Characteristic classes on Grassmannians

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Abstract: In this paper, we study the geometry and topology on the oriented Grassmann manifolds. In particular, we use characteristic classes and the Poincaré duality to study the homology groups of Grassmann manifolds. We show that for \( k = 2 \) or \( n \leq 8 \), the cohomology groups \( H^*(G(k,n),\mathbb{R}) \) are generated by the first Pontrjagin class, the Euler classes of the canonical vector bundles. In these cases, the Poincaré duality: \( H^q(G(k,n),\mathbb{R}) \to H^{n-k-q}(G(k,n),\mathbb{R}) \) can be expressed explicitly.

Key words: Grassmann manifold, fibre bundle, characteristic class, homology group, Poincaré duality

1. Introduction

Let \( G(k,n) \) be the Grassmann manifold formed by all oriented \( k \)-dimensional subspaces of Euclidean space \( \mathbb{R}^n \). For any \( \pi \in G(k,n) \), there are orthonormal vectors \( e_1, \ldots, e_k \) such that \( \pi \) can be represented by \( e_1 \wedge \cdots \wedge e_k \). Thus \( G(k,n) \) becomes a submanifold of the space \( \Lambda^k(\mathbb{R}^n) \); then we can use moving frame to study the Grassmann manifolds.

There are 2 canonical vector bundles \( E = E(k,n) \) and \( F = F(k,n) \) over \( G(k,n) \) with fibres generated by vectors of the subspaces and the vectors orthogonal to the subspaces, respectively. Then we have Pontrjagin classes \( p_i(E) \) and \( p_j(F) \) with the relationship

\[
(1 + p_1(E) + \cdots)(1 + p_1(F) + \cdots) = 1.
\]

If \( k \) or \( n - k \) is an even number, we have Euler class \( e(E) \) or \( e(F) \).

The oriented Grassmann manifolds are classifying spaces for oriented vector bundles. For any oriented vector bundle \( \tau: \xi \to M \) with fibre type \( \mathbb{R}^k \), there is a map \( g: M \to G(k,n) \) such that \( \xi \) is isomorphic to the induced bundle \( g^*E \). If the maps \( g_1, g_2: M \to G(k,n) \) are homotopic, the induced bundles \( g_1^*E \) and \( g_2^*E \) are isomorphic. Then the characteristic classes of the vector bundle \( \xi \) are the pullback of the characteristic classes of the vector bundle \( E \).

In this paper, we study the geometry and topology on the oriented Grassmann manifolds. In particular, we use characteristic classes and the Poincaré duality to study the homology groups of oriented Grassmann manifolds. The characteristic classes of the canonical vector bundles can be represented by curvature and

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are the harmonic forms, see [5, 7, 8, 15, 20]. For \( k = 2 \) or \( n \leq 8 \), we show that the cohomology groups \( H^*(G(k, n), \mathbb{R}) \) are generated by the first Pontrjagin class \( p_1(E) \) and the Euler classes \( e(E) \), \( e(F) \) if \( k \) or \( n-k \) is even. In these cases, the Poincaré duality: \( H^q(G(k, n), \mathbb{R}) \to H_{k(n-k)-q}(G(k, n), \mathbb{R}) \) can be given explicitly.

In §2, we compute volumes of some homogeneous spaces that are needed in the later discussion. In §3, we study the Poincaré duality on oriented compact Riemannian manifolds. The results are Theorem 3.1.

The Poincaré polynomials of Grassmann manifolds \( G(k, n) \) for \( k = 2 \) or \( n \leq 8 \) are listed at the end of §3, which give the real homology groups of Grassmann manifolds. From [12], we know that the tangent space of Grassmann manifolds is isomorphic to tensor products of the canonical vector bundles. In §4 we use the splitting principle of the characteristic class to study the relationship among these vector bundles, and show that the characteristic classes of the tangent bundle on Grassmann manifolds can be represented by that of canonical vector bundles.

In §5, we study \( G(2, N) \); the main results are Theorem 5.5. In §6, we study the Grassmann manifold \( G(3, 6) \); the main results are Theorem 6.1.

In §7, §8 we study the Grassmann manifold \( G(3, 7) \) and \( G(3, 8) \); the main results are Theorem 7.5 and 8.4. In §9, we study \( G(4, 8) \); the main results are Theorem 9.4, 9.5.

As an application, in §5 and §9, we consider the Gauss maps of submanifolds in Euclidean spaces. The results generalize the work by Chern and Spanier [4]. For example, if \( g: M \to G(4, 8) \) is the Gauss map of an immersion \( f: M \to \mathbb{R}^8 \) of a compact oriented 4-dimensional manifold, we have

\[
g_*[M] = \frac{1}{2} \lambda(M)[G(4, 5)] + \lambda[G(1, 5)] + \frac{3}{2} \tau(M)[G(2, 4)],
\]

where \( \lambda = \frac{1}{2} \int_M e(F(4, 8)) \) and \( \tau(M) \) is the signature of \( M \). \( \lambda = 0 \) if \( f \) is an imbedding.

In §10 we use Gysin sequence to compute the cohomology of the homogeneous space \( ASSOC = G_2/\text{SO}(4) \), which was studied by Borel and Hirzebruch [6].

The cohomology groups of infinite Grassmann manifold \( G(k, \mathbb{R}^\infty) \) are simple; they are generated by Pontrjagin classes and the Euler class (if \( k \) is even) of the canonical vector bundle freely; see [13], p.179.

The computations on specific Grassmann manifolds like \( G(3, 7) \) or \( G(4, 8) \) have important implications on the theory of calibrated submanifolds like associative, coassociative, or Cayley submanifolds of Riemannian 7-8-manifolds of \( G_2 \) or \( \text{Spin}_7 \) holonomy. This work has many applications like [1, 11] among potential others.

In [1, 11], there are applications to associative, coassociative submanifolds of \( G_2 \) manifolds.

2. The volumes of homogeneous spaces

For any \( \pi \in G(k, n) \), there are orthonormal vectors \( e_1, \cdots, e_k \) such that \( \pi \) can be represented by \( e_1 \wedge \cdots \wedge e_k \).

