Gonality of curves with a singular model on an elliptic quadric surface

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Abstract: Let \( W \subset \mathbb{P}^3 \) be a smooth quadric surface defined over a perfect field \( K \) and with no line defined over \( K \) (e.g., an elliptic quadric surface over a finite field). In this note we study the gonality over \( K \) of smooth curves with a singular model contained in \( W \) and with mild singularities.

Key words: Gonality, curve over a perfect field, \( K \)-gonality, elliptic quadric surface

1. Introduction
Let \( K \) be a perfect field such that there is a degree 2 extension \( L \) of \( K \). Let \( f(x_0, x_1) \in K[x_0, x_1] \) denote any degree 2 homogeneous polynomial such that \( L = K(\alpha) \) with \( \alpha \) a root of \( f(1, t) \), i.e. take as \( f \) any degree 2 homogeneous polynomial that is irreducible over \( K \) but reducible over \( L \). The main examples are the case \( K = \mathbb{R}, L = \mathbb{C} \) and the case \( K = \mathbb{F}_q \) and \( L = \mathbb{F}_q^2 \). Take homogeneous coordinates \( x_0, x_1, x_2, x_3 \) of \( \mathbb{P}^3 \) (over \( K \) and hence over \( K') \). Let \( W \subset \mathbb{P}^3 \) denote the smooth quadric surface with \( x_2x_3 + f(x_0, x_1) \) as its equation. If \( K = \mathbb{R} \), then these types of surfaces are just ellipsoids. If \( K = \mathbb{F}_q \), then \( W \) is an elliptic quadric surface [4]. In this paper we study the \( K \)-gonality of smooth curves \( C \) either contained in \( W \) or with a singular model \( Y \subset W \), but with a small number of singularities. We prove the following result.

Corollary 1 Let \( Y \subset W \) be a geometrically integral curve defined over \( K \) and let \( u : C \to Y \) be the normalization of \( Y \). Let \( a > 0 \) be the positive integer such that \( Y \in |O_W(a)| \). Assume that \( Y(\overline{K}) \) has only ordinary nodes and ordinary cusps as singularities and set \( J := \text{Sing}(Y(\overline{K})) \). Assume \( \sharp(J) \leq a - 5 \) and that no line of \( W(\overline{K}) \) contains at least 2 points of \( J \). Let \( R \in \text{Pic}^0(C)(K) \) be a spanned line bundle on \( C \) defined over \( K \) and with minimal positive degree. Then \( 2a - 4 \leq y \leq 2a \) and \( R \) is induced by a subseries of \( |O_W(1)| \).

We have \( y = 2a - 4 \) if and only if there is a degree 2 extension \( K' \) of \( K \) such that \( \sharp(J(K')) \geq 2 \).

We have \( y = 2a \) if and only if \( Y(K') = \emptyset \) for each degree 2 extension \( K' \) of \( K \).

See Theorem 1 for spelling out the possible cases of \( y \). For the foundational results on the gonality of curves over algebraically closed fields, see [8], [5], [9].

Since we work in arbitrary characteristic we cannot use some of the strongest tools in the literature. In our opinion in characteristic zero the best results are still obtained using [7] or the case \( e = 0 \) of [10] and [6].
Remark 2 on page 351. To get Corollary 1 and related results we need first to work over an algebraically closed field \( \mathbb{K} \) and study low degree linear series on smooth models of singular curves on a smooth quadric surface \( Q \) (see section 2). As stressed above, in characteristic zero stronger tools are available.

We discuss our method and possible improvements in Subsection 2.1.

Many thanks are due to a referee who improved the exposition.

