappa)\)-permission weight, \((p, \kappa)\)-specific permission weight, and a doubling weight, then \((u, \omega) \in \mathcal{S}_{p, \kappa} \) is the necessary condition of \( \|Mf\|_{L^{p, \kappa}(u)} \leq C\|f\|_{L^{p, \kappa}(\omega)} \).

Proof Suppose \( Q_1, Q_2, \ldots, Q_{2^n} \) are any neighboring cubes that have the same edge length but no intersecting interior whose union is a new big cube, which is denoted by \( Q_0 \). Let \( x \in Q_i, i \in \{1, 2, \ldots, 2^n\} \). Then for \( j \neq i \):

\[
M(\chi_{Q_j})(x) = \sup_{x \in Q} \left( \frac{1}{|Q|} \int_Q \chi_{Q_j}(y) dy \right) \geq \frac{1}{|Q|} \int_{Q_j} dy = \frac{1}{2^n}.
\]

Hence:

\[
\sup_Q \left( \frac{1}{u(Q)^c} \int_Q \chi_{Q_j}(y) ud\gamma \right) \leq 2^{np} \sup_Q \left( \frac{1}{u(Q)^c} \int_{Q \cap Q_i} (M(\chi_{Q_j})(y))^p ud\gamma \right)
\]

\[
\leq 2^{np} C^p \|\chi_{Q_j}\|_{L^{p, \kappa}(\omega)}^p.
\]

Note that \( Q_j \subseteq 3Q_i \),

\[
\|\chi_{Q_j}\|_{L^{p, \kappa}(u)} \leq 2^n C \|\chi_{Q_j}\|_{L^{p, \kappa}(\omega)} \leq C \|3Q_i\|_{L^{p, \kappa}(\omega)}.
\]

On the other hand, for every \( x \in Q_j \), we have

\[
M(\chi_{Q_j}, \sigma)(x) = \sup_{x \in Q_j} \frac{1}{|Q_j|} \int_{Q \cap Q_j} \sigma dx \geq \frac{1}{|Q_j|} \int_{Q_j} \frac{1}{2^n |Q_j|} \int_{Q_j} \sigma dx.
\]

Then

\[
\left( \frac{1}{|Q_j|} \int_{Q_j} \sigma dx \right)^p \sup_Q \frac{1}{u(Q)^c} \int_{Q \cap Q_j} ud\gamma
\]

\[
\leq 2^{np} \sup_Q \frac{1}{u(Q)^c} \int_{Q \cap Q_j} (M(\chi_{Q_j}, \sigma)(y))^p ud\gamma
\]

\[
\leq C \|\chi_{Q_j}, \sigma\|_{L^{p, \kappa}(\omega)}^p.
\]

Since \( \sigma \) is a doubling weight and \( 3Q_i \subseteq 5Q_j \), we have \( \sigma(3Q_i) \leq \sigma(5Q_j) \leq C \sigma(Q_j) \) and

\[
\frac{\|\chi_{Q_j}\|_{L^{p, \kappa}(u)}}{\|\chi_{3Q_i}, \sigma\|_{L^{p, \kappa}(\omega)}} \leq C \frac{|3Q_i|}{\sigma(3Q_i)} \leq C \frac{|Q_i|}{\sigma(3Q_i)}.
\]

This completes the proof. \( \square \)

References