These give an imbedding of \( G(k, n) \) in Euclidean space \( \bigwedge^k(\mathbb{R}^n) \); see [2, 20]. Let \( e_1, e_2, \cdots, e_n \) be orthonormal frame fields on \( \mathbb{R}^n \) such that \( G(k, n) \) is generated by \( e_1 \wedge \cdots \wedge e_k \) locally. The vectors \( e_1, e_2, \cdots, e_n \) can be viewed as functions on Grassmann manifolds. Let \( de_A = \sum_{B=1}^{n} \omega_B^A e_B \), \( \omega_A^B = \langle de_A, e_B \rangle \) be 1 forms on \( G(k, n) \).

From \( d^2 e_A = 0 \), we have \( d\omega_A^B = \sum_{C=1}^{n} \omega_A^C \wedge \omega_C^B \). By

\[
d(e_1 \wedge \cdots \wedge e_k) = \sum_{i=1}^{k} \sum_{\alpha=k+1}^{n} \omega_i^\alpha E_{i\alpha},
\]

493
Then $E_{\alpha} = e_1 \cdots e_{i-1} e_{i+1} \cdots e_k$, $i = 1, \ldots, k$, $\alpha = k + 1, \ldots, n$, we know $E_{\alpha}$ forms a basis of $T_{e_{i-1} e_i} G(k,n)$ and $\omega^\alpha_i$ is their dual basis.

$$ds^2 = \langle d(e_1 \wedge \cdots \wedge e_k), d(e_1 \wedge \cdots \wedge e_k) \rangle = \sum_{\alpha, \beta} (\omega^\alpha_i)^2$$

is the induced metric on $G(k,n)$. Differential $E_{\alpha} = e_1 \cdots e_{i-1} e_{i+1} \cdots e_k$, by Gauss equation, we get the Riemannian connection $\nabla$ on $G(k,n)$,

$$\nabla E_{\alpha} = \sum_{j=1}^{k} \omega^\beta_j E_{\alpha} + \sum_{\beta = k+1}^{n} \omega^\beta_{\alpha} E_{\beta}.$$

Grassmann manifold $G(k,n)$ is oriented; the orientation is given by the volume form

$$\omega^{k+1} \wedge \omega^{k+1} \wedge \cdots \wedge \omega^{k+1} \wedge \cdots \wedge \omega^{n+1} \wedge \cdots \wedge \omega^{n+1}.$$ 

For later use we compute the volumes for some homogeneous spaces. We first compute the volume of special orthogonal group $SO(n)$.

Let $gl(n, \mathbb{R})$ be the set of all $n \times n$ real matrices with the inner product

$$\langle X, Y \rangle = \text{tr} (XY^t) = \sum_{A,B} X_{AB}Y_{AB}, \quad X = (X_{AB}), Y = (Y_{CD}) \in gl(n, \mathbb{R}).$$

Then $gl(n, \mathbb{R})$ is a Euclidean space and $SO(n)$ is a Riemannian submanifold of $gl(n, \mathbb{R})$. Represent the elements of $SO(n)$ by $G = (e_1, \ldots, e_n)^t$, where $e_A$ is the $A$-th row of $G$. The vectors $e_1, \ldots, e_n$ can be viewed as functions of $SO(n)$; then $\omega^B_A = \langle de_A, e_B \rangle = de_A \cdot e_B$ are 1 forms on $SO(n)$, $\omega^B_A + \omega^A_B = 0$. Let $E_{BC}$ be the matrix with 1 in the $B$-th row, $C$-th column, the others being zero. We have

$$dGG^{-1} = (\omega^B_A) = \sum_{A,B} \omega^B_A E_{AB}, \quad dG = \sum_{A < B} \omega^B_A (E_{AB} - E_{BA})G.$$ 

Then $\{E_{AB} - E_{BA}\}$ is a basis of $T_G SO(n)$ and

$$ds^2 = \langle dG, dG \rangle = 2 \sum_{A < B} \omega^B_A \otimes \omega^B_A$$

is a Riemannian metric on $SO(n)$.

**Proposition 2.1** The volume of $SO(n)$ is

$$V(SO(n)) = 2^{\frac{k}{2}(n-1)} V(S^{n-1}) V(SO(n-1)) = 2^{\frac{k}{2}(n-1)} V(S^{n-1}) \cdots V(S^1).$$

**Proof** Let $\tilde{e}_n = (0, \ldots, 0, 1)$ be a fixed vector. The map $\tau(G) = \tilde{e}_n G = e_n$ defines a fibre bundle $\tau: SO(n) \rightarrow S^{n-1}$ with fibres $SO(n-1)$. By $de_n = \sum \omega^A_n e_A$,

$$dV_{S^{n-1}} = \omega^1_n \cdots \omega^n_{n-1}$$
is the volume element of $S^{n-1}$. The volume element of $SO(n)$ can be represented by
\[ dV_{SO(n)} = (\sqrt{2})^{\frac{1}{2}(n-1)} \prod_{A<B} \omega_A^B = 2^{\frac{1}{2}(n-1)}(\sqrt{2})^{\frac{1}{2}(n-1)(n-2)} \prod_{A<B<n} \omega_A^B \cdot \tau^* dV_{S^{n-1}}, \]
restricting $(\sqrt{2})^{\frac{1}{2}(n-1)(n-2)} \prod_{A<B<n} \omega_A^B$ on the fibres of $\tau$ are the volume elements of the fibres. Integration $dV_{SO(n)}$ along the fibre of $\tau$ first, then on $S^{n-1}$, shows
\[ V(SO(n)) = 2^{\frac{1}{2}(n-1)} V(S^{n-1}) V(SO(n-1)). \]
\[ \square \]
As we know $V(S^m) = \frac{2\pi^{m+1}}{\Gamma(m+2)}$,
\[ V(S^{2n-1}) = \frac{2\pi^n}{(n-1)!}, \quad V(S^{2n}) = \frac{2^{2n+1} n! \pi^n}{(2n)!}. \]

To compute the volume of $G(k,n)$, we use principle bundle $SO(n) \to G(k,n)$ with the Lie group $SO(k) \times SO(n-k)$ as fibres.