2. Over an algebraically closed field \( \mathbb{K} \)

Let \( Q \subset \mathbb{P}^3 \) be a smooth quadric surface defined over an algebraically closed field \( \mathbb{K} \). For any coherent sheaf \( \mathcal{F} \) on \( Q \) and any integer \( i \geq 0 \) set \( H^i(\mathcal{F}) := H^i(Q, \mathcal{F}) \) and \( h^i(\mathcal{F}) := \dim(H^i(\mathcal{F})) \). For all \((a, b) \in \mathbb{Z}^2\) let \( \mathcal{O}_Q(a, b) \) denote the line bundle on \( Q \) with bidegree \((a, b)\). We have \( h^0(\mathcal{O}_Q(a, b)) = (a+1)(b+1) \) and \( h^1(\mathcal{O}_Q(a, b)) = 0 \) if \( a \geq 0 \) and \( b \geq 0 \), while \( h^0(\mathcal{O}_Q(a, b)) = 0 \) if either \( a < 0 \) or \( b < 0 \). If \( a \geq 0 \), \( b \geq 0 \) and \( T \in |\mathcal{O}_Q(a, b)| \), then we say that \( T \) has type \((a, b)\). The lines contained in \( Q \) are the curves \( D \subset Q \) with either type \((1, 0)\) or type \((0, 1)\). For any zero-dimensional scheme \( Z \subset Q \) and any \( T \in |\mathcal{O}_Q(u, v)| \), let \( \text{Res}_T(Z) \) denote the residual scheme of \( Z \) with respect to \( T \), i.e. the closed subscheme of \( Q \) with \( \mathcal{I}_Z : \mathcal{I}_T \) as its ideal sheaf. We have \( \text{Res}_T(Z) \subset Z \), \( \text{deg}(Z) = \text{deg}(\text{Res}_T(Z)) + \text{deg}(Z \cap T) \) and for all \((a, b) \in \mathbb{Z}^2\) we have an exact sequence (often called the residual exact sequence)

\[
0 \to \mathcal{I}_{\text{Res}_T(Z)}(a - u, b - v) \to \mathcal{I}_Z(a, b) \to \mathcal{I}_{Z \cap T, T}(a, b) \to 0
\]

(1)

2.1. Outline of the proof and of possible improvements

Take an integral curve \( Y \subset Q \) with bidegree \((a, a)\). Let \( u : C \to Y \) be the normalization map and \( w : C \to Q \) the composition of \( u \) with the inclusion \( Y \to Q \). Let \( \mathcal{J} \subset \mathcal{O}_Q \) be the conductor of \( w \) and \( J \subset Q \) the zero-dimensional subscheme of \( Q \) with \( \mathcal{J} \) as its ideal sheaf. Let \( J_{\text{red}} \) be the support of \( J \). We assume for instance \( \text{deg}(J) \leq a - 5 \). Let \( \mathcal{F} \) be the set of all irreducible \( E \in |\mathcal{O}_Q(1, 1)| \) such that \( 1 \leq \sharp(E \cap J_{\text{red}}) \leq 2 \). Let \( \mathcal{G} \) be the set of all irreducible \( E \in |\mathcal{O}_Q(1, 1)| \) such that \( \sharp(E \cap J_{\text{red}}) \geq 3 \). Let \( \mathcal{H} \) be the set of all reducible \( E \in |\mathcal{O}_Q(1, 1)| \) such that each component of \( E \) meets \( J_{\text{red}} \). Take \( B \) as in the proof of Lemma 5. Since \( \mathcal{G} \cup \mathcal{H} \) is finite, while \( B \) is general, we have \( E \cap B = \emptyset \) for all \( E \in (\mathcal{G} \cup \mathcal{H}) \). To apply Lemmas 1 and 2 to the scheme \( Z = J \cup B \) it is sufficient to assume \( \text{deg}(J \cap E) + y \leq 2a - 5 \) for all \( E \in |\mathcal{O}_Q(1, 1)| \). With this assumption steps (ii), (iii), (iv) of the proof of Lemma 5 carry over, because \( \text{deg}(J \cap E) \leq 2a - 5 - y \) for all \( E \in \mathcal{F} \) and \( \text{deg}(D \cap B) \leq 2 \) if \( D \in |\mathcal{O}_Q(1, 1)| \) is reducible and \( b_1 = b_2 = 1 \). Step (i) of the proof of Lemma 5 requires the following modifications for arbitrary singularities. For each \( P \in J_{\text{red}} \) let \( u_P \) be the degree of the effective divisor \( w^{-1}(P) \subset C \). For each connected degree 2 zero-dimensional scheme \( Z \subset Q \) whose support is a point \( P \in J_{\text{red}} \) let \( u_{Z, P} \) be the degree of the effective divisor \( w^{-1}(Z) \subset C \). We say that \( Y \) has either an ordinary node or an ordinary cusp at \( P \) if \( u_P = 2 \) and for each connected degree 2 scheme \( Z \subset Q \) with \( P \) as its support either \( u_{Z, P} = 3 \) (if and only if in the plane \( T_PQ \) the line through \( Z \) is in the tangent cone of \( Y \) at \( P \)) or \( u_{Z, P} = 2 \). In the description of step (i) of the proof of Lemma 5 we use the integers \( u_P \) (with \( u_P = 2 \) for double points) and \( u_{Z, P} \) (which are 2 or 3 for ordinary nodes and cusps with 3 if and only if \( Z \) corresponds to a branch of \( Y \) at \( P \). See for instance [1], [2], [3] for the formal theory of plane and space curves.

Now assume \( Y \subset W \) and that \( Y \) is defined over \( K \). To extend Theorem 1 one needs to know the integers \( u_P, P \in J_{\text{red}}(K') \) for any degree 2 extension \( K' \) of \( K \) and the integers \( u_{Z, P} \) with \( P \in J_{\text{red}}(K) \) and \( Z \) defined

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over $K$. The tools work for all spanned $R \in \text{Pic}^h(C)(K)$ with $\deg(J) + y \leq 3a - 5$, without assuming that $y$ is the $K$-gonality of $C$.

2.2. Proofs over $K$

Lemma 1  Fix an integer $c \geq 2$ and a zero-dimensional scheme $Z \subset Q$. Assume $\deg(Z \cap L) \leq 1$ for each line $L \subset Q$, $h^1(I_Z(c,e)) > 0$ and $\deg(Z) \leq 3c + 1$. Then there is an integral $D \in |\mathcal{O}_Q(1,1)|$ such that $\deg(D \cap Z) \geq 2c + 2$.

Proof  Set $Z_0 := Z$. Let $T_1 \subset Q$ be any element of $|\mathcal{O}_Q(1,1)|$ such that $e_1 := \deg(T_1 \cap Z)$ is maximal. Set $Z_1 := \text{Res}_{T_1}(Z_0)$. For each integer $i \geq 2$ define recursively the integer $e_i$, the curve $T_i \in |\mathcal{O}_Q(1,1)|$, and the scheme $Z_i \subset Z_{i-1}$ in the following way. Let $T_i \subset Q$ be any element of $|\mathcal{O}_Q(1,1)|$ such that $e_i := \deg(T_i \cap Z_{i-1})$ is maximal. Set $Z_i := \text{Res}_{T_i}(Z_{i-1})$. The sequence $\{e_i\}_{i \geq 1}$ is nonincreasing. Since $h^0(\mathcal{O}_Q(1,1)) = 4$, we have $e_{i+1} = 0$ and $Z_i = \emptyset$ if $e_i \leq 2$. Since $\deg(Z \cap L) \leq 1$ for each line $L \subset Q$, we may take $T_i$ as above and with the additional restriction that each $T_i$ is irreducible. Since $\deg(Z) \leq 3c + 1$, we get $e_{c+1} \leq 1$ and $Z_{c+1} = \emptyset$. From (1) for each $i \in \{1, \ldots, c\}$ we get the exact sequences

$$0 \to I_{Z_i}(c-i,c-i) \to I_{Z_{i-1}}(c-i+1,c-i+1) \to I_{Z_{i-1},T_i}(c-i+1,c-i+1) \to 0$$

(2)

Since $\deg(Z_c) \leq 1$, we have $h^1(I_{Z_c}) = 0$. Since $h^1(I_Z(c,e)) > 0$, we get the existence of an integer $i \in \{1, \ldots, c\}$ such that $h^1(T_i, I_{Z_i}, T_i(c-i+1,c-i+1)) > 0$. Let $f$ be the minimal such integer. Since $T_f$ is irreducible, we have $T_f \cong \mathbb{P}^1$. Since $\deg(\mathcal{O}_{T_f}(c-f+1,c-f+1)) = 2c-2f+2$, we have $h^1(T_f, I_{Z_{i-1},T_i}(c-f+1,c-f+1)) > 0$ if and only if $e_f \geq 2c-2f+4$. If $f = 1$, then we may take $D := T_1$. Now assume $f \geq 2$. Since $e_i \geq e_f$ for all $i < f$, we get $\deg(Z) \geq 2f(c-f+2)$. The function $\psi(t) := 2t(c+2-t)$ is increasing in the interval $[2,(c+2)/2]$ and decreasing for $t > (c+2)/2$. Since $\psi(2) = \psi(c) = 4c$, we get $\deg(Z) \geq 4c$, a contradiction. □

Lemma 2  Fix integers $k \geq c \geq 0$ and a zero-dimensional scheme $Z \subset Q$ such that $\deg(Z) \leq k + c + 1$ and $\deg(Z \cap L) \leq 1$ for each line $L \subset Q$. Then $h^1(I_Z(k,c)) = 0$.