**Proposition 2.2** The volume of Grassmann manifold $G(k,n)$ is
\[ V(G(k,n)) = \frac{V(SO(n))}{2^{\frac{1}{2}k(n-k)} V(SO(k)) V(SO(n-k))} = \frac{V(S^{n-k})}{V(S^{k-1}) V(S^1)}. \]
The proof is similar to that of Proposition 2.1. By simple computation, we have
\[ V(G(2, n+2)) = \frac{2(2\pi)^n}{n!}, \quad V(G(3, 6)) = \frac{2}{3} \pi^5, \]
\[ V(G(3, 7)) = \frac{16}{45} \pi^6, \quad V(G(3, 8)) = \frac{2}{45} \pi^8, \quad V(G(4, 8)) = \frac{8}{135} \pi^8. \]

Now we compute the volume of complex Grassmann manifold $G_C(k,n)$. Let $J$ be the natural complex structure on $\mathbb{C}^n = \mathbb{R}^{2n}$ and $s_1, \ldots, s_k$ be Hermitian orthonormal basis of $\pi \in G_C(k,n)$. Let $e_{2i-1}, e_{2i} = J e_{2i-1} \in \mathbb{R}^{2n}$ be the realization vectors of $s_i$, $\sqrt{-1} s_i$ respectively. Then $e_1 e_2 \cdots e_{2k-1} e_{2k} \in G(2k, 2n)$, and $G_C(k,n)$ becomes a submanifold of $G(2k, 2n)$.

Let $U(n) = \{ G \in gl(n, \mathbb{C}) \mid G \cdot \mathbb{C}^n = I \}$ be the unitary group and the Hermitian inner product of $X = (X_{AB}), Y = (Y_{CD}) \in gl(n, \mathbb{C})$ be
\[ \langle X, Y \rangle = \text{tr}(X \bar{Y}^t) = \sum_{A,B} X_{AB} Y_{AB}. \]

Let $G = (s_1, \ldots, s_n)^t \in U(n)$ represented by the rows of $G$, $\omega_A^B = \langle ds_A, s_B \rangle = ds_A \cdot \bar{s}_B$ be 1 forms on $U(n)$. Let $\omega_A^B = \varphi_A^B + \sqrt{-1} \psi_A^B$. From $\omega_A^B + \omega_B^A = 0$ we have $\varphi_A^B + \varphi_B^A = 0$, $\psi_A^B - \psi_B^A = 0$. Then
\[ dG = \sum_{A,B} \omega_A^B E_{AB} G \]
\[ = \sum_{A<B} \varphi_A^B (E_{AB} - E_{BA}) G + \sqrt{-1} \sum_{A<B} \psi_A^B (E_{AB} + E_{BA}) G + \sum_A \psi_A^A (E_{AA} G), \]

495
and
\[ ds^2 = \langle dG, dG \rangle = 2 \sum_{A < B} (\varphi^B_A \otimes \varphi^B_A + \psi^B_A \otimes \psi^B_A) + \sum_A \psi^A_A \otimes \psi^A_A \]
is a Riemannian metric on \( U(n) \). The volume element is
\[ dV_{U(n)} = 2^{4n(n-1)} \psi_1^1 \cdots \psi_n^n \prod_{A < B} \varphi^B_A \varphi^B_A. \]

**Proposition 2.3** (1) The volume of \( U(n) \) is
\[ V(U(n)) = 2^{n-1} V(S^{2n-1}) V(U(n-1)) = 2^{4n(n-1)} V(S^{2n-1}) V(S^{2n-3}) \cdots V(S^1); \]
(2) As Riemannian submanifold of \( G(2k, 2n) \), the volume of \( G_C(k, n) \) is
\[ V(G_C(k, n)) = \frac{V(U(n))}{V(U(k)) V(U(n-k))}; \]
(3) The volume of \( \mathbb{C}P^n = G_C(1, n+1) \) is
\[ V(\mathbb{C}P^n) = \frac{(2\pi)^n}{n!}. \]

**Proof** Let \( \bar{e}_n = (0, \cdots, 0, 1) \) be a fixed vector. The map \( \tau(G) = \bar{e}_n G = s_n \) defines a fibre bundle \( \tau: U(n) \to S^{2n-1} \) with fibre type \( U(n-1) \). From \( ds_n = \sum \omega^n_A s_A \) and \( \omega^n_A = \varphi^n_A + \sqrt{-1} \psi^n_A \), \( \varphi^n_A = 0 \), we have the volume element of \( S^{2n-1} \),
\[ dV_{S^{2n-1}} = \varphi_{n-1}^1 \psi_1^1 \cdots \varphi_{n-1}^n \psi_{n-1}^n \psi_n^n. \]
Then the volume element of \( U(n) \) can be represented by
\[ dV_{U(n)} = 2^{n-1} \tau^* dV_{S^{2n-1}} \cdot dV_{U(n-1)}. \]
These prove (1).

As noted above, the map \([s_1 \cdots s_k] \to e_1 e_2 \cdots e_{2k-1} e_{2k}\) gives an embedding of \( G_C(k, n) \) in \( G(2k, 2n) \). From \( ds_i = \sum \omega^i_j s_j + \sum \omega^i_0 s_0 \), \( \omega^i_0 = \varphi^i_0 + \sqrt{-1} \psi^i_0 \), \( \omega^i_1 = \varphi^i_1 + \sqrt{-1} \psi^i_1 \), we have
\[ d_{e_{2i-1}} = \sum (\varphi^i_1 e_{2j-1} + \psi^i_1 e_{2j}) + \sum (\varphi^i_0 e_{2a-1} + \psi^i_0 e_{2a}), \]
\[ d_{e_{2i}} = \sum (\varphi^i_0 e_{2j-1} - \psi^i_0 e_{2j}) + \sum (\varphi^i_0 e_{2a-1} - \psi^i_0 e_{2a-1}). \]
Then
\[ d(e_1 e_2 \cdots e_{2k-1} e_{2k}) = \sum_{i, \alpha} \varphi^i_\alpha (E_{2i-1 - 2\alpha} + E_{2i - 2\alpha}) + \sum_{i, \alpha} \psi^i_\alpha (E_{2i-1 + 2\alpha} - E_{2i + 2\alpha}), \]
\[ dV_{G_C(k, n)} = 2^{k(n-k)} \varphi_1^{k+1} \psi_1^{k+1} \cdots \varphi_k^n \psi_k^n. \]
The rest is similar to that of Proposition 2.1.