Proof  If $c = 0$, then one may use $k-c$ residual exact sequences, each time with respect to some $L \in |\mathcal{O}_Q(1,0)|$. If $k = c = 1$, then the lemma is obvious. If $k = c \geq 2$, then we may apply Lemma 1. Now assume $k > c > 0$. By the case $c = 0$ we may assume $\deg(Z) \geq k - c$. Since $h^0(Q, \mathcal{O}_Q(k-c,0)) = k - c + 1$, there is $F \in |\mathcal{O}_Q(k-c,0)|$ such that $\deg(F \cap Z) \geq k - c$. Since $\deg(L \cap Z) \leq 1$ for each $L \in |\mathcal{O}_Q(1,0)|$, we have $\deg(F \cap Z) = k - c$. Hence $\deg(\text{Res}_F(Z)) = \deg(Z) - k + c \leq 2c + 1$. Lemma 1 gives $h^1(I_{\text{Res}_F(Z)}(c,c)) = 0$. We saw that $h^1(I_{F \cap Z}(k,0)) = 0$ and hence $h^1(I_{F \cap Z}(k,c)) = 0$. Therefore $h^1(F, I_{F \cap Z}(k,c)) = 0$. A residual exact sequence gives $h^1(I_Z(k,c)) = 0$. □

Lemma 3  Let $T \subset Q$ be an integral element of $|\mathcal{O}_Q(a,a)|$ and $u : C \to T$ its normalization. Let $J \subset \mathcal{O}_Q$ be the conductor of $u$ and $I \subset Q$ the closed subscheme with $J$ as its ideal sheaf. Fix integers $x \in \{0, \ldots, a-2\}$ and $y \in \{0, \ldots, a-2\}$. We have $h^0(C, u^*(\mathcal{O}_T(x,y))) = (x+1)(y+1)$ if and only if $h^1(I_J(a-2-x,b-2-y)) = 0$.

Proof  Since $a > x$, $a > y$ and $T$ has type $(a,a)$, we have $h^0(I_T(x,y)) = 0$. Since $h^1(Q, \mathcal{O}_Q(a-x,b-y)) = 0$, the exact sequence (1) for $Z = \emptyset$ gives $h^0(T, \mathcal{O}_T(x,y)) = (x+1)(y+1)$. Hence we have
Proof

Lemma 4 Fix positive integers \( c, b_1, b_2 \) such that \( \max\{b_1, b_2\} \leq c + 1 \). Fix a zero-dimensional scheme \( J \subset Q \) and a finite set \( B \subset Q \) such that \( B \cap J = \emptyset \), \( \deg(J \cap I) \leq 1 \) for every line \( I \subset Q \), no line of \( Q \) intersects both \( J \) and \( B \), either \( I \cap B = \emptyset \) or \( I \cap B = b_1 \) for each \( I \in |O_Q(1, 0)| \) and either \( I \cap B = \emptyset \) or \( I \cap B = b_2 \) for each \( I \in |O_Q(0, 1)| \). Assume \( h^1(I_{J \cup B}(c, c)) > 0 \).

(a) If \( b_1 = b_2 = 1 \) and \( \deg(J \cup B) \leq 3c + 1 \), then there is an integral \( D \in |O_Q(1, 1)| \) such that \( \sharp(D \cap (J \cup B)) \geq 2c + 2 \).

(b) If \( \delta := \max\{b_1, b_2\} \geq 2 \), then \( \deg(J) \geq 2c + 2 - \sharp(B)/\delta \).

Proof

Set \( Z = J \cup B \). The case \( b_1 = b_2 = 1 \) is true by Lemma 1. Hence we may assume \( b_1 \geq 2 \). We have \( \sharp(B) = xb_1 = yb_2 \) for some positive integers \( x, y \). Without losing generality we may assume \( b_1 \geq b_2 \). Let \( F \in |O_Q(x, 0)| \) be the union of all lines containing at least one point of \( B \). By assumption \( F \cap J = \emptyset \). Since \( \sharp(B \cap I) = b_1 \leq c + 1 \) for each component \( I \) of \( F \), we have \( h^1(F, I_{Z \cap F}(c, c)) = 0 \). Hence the exact sequence

\[
0 \to I_J(c - x, c) \to I_Z(c, c) \to I_{F \cap Z, F}(c, c) \to 0
\]
gives \( h^1(I_J(c - x, c)) > 0 \). Lemma 2 gives \( \deg(J) \geq 2c - x + 2 \). 

Remark 1 In the next lemma the integers \( b_1 \) and \( b_2 \) are positive integers dividing \( y \) (they may be 1). In the applications to \( W \) (Corollary 1 and Theorem 1) \( b_1 = b_2 \) and \( b_1 \) divides \( a \). Hence when one needs to apply Lemma 5 to curves in \( W \) there is a very small number of possible pairs \( (b_1, b_2) \neq (1, 1) \).

Lemma 5 Let \( T \subset Q \) be an integral element of \( |O_Q(a, a')| \), \( a' \geq a \geq 2 \), and \( u : C \to T \) its normalization. Let \( w : C \to Q \) be the composition of \( u \) with the inclusion \( T \to Q \). Assume that \( T \) has only ordinary nodes and ordinary cusps as singularities and set \( J := \text{Sing}(T) \). Assume \( \deg(J \cap L) \leq 1 \) for each line \( L \subset Q \). Fix \( R \in \text{Pic}^R(C) \), \( y > 0 \), such that \( R \) has no base points and \( R \) is neither \( u^*(O_C(1, 0)) \) nor \( u^*(O_C(0, 1)) \).

Let \( h : C \to \mathbb{P}^1 \) be the morphism associated to a general 2-dimensional linear subspace of \( H^0(C, R) \). Let \( u_1 : C \to \mathbb{P}^1 \) and \( u_2 : C \to \mathbb{P}^1 \) be the morphisms associated to the 2 projections \( Q \to \mathbb{P}^1 \). Let \( b_i \) be the degree of the morphism \( (h, u_i) \).
(a) Assume $b_1 = b_2 = 1$ and $y + \sharp(J) \leq 2a + a' - 5$. There is a zero-dimensional scheme $\Gamma \subset Q$ with $0 \leq \deg(\Gamma) \leq 2$ such that $h^0(R) = 4 - \deg(\Gamma)$ and $R$ is induced by the linear system $|\mathcal{I}_{\Gamma}(1,1)|$. We have $\deg(R) = a + a' - \deg(\Gamma')$, where $\Gamma' := w^{-1}(\Gamma)$.

(b) Assume $(b_1, b_2) \neq (1, 1)$ and set $\delta := \max\{b_1, b_2\}$. We have $\sharp(J) \geq a' + a - 2 - y/\delta$.

**Proof** Set $R' := u^*(\mathcal{O}_Q(1,1))$. Lemma 3 gives $h^0(C, R') = 4$. Hence $|R'|$ is induced by $|\mathcal{O}_Q(1,1)|$.

(i) Assume for the moment that $|R|$ is induced by a linear subseries $M$ of $|\mathcal{O}_Q(1,1)|$, after deleting a base locus. Let $\Gamma \subset Q$ be the base locus of $M$. Since $R$ is neither $u^*(\mathcal{O}_C(1,0))$ nor $\mathcal{O}_C(0,1)$, $\Gamma$ is not a line. Hence $\Gamma$ is a zero-dimensional scheme (it may be empty). Set $\Gamma' := w^{-1}(\Gamma)$. Since $\mathcal{O}_Q(1,1)$ is very ample, we have $h^0(\mathcal{I}_E(1,1)) = 4 - \deg(E)$ for all zero-dimensional schemes $E \subset Q$ with $\deg(E) \leq 2$. Notice that $h^0(\mathcal{I}_E(1,1)) = 1$ for each degree $3$ scheme $E \subset Q$ not contained in a line of $Q$. Since every line $L \subset \mathbb{P}^3$ with $\deg(L \cap Q) \geq 3$ is contained in $Q$, we get $\deg(\Gamma) \leq 2$ and $h^0(R) = 4 - \deg(\Gamma)$. Moreover, $\mathcal{I}_\Gamma(1,1)$ is spanned, unless $\deg(\Gamma) = 2$ and $\Gamma$ is contained in a line of $Q$. The latter case does not occur for $R$, because the line would be in the base locus $\Gamma$, while $\dim(\Gamma) = 0$. Hence $\mathcal{I}_\Gamma(1,1)$ is spanned. Since $\mathcal{I}_\Gamma(1,1)$ and $R$ are spanned, we have $R \cong R'(-\Gamma')$.