The symmetric space \( SLAG = SU(n)/SO(n) \) can be imbedded in \( G(n, 2n) \) as follows. Let \( \tilde{e}_{2i-1}, \tilde{e}_{2i} = J\tilde{e}_{2i-1}, i = 1, \cdots, n \), be a fixed orthonormal basis of \( \mathbb{C}^n = \mathbb{R}^{2n} \); the subspace \( \{G(\tilde{e}_1 \tilde{e}_3 \cdots \tilde{e}_{2n-1}) \mid G \in SU(n) \subset SO(2n) \} \) is diffeomorphic to \( SLAG = SU(n)/SO(n) \).
Proposition 2.4 (1) The volume of special unitary group \( SU(n) \) is

\[
V(SU(n)) = 2^{n-1} \sqrt{\frac{n}{n-1}} V(S^{2n-1}) V(SU(n-1));
\]

(2) The volume of SLAG is

\[
V(SLAG) = \frac{V(SU(n))}{V(SO(n))}.
\]

**Proof** The proof is similar to that of Proposition 2.1. Let \( G = (s_1, \cdots, s_n)^t \in SU(n) \), \( \omega_A^B = ds_A \cdot \vec{s}_B \). From det \( G = 1 \) we have \( \sum_{A=1}^{n} \omega_A^A = 0 \); then \( \psi_n^A = - \sum_{B \neq n} \psi_B^A \). The Riemannian metric on \( SU(n) \) is

\[
ds^2 = 2 \sum_{A<B} (\varphi_A^B \otimes \varphi_A^B + \psi_A^B \otimes \psi_A^B) + \sum_{B \neq n} \psi_B^B \otimes \psi_B^B + \psi_n^B \otimes \psi_n^B
\]

\[
+ (\psi_1^1, \cdots, \psi_{n-1}^{n-1}) \begin{pmatrix}
2 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 2
\end{pmatrix}
\begin{pmatrix}
\psi_1^1 \\
\vdots \\
\psi_{n-1}^{n-1}
\end{pmatrix}.
\]

Then

\[
dV_{SU(n)} = 2^{\frac{3}{4}n(n-1)} \sqrt{n\psi_1^1 \cdots \psi_{n-1}^{n-1}} \prod_{A<B} \varphi_A^B \psi_A^B.
\]

The volume of special unitary group \( SU(n) \) is

\[
V(SU(n)) = 2^{n-1} \sqrt{\frac{n}{n-1}} V(S^{2n-1}) V(SU(n-1)).
\]

Let \( e_{2A-1}, e_{2A} = J e_{2A-1} \) be the realization vectors of \( s_A, \sqrt{-1} s_A \) respectively. SLAG is generated by \( G(e_1 e_3 \cdots e_{2n-1}) = e_1 e_3 \cdots e_{2n-1} \),

\[
d(e_1 e_3 \cdots e_{2n-1}) = \sum_{A<B} \psi_B^B (E_{2B-12B} - E_{2n-12n}) + \sum_{A<B} \psi_A^B (E_{2A-12B} + E_{2B-12A}),
\]

\[
ds^2 = 2 \sum_{A<B} \psi_A^B \otimes \psi_A^B + 2 \sum_{B \neq n} \psi_B^B \otimes \psi_B^B + \sum_{B \neq C \neq n} \psi_B^B \otimes \psi_C^C.
\]

Then

\[
dV_{SLAG} = 2^{\frac{3}{4}n(n-1)} \sqrt{n\psi_1^1 \cdots \psi_{n-1}^{n-1}} \prod_{A<B} \psi_A^B.
\]

Let \( \tau: SU(n) \to SLAG \) be the projection with fibres \( SO(n) \). Restricting \( ds_i = \sum \omega_i^j s_j + \sum \omega_i^\alpha s_\alpha \) on the fibre of \( \tau \), we have \( \omega_i^\alpha = 0 \) and \( \psi_i^j = 0 \); then \( dV_{SO(n)} = 2^{\frac{3}{4}n(n-1)} \prod_{A<B} \varphi_A^B \) is the volume element of the fibres. This completes the proof. \( \square \)
Let $Sp(n) = \{G \in gl(n, \mathbb{H}) \mid G \cdot G^t = I\}$ be the symplectic group, and $G_H(k, n) = \frac{Sp(n)}{Sp(k) \times Sp(n-k)}$ be the quaternion Grassmann manifold which can also be imbedded in $G(4k, 4n)$. The following proposition can be proved as Proposition 2.3.

**Proposition 2.5**

1. The volume of $Sp(n)$ is
   
   $$V(Sp(n)) = 4^{n-1}V(S^{4n-1})V(Sp(n-1)) = 2^{n(n-1)}V(S^{4n-1})V(S^{4n-5}) \cdots V(S^3);$$