(ii) Fix a general $A \in |R|$ and set $B := u(A)$. Let $f : C \to \mathbb{P}^1$ be the degree $y$ morphism induced by $|R|$. Since $f$ is induced by a general pencil of the complete linear system $|R|$, it cannot factor through the Frobenius of order $p$. Since $\mathbb{K}$ is perfect, we get that $f$ is separable. Since $A$ is general, $A$ is a reduced set of $y$ points. Since $|R|$ is spanned, we may also assume $A \cap u^{-1}(\text{Sing}(T)) = \emptyset$. Hence $B \cap J = \emptyset$ and $\sharp(B) = y$.

*Claim:* We have $h^1(\mathcal{I}_{J \cup B}(a - 2, a' - 2)) > 0$.

*Proof of the Claim:* Fix $O \in A$. Since $R$ is spanned, we have $h^0(R(-O)) = h^0(R) - 1$, i.e. $h^0(\mathcal{O}_C(-(A \setminus \{O\}))) = h^0(\mathcal{O}_C(-A))$ (Riemann–Roch and Serre duality). Hence $h^1(\mathcal{O}_C(-A)) > 0$. We have $\mathcal{O}_Q \cong \mathcal{O}_Q(-2, -2)$. Hence the adjunction formula gives $\omega_T \cong \mathcal{O}_T(a - 2, a' - 2)$. Since $h^i(\mathcal{O}_Q(-2, -2)) = 0$, $i = 0, 1$, the restriction map $H^0(\mathcal{O}_Q(a - 2, a' - 2)) \to H^0(T, \omega_T)$ is bijective. Since $T$ has only ordinary nodes and ordinary cusps as singularities, we have $H^0(C, \omega_C) \cong H^0(\mathcal{I}_J(a - 2, a' - 2))$. Hence $h^1(\mathcal{I}_{J \cup B}(a - 2, a' - 2)) > 0$.

(iii) In this step we assume $a' = a$ and $h^0(R) = 2$. We first prove that $R$ is a subsheaf of $u^*(\mathcal{O}_T(1,1))$.

(a) Assume $b_1 = b_2 = 1$. Since $y + \sharp(J) \leq 3a - 5$ and $h^i(\mathcal{I}_{J \cup B}(a - 2, a' - 2)) > 0$ by the Claim, Lemma 4 gives the existence of a divisor $D \in |\mathcal{O}_Q(1,1)|$ such that $\deg(D \cap (J \cup B)) \geq 2a - 2$. Since $R$ has no base points and $h^0(R) = 2$, we get $B = B \cap D$. Moving $A \in |R|$ the set $B$ moves and hence $D$ moves, but $Y$ and the set $J \cap D$ are the same for all general $A$. Hence $|R|$ is induced by a subsheaf $M$ of the linear system $|\mathcal{O}_Q(1,1)|$. Let $\Gamma \subset Q$ be the base locus of $M$. Since $h^0(R) = 2$, step (i) gives $\deg(\Gamma) = 2$. Step (i) gives $y = 2a - \deg(\Gamma')$.

(b) Assume $\delta \geq 2$ and say $b_1 \geq b_2$. Since $B$ is general, either $I \cap B = \emptyset$ or $\sharp(I \cap B) = b_1$ for each $I \in |\mathcal{O}_Q(1,0)|$ and either $I \cap B = \emptyset$ or $\sharp(I \cap B) = b_2$ for each $I \in |\mathcal{O}_Q(0,1)|$. Since $R \neq u^*(\mathcal{O}_T(1,0))$, we have $\delta < a$. Lemma 4 gives $\sharp(J) \geq 2a - 2 - y/\delta$.