2. As Riemannian submanifold of $G(4k, 4n)$, the volume of $G_H(k, n)$ is
   
   $$V(G_H(k, n)) = \frac{2^{2k(n-k)}V(Sp(n))}{V(Sp(k))V(Sp(n-k))}.$$

As $\mathbb{H}P^n = G_H(1, n+1)$, we have

$$V(\mathbb{H}P^n) = \frac{(4\pi)^{2n}}{(2n+1)!}.$$

3. The Poincaré duality

Let $M$ be a compact oriented Riemannian manifold and $H_q(M) = H_q(M, \mathbb{R})$ its $q$-th singular homology group, and $H^q(M) = H^q(M, \mathbb{R})$ be the $q$-th de Rham cohomology group. For any $[\xi] \in H^q(M)$ and $[z] = \sum \lambda_i \sigma_i \in H_q(M)$, we can define

$$[\xi](z) = \int z : = \sum \lambda_i \int_{\sigma_i} \xi = \sum \lambda_i \int_{\Delta^n} \sigma_i^* \xi,$$

where every singular simplex $\sigma_i : \Delta^n \to M$ is differentiable. If $[\xi] \in H^q(M, \mathbb{Z})$ and $[z] \in H_q(M, \mathbb{Z})$, we have $[\xi](z) \in \mathbb{Z}$. By universal coefficients theorem, we have

$$H^q(M, \mathbb{R}) \cong \text{Hom}(H_q(M, \mathbb{R}), \mathbb{R}),$$

and

$$H^q(M, \mathbb{Z}) \cong \text{Hom}(H_q(M, \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_{q-1}(M, \mathbb{Z}), \mathbb{Z}).$$

On the other hand, we have Poincaré duality

$$D : H^q(M, \mathbb{R}) \to H_{n-q}(M, \mathbb{R}), \quad n = \dim M.$$ 

For any $[\xi] \in H^q(M)$, $D[\xi] \in H_{n-q}(M)$, we have

$$[\eta](D[\xi]) = \int_{D[\xi]} \eta = \int_M \xi \wedge \eta$$

for any $[\eta] \in H^{n-q}(M)$.

In the following, we use harmonic forms to represent the Poincaré duality. Let $\varphi_1, \cdots, \varphi_k$ be the basis of $H^q(M)$ and $[T_i] = D(\varphi_i)$ be their Poincaré duals. By Hodge Theorem, we can assume that $\varphi_1, \cdots, \varphi_k$ are
all the harmonic forms on $M$. Then $*\varphi_1, \ldots , *\varphi_k$ are also the harmonic forms and form a basis of $H^{n-q}(M)$. Let

$$a_{ij} = (\varphi_i, \varphi_j) = \int_M (\varphi_i, \varphi_j) dV_M = \int_M \varphi_i \wedge *\varphi_j$$

be the inner product of differential forms $\varphi_i, \varphi_j$. Let $\psi_1, \ldots , \psi_k$ be the dual basis of $[T_1], \ldots , [T_k]$, also represented by harmonic forms. Assuming $\psi_j = \sum *\varphi_i b_{ij}$, by Poincaré duality,

$$\delta_{ij} = \int_{T_i} \psi_j = \int_M \varphi_i \wedge \psi_j = \int_M \sum \varphi_i \wedge *\varphi_j b_{ij} = \sum a_{il} b_{lj}.$$ 

This shows $(b_{ij}) = (a_{ij})^{-1}$, and we have

$$(\psi_1, \ldots , \psi_k) = (*\varphi_1, \ldots , *\varphi_k)(a_{ij})^{-1}.$$ 

**Theorem 3.1** Let $\varphi_1, \ldots , \varphi_k$ be a basis of the cohomology group $H^q(M)$ represented by harmonic forms. Let $[T_1], \ldots , [T_k] \in H_{n-q}(M)$ be the dual of $(\psi_1, \ldots , \psi_k) = (*\varphi_1, \ldots , *\varphi_k)(a_{ij})^{-1}$, where $a_{ij} = (\varphi_i, \varphi_j)$. The

Poincaré duality $D: H^q(M) \to H_{n-q}(M)$ is given by

$$D(\varphi_i) = [T_i].$$

Furthermore, if $[S_1], \ldots , [S_k]$ are the dual basis of $\varphi_1, \ldots , \varphi_k$, then

$$D(\psi_i) = (-1)^{q(n-q)}[S_i].$$

**Proof** The equations $D(\psi_i) = (-1)^{q(n-q)}[S_i]$ follow from $* * \varphi_i = (-1)^{q(n-q)} \varphi_i$ and $(\varphi_i, \varphi_j) = (*\varphi_i, *\varphi_j)$. □

Theorem 3.1 can be applied to the Poincaré duality $D: H^q(M, \mathbb{Z}) \to H_{n-q}(M, \mathbb{Z})$ if we ignore the torsion elements of $H^q(M, \mathbb{Z})$.

The $q$-th Betti number is the common dimension of the real homology and cohomology groups $H_q(G(k, n))$ and $H^q(G(k, n))$ (and is also the rank of $H_q(G(k, n), \mathbb{Z})$ and $H^q(G(k, n), \mathbb{Z})$). The Poincaré polynomials, with the Betti numbers as coefficients, are given by the following Table (see [7, 8, 18]).

<table>
<thead>
<tr>
<th>Grassmannian</th>
<th>Poincaré polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G(1, n + 1)$</td>
<td>$1 + t^n$</td>
</tr>
<tr>
<td>$G(2, 2n + 1)$</td>
<td>$1 + t^2 + t^3 + \cdots + t^{2n-2}$</td>
</tr>
<tr>
<td>$G(2, 2n + 2)$</td>
<td>$(1 + t^{2n})(1 + t^2 + \cdots + t^{2n})$</td>
</tr>
<tr>
<td>$G(3, 6)$</td>
<td>$(1 + t^4)(1 + t^5)$</td>
</tr>
<tr>
<td>$G(3, 7)$</td>
<td>$(1 + t^4 + t^5)(1 + t^4)$</td>
</tr>
<tr>
<td>$G(3, 8)$</td>
<td>$(1 + t^4 + t^5)(1 + t^4)$</td>
</tr>
<tr>
<td>$G(4, 8)$</td>
<td>$(1 + t^4 + t^5)(1 + t^4)^2$</td>
</tr>
</tbody>
</table>

4. The vector bundles on $G(k, n)$

Let $\tau_1: E(k, n) \to G(k, n)$ be the canonical vector bundle on Grassmann manifold $G(k, n)$, and the fibre over $\pi \in G(k, n)$ be the vectors of $\pi$. $E = E(k, n)$ is a Riemannian vector bundle with the induced metric. Let $e_1, \ldots , e_{k+1}, \ldots , e_n$ be orthonormal frame fields on $\mathbb{R}^n$, $G(k, n)$ is locally generated by
The total Pontrjagin classes of the vector bundle $F$ over $G$. Let $E$ be over $d$. Then Pontrjagin classes of $F$ are determined by that of $F$ are generated by $e_{1} \cdot e_{k} = e_{1} \wedge \cdots \wedge e_{k}$. Then $e_{1}, \cdots, e_{k}$ are local orthonormal sections of the vector bundle $\tau_{1}$. From $\text{de}_{i} = \sum_{j=1}^{k} \omega_{i}^{j} e_{j} + \sum_{\alpha=k+1}^{n} \omega_{\alpha}^{i} e_{\alpha}$, we know that $\nabla e_{i} = \sum_{j} \omega_{i}^{j} e_{j}$ defines a Riemannian connection on $\tau_{1}$. From $\nabla^{2} e_{i} = \sum (d\omega_{i}^{j} - \sum \omega_{i}^{\alpha} \wedge \omega_{\alpha}^{j}) e_{j}$, we have curvature forms

$$\Omega_{i}^{j} = d\omega_{i}^{j} - \sum \omega_{i}^{\alpha} \wedge \omega_{\alpha}^{j} = \sum \omega_{i}^{\alpha} \wedge \omega_{\alpha}^{j}.$$ 

The total Pontrjagin classes of the vector bundle $\tau_{1}: E \to G(k,n)$ are

$$p(E) = 1 + p_{1}(E) + p_{2}(E) + \cdots = \det(I + \frac{1}{2\pi}(\Omega_{i}^{j})).$$

If $k$ is even, we have Euler class

$$e(E) = \frac{(-1)^{\frac{k}{2}}}{(4\pi)^{\frac{k}{2}}\frac{k!}{2^{k}}} \sum_{i_{1}, \cdots, i_{k}} \varepsilon(i_{1}i_{2}\cdots i_{k}) \Omega_{i_{1}i_{2}} \Omega_{i_{3}i_{4}} \cdots \Omega_{i_{k-1}i_{k}}.$$ 

Similarly, we can define vector bundle $\tau_{2}: F = F(k,n) \to G(k,n)$ on Grassmann manifold $G(k,n)$; the fibre over $e_{1} \cdot e_{k} \in G(k,n)$ is the vectors orthogonal to $e_{1}, \cdots, e_{k}$. Then $e_{k+1}, \cdots, e_{n}$ are local orthonormal sections of $F$. From $\text{de}_{\alpha} = \sum \omega_{\alpha}^{\beta} e_{\beta} + \sum \omega_{\alpha}^{i} e_{i}$, we have Riemannian connection $\nabla e_{\alpha} = \sum \omega_{\alpha}^{\beta} e_{\beta}$. The curvature forms are given by

$$\nabla^{2} e_{\alpha} = \sum \Omega_{\alpha}^{\beta} e_{\beta}, \quad \Omega_{\alpha}^{\beta} = \sum \omega_{\alpha}^{i} \wedge \omega_{i}^{\beta}.$$ 

The total Pontrjagin classes of the vector bundle $\tau_{2}: F \to G(k,n)$ are

$$p(F) = 1 + p_{1}(F) + p_{2}(F) + \cdots = \det(I + \frac{1}{2\pi}(\Omega_{\alpha}^{\beta})).$$

The direct sum $E(k,n) \oplus F(k,n) = G(k,n) \times \mathbb{R}^{n}$ is trivial, and we have

$$(1 + p_{1}(E) + p_{2}(E) + \cdots) \cdot (1 + p_{1}(F) + p_{2}(F) + \cdots) = 1.$$ 

Then Pontrjagin classes of $F$ are determined by that of $E$. For example, we have

$$p_{1}(F) = -p_{1}(E), \quad p_{2}(F) = p_{1}^{2}(E) - p_{2}(E), \quad p_{3}(F) = -p_{1}(E) + 2p_{1}(E)p_{2}(E) - p_{3}(E).$$

Let $*: \bigwedge^{k}(\mathbb{R}^{n}) \to \bigwedge^{n-k}(\mathbb{R}^{n})$ be the star operator, $*G(k,n) = G(n-k,n)$, and the canonical vector bundles $E(k,n), F(k,n)$ are interchanged under the map $*$. 

**Proposition 4.1** The tangent space $TG(k,n)$ of a Grassmann manifold is isomorphic to tensor product $E(k,n) \otimes F(k,n)$. If $k(n-k)$ is even, we have

$$e(G(k,n)) = e(E(k,n) \otimes F(k,n)).$$

**Proof** Let $e_{1}, e_{2}, \cdots, e_{n}$ be an oriented orthonormal basis of $\mathbb{R}^{n}$, the fibre of $E(k,n)$ over $x = e_{1} \wedge \cdots \wedge e_{k} \in G(k,n)$ is generated by $e_{1}, \cdots, e_{k}$ and the fibre of $F(k,n)$ over $x$ is generated by $e_{k+1}, \cdots, e_{n}$. On the other
hand, the tangent space $T_xG(k,n)$ is generated by $E_{i\alpha} = e_1 \wedge \cdots \wedge e_{i-1} \wedge \eta_{\alpha} \wedge e_{i+1} \cdots \wedge e_k$. It is easy to see that the map $E_{i\alpha} \mapsto e_i \otimes \eta_{\alpha}$ gives an isomorphism from tangent bundle $TG(k,n)$ to tensor product $E \otimes F$. See also [12].