(iv) Assume $a' > a$ and $h^0(R) = 2$. Let $F \subset Q$ be a union of $a' - a$ lines of type $(0,1)$, each of them meeting $B$. Notice that $F \cap J = \emptyset$ and $\sharp(L \cap B) = b_1$ for each component $L$ of $F$. Since $b_1 \leq a + 1$, we have $h^1(F, \mathcal{I}_{\mathcal{F}(B \cup J), F}(a, a')) = 0$. Hence $h^1(\mathcal{I}_{J \cup B}(a, a')) \leq h^1(\mathcal{I}_{J \cup B \cap F}(a, a))$ by a residual exact sequence like (1). Apply step (iii).
(v) Assume $h^0(R) > 2$. By steps (iii) and (iv) a general pencil of $R$ is induced by a 2-dimensional linear subspace of $|O_Q(1,1)|$. Hence $R$ is induced by a subspecies of $|O_Q(1,1)|$ after deleting the base points. Use step (i).

**Corollary 3** In the set-up of Lemma 5 assume $a = a'$. Then $y \geq 2a - 2 - \min\{2, \deg(J)\}$ and for each $y$ with $2a - 2 - \min\{2, \sharp(J)\} \leq y \leq 2a$ there is a spanned $R \in \text{Pic}^y(C)$ with $|R|$ induced by a linear subspace of $|O_Q(1,1)|$.

### 3. The quadric surface $W$

Let $K$ be a perfect field having a quadratic extension. Fix homogeneous coordinates $x_0, x_1, x_2, x_3$ on $\mathbb{P}^3$. Fix $f \in K[x_0, x_1]$ with $f$ homogeneous of degree 2 and with no nontrivial zero in $K$. Set $W := \{x_2x_3 + f(x_0, x_1) = 0\} \subset \mathbb{P}^3$. $W$ is a geometrically smooth quadric surface containing no line defined over $K$. Hence $\text{Pic}(W)(K)$ is freely generated by $\mathcal{O}_W(1)$. Let $Y \subset W$ be a geometrically irreducible curve defined over $K$ and $u : C \to Y$ the normalization map. $C$ is a geometrically connected smooth curve and $C$ and $u$ are defined over $K$. Let $a$ be the only integer such that $Y \in |\mathcal{O}_W(a,a)|$. Set $Q := W(K)$.

In the set-up of Remark 1 and Corollary 3 the curve $Y(K)$ has $b_1 = b_2$. For any field $K' \supseteq K$ let $J(K')$ denote the set of all $P \in J$ defined over $K'$.

The following statement implies Corollary 1.

**Theorem 1** Take the set-up of Corollary 1.

(a) If $\sharp(J(K')) \geq 2$ for some quadratic extension $K'$ of $K$, then $y = 2a - 4$.

(b) If $\sharp(J(K)) = 1$, $J(K) = J(K')$ for every quadratic extension $K'$ of $K$ and $Y(K) \setminus J(K) \neq \emptyset$, then $y = 2a - 3$.

(c) Assume $\sharp(J(K)) = 1$, $J(K) = J(K')$ for every quadratic extension $K'$ of $K$ and $Y(K) = J(K)$. Set $\{P\} := J(K)$. If $Y$ has an ordinary node at $P$ and the formal branches of $Y$ at $P$ are not defined over $K$, then $y = 2a - 2$; otherwise, $y = 2a - 3$.

(d) If $J(K'') = \emptyset$ for every quadratic extension $K''$ of $K$ and there is a quadratic extension $K'$ of $K$ with $\sharp(Y(K')) \geq 2$, then $y = 2a - 2$.

(e) If $Y(K)$ has a unique point $P$, $P \notin J$ and $Y(K') = \{P\}$ for every quadratic extension $K'$ of $K$, then $y = 2a - 1$.

(f) If $J(K') = Y(K') = \emptyset$ for every quadratic extension $K'$ of $K$, then $y = 2a$.

In case (e) the only line bundle evincing $y$ is the pull-back of $\mathcal{O}_Y(1)(-P)$ and we have $h^0(R) = 3$.

In case (f) the only line bundle $R$ evincing $y$ is the one induced by the pull-back of $\mathcal{O}_W(1)$ and we have $h^0(R) = 4$.

**Proof** Since $\mathcal{O}_W(1)$ is spanned, we have $y \leq 2a$. Part (b) of Lemma 5 shows that $b_1 = b_2 = 1$. Theorem 1 follows from Corollary 3 and step (i) of the proof of Lemma 5.

Notice that if $J(K') \supseteq J(K)$ for some quadratic extension $K'$ of $K$, then $J(K') \setminus J(K)$ contains at least 2 elements and hence we are in case (a) with $y = 2a - 4$. 

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References