The isomorphism $TG(k,n) \to E(k,n) \otimes F(k,n)$ preserves the connections on $TG(k,n)$ and $E \otimes F$, respectively, where the connection on $E \otimes F$ is

$$\nabla (e_i \otimes \eta_{\alpha}) = \sum \omega^j_i e_j \otimes \eta_{\alpha} + \sum \omega^j_{\alpha} e_i \otimes \eta_j.$$

In the following we use the splitting principle of the characteristic class to study the relationship among these vector bundles. We study the oriented Grassmann manifold $G(2k, 2n)$; the other cases can be discussed similarly. Let $s_1, \cdots, s_{2k}$ be the orthonormal sections of vector bundle $E(2k, 2n)$ such that the curvature of Riemannian connection has the form

$$\frac{1}{2\pi} \nabla^2 \left( \begin{array}{c} s_1 \\ s_2 \\ \vdots \\ s_{2k-1} \\ s_{2k} \end{array} \right) = \left( \begin{array}{ccccc} 0 & -x_1 \\ x_1 & 0 & \ddots & \vdots \\ & \ddots & 0 & -x_k \\ & & \ddots & 0 & -x_1 \\ & & & \ddots & 0 \\ & & & & 0 & -x_k \end{array} \right) \left( \begin{array}{c} s_1 \\ s_2 \\ \vdots \\ s_{2k-1} \\ s_{2k} \end{array} \right).$$

The total Pontrjagin classes and the Euler class of $E = E(2k, 2n)$ are

$$p(E) = \prod_{i=1}^k (1 + x_i^2), \quad e(E) = x_1 \cdots x_k.$$

Similarly, assuming $t_{2k+1}, t_{2k+2}, \cdots, t_{2n}$ are the orthonormal sections of vector bundle $F(2k, 2n)$, the curvature of the Riemannian connection has the form

$$\frac{1}{2\pi} \nabla^2 \left( \begin{array}{c} t_{2k+1} \\ t_{2k+2} \\ \vdots \\ t_{2n-1} \\ t_{2n} \end{array} \right) = \left( \begin{array}{cccc} 0 & -y_{k+1} \\ y_{k+1} & 0 & \ddots & \vdots \\ & \ddots & 0 & -y_n \\ & & \ddots & 0 \\ & & & \ddots & 0 \\ & & & & 0 & -y_k \end{array} \right) \left( \begin{array}{c} t_{2k+1} \\ t_{2k+2} \\ \vdots \\ t_{2n-1} \\ t_{2n} \end{array} \right).$$

The total Pontrjagin classes and the Euler class of $F = F(2k, 2n)$ are

$$p(F) = \prod_{\alpha=k+1}^n (1 + y_{\alpha}^2), \quad e(F) = y_{k+1} \cdots y_n.$$

$s_{2i-1} \otimes t_{2\alpha-1}, s_{2i} \otimes t_{2\alpha-1}, s_{2i-1} \otimes t_{2\alpha}, s_{2i} \otimes t_{2\alpha}$ are the local orthonormal sections of vector bundle $E \otimes F \cong TG(2k, 2n)$. The curvature of Riemannian connection on $E \otimes F$ is given by

$$\frac{1}{2\pi} \nabla^2 \left( \begin{array}{cccc} s_{2i-1} \otimes t_{2\alpha-1} \\ s_{2i} \otimes t_{2\alpha-1} \\ s_{2i-1} \otimes t_{2\alpha} \\ s_{2i} \otimes t_{2\alpha} \end{array} \right) = \left( \begin{array}{cccc} 0 & -x_i & -y_{\alpha} & 0 \\ x_i & 0 & 0 & -y_{\alpha} \\ y_{\alpha} & 0 & 0 & -x_i \\ 0 & y_{\alpha} & x_i & 0 \end{array} \right) \left( \begin{array}{c} s_{2i-1} \otimes t_{2\alpha-1} \\ s_{2i} \otimes t_{2\alpha-1} \\ s_{2i-1} \otimes t_{2\alpha} \\ s_{2i} \otimes t_{2\alpha} \end{array} \right).$$

Then we have
Lemma 4.2 \( (1) \) \( e(TG(2k,2n)) = e(E \otimes F) = \prod_{i,\alpha} (x_i^2 - y_{\alpha}^2); \)

\( (2) \) \( p(TG(2k,2n)) = p(E \otimes F) = \prod_{i,\alpha} (1 + 2(x_i^2 + y_{\alpha}^2) + (x_i^2 - y_{\alpha}^2)^2). \)

By simple computation, we have

\[
p_1(TG(2k,2n)) = (2n - 2k)p_1(E) + 2kp_1(F) = 2(n - 2k)p_1(E).
\]

In particular, \( p_1(TG(2k,4k)) = 0. \)

In the next section, we shall show

\[
e(TG(2,2n + 2)) = (n + 1)e^{2n}(E(2,2n + 2)),
\]

\[
e(TG(2,2n + 3)) = (n + 1)e^{2n+1}(E(2,2n + 3)).
\]

We can also show

\[
e(TG(3,7)) = 3e^{3}(F(3,7)), \quad e(TG(4,8)) = 6e^{4}(E(4,8)) = 6e^{4}(F(4,8)).
\]

5. The cases of \( G(2, N) \)

In this section, we study the real homology of Grassmann manifold \( G(2, N) \).

As is well known, the oriented Grassmann manifold \( G(2, N) \) is a Kähler manifold and can be imbedded in a complex projective space. Here we give a new proof. Let \( e_1, e_2 \) be the oriented orthonormal basis of \( \pi \in G(2, N) \), \( e_1 \mapsto e_2, \ e_2 \mapsto -e_1 \) defines an almost complex structure

\[
J: T_\pi G(2, N) \to T_\pi G(2, N),
\]

\[
E_1 = e_1 \wedge e_2 \mapsto -e_1 \wedge e_1 = E_2, \ E_2 = e_1 \wedge e_2 \mapsto e_2 \wedge e_2 = -E_1.
\]

It is easy to see that \( J \) is well defined and preserves the metric on \( G(2, N) \).

**Proposition 5.1** \( G(2, N) \) is a Kähler manifold with complex structure \( J \).

**Proof** Let \( \nabla \) be the Riemannian connection on \( TG(2, N) \) defined above. We have

\[
(\nabla J)E_i = \nabla (JE_i) - J(\nabla E_i) = 0, \quad i = 1, 2.
\]

Hence, \( \nabla J = 0 \), \( J \) is a complex structure and \( G(2, N) \) is a Kähler manifold. \( \square \)

The Euler classes of canonical vector bundles \( E = E(2, 2n + 2) \) and \( F = F(2, 2n + 2) \) can be represented by

\[
e(E) = \frac{1}{2\pi} \sum_{\alpha = 3}^{2n+2} \omega^\alpha_1 \wedge \omega^\alpha_2,
\]

\[
e(F) = \frac{(-1)^n}{(4\pi)^n n!} \sum \varepsilon(\alpha_1 \alpha_2 \cdots \alpha_{2n}) \Omega_{\alpha_1 \alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2n-1} \alpha_{2n}}.
\]

For \( k < 2n \), \( G(2, k + 2) \) is a submanifold of \( G(2, 2n + 2) \) whose elements are contained in a fixed \( k + 2 \)-dimensional subspace of \( \mathbb{R}^{2n+2} \), \( i: \ G(2, k + 2) \to G(2, 2n + 2) \) the inclusion. Then, \( E(2, k + 2) = i^*E(2, 2n + 2) \) and \( e(E(2, k + 2)) = i^*e(E(2, 2n + 2)) \). Let \( G(1, 2n + 1) \) be a submanifold of \( G(2, 2n + 2) \) with elements \( e_1 \wedge e_2, \ e_2 = (0, \cdots, 0, 1), \ j: \ G(1, 2n + 1) \to G(2, 2n + 2) \) be the inclusion.
Theorem 5.2 For Grassmann manifold $G(2, 2n + 2)$, we have

(1) $p_q(F) = (-1)^q p_1^q (E) = (-1)^q e^{2q}(E), \quad q = 1, \cdots, n$;

(2) The Pontrjagin classes and Euler class of tangent bundle $TG(2, 2n + 2)$ can be represented by the Euler class of $E$,

$$p_1(G(2, 2n + 2)) = 2(n - 1)e^2(E), \quad p_2(G(2, 2n + 2)) = (2n^2 - 5n + 9)e^4(E), \cdots,$$

$$e(G(2, 2n + 2)) = (n + 1)e^{2n}(E);$$

(3) $\int_{G(1, 2n+1)} e(F) = \int_{G(2, k+2)} e^k(E) = 2, \quad k = 1, \cdots, 2n; \quad \int_{G(1, 2n+1)} e^n(E) = \int_{G(2, n+2)} e(F) = 0$.

Proof For Grassmann manifold $G(2, 2n + 2)$, we have $p_1(E) = e^2(E), \quad p_n(F) = e^2(F)$. From $(1 + p_1(E)) \cdot (1 + p_1(F) + p_2(F) + \cdots + p_n(F)) = 1$, we have

$$1 + p_1(F) + p_2(F) + \cdots + p_n(F) = \frac{1}{1 + p_1(E)} = 1 + \sum_{q=1}^{n} (-1)^q p_q^2 (E).$$

Hence, $p_q(F) = (-1)^q p_q^2 (E) = (-1)^q e^{2q}(E), \quad p_n(F) = e^2(F) = (-1)^n e^{2n}(E)$. This proves (1).

By Lemma 4.2, note that $x_1 = e(E)$, the Euler class of $G(2, 2n + 2)$ is

$$e(TG(2, 2n + 2)) = (x_1^2 - y_2^2)(x_2^2 - y_3^2) \cdots (x_n^2 - y_{n+1}^2)$$

$$= x_1^{2n} - x_1^{2n-2} p_1(F) + x_1^{2n-4} p_2(F) - \cdots + (-1)^n p_n(F)$$

$$= (n + 1)e^{2n}(E).$$

By Gauss–Bonnet formula, we have

$$\chi(G(2, 2n + 2)) = \int_{G(2, 2n+2)} e(G(2, 2n + 2)) = 2n + 2.$$

From (1) and

$$p(G(2, 2n + 2)) = \prod_{\alpha=2}^{n+1} (1 + 2(x_1^2 + y_\alpha^2) + (x_1^2 - y_\alpha^2)^2),$$

we can prove (2).

Restricting the Euler class $e(E)$ on $G(2, k + 2)$, we have

$$i^* e(E) = \frac{1}{2\pi} \sum_{\alpha=3}^{k+2} \omega_{\alpha}^1 \wedge \omega_{\alpha}^2.$$

Then

$$i^* e^k(E) = \frac{k!}{(2\pi)^k} \omega_1^3 \wedge \omega_2^3 \wedge \cdots \wedge \omega_1^{k+2} \wedge \omega_2^{k+2} = \frac{k!}{(2\pi)^k} d V_{G(2, k+2)},$$

where $d V_{G(2, k+2)}$ is the volume element of $G(2, k + 2)$. Then

$$\int_{G(2, k+2)} e^k(E) = 2, \quad k = 1, \cdots, 2n.$$
Restricting on $G(1, 2n + 1)$, $\omega_2^n = 0$, we have
\[
e(F(1, 2n + 1)) = j^* e(F(2, 2n + 2)) = \frac{(2n)!}{2^{2n}n!\pi^n} \omega_1^3 \wedge \omega_1^4 \wedge \cdots \wedge \omega_1^{2n+2}.
\]

It is easy to see that $e(F(1, 2n + 1))$ is the Euler class of the tangent bundle of $S^{2n} = G(1, 2n + 1)$ and $\omega_1^3 \wedge \omega_1^4 \wedge \cdots \wedge \omega_1^{2n+2}$ is the volume element. Hence by Gauss–Bonnet formula or by direct computation, we have
\[
\int_{G(1, 2n+1)} j^* e(F(2, 2n + 2)) = 2.
\]

Furthermore, from $\omega_2^n|_{G(1, 2n+1)} = 0$ and $\Omega_{\alpha, \beta}|_{G(2, n+2)} = 0$ for $\alpha, \beta > n + 2$, we have
\[
\int_{G(1, 2n+1)} j^* e^n(E) = 0, \quad \int_{G(2, n+2)} i^* e(F) = 0.
\]

By $p_k^1(E) = e^{2k}(E)$, we have
\[
\int_{G(2, 2n+2)} p_k^1(E) = 2.
\]

The Poincaré polynomial of $G(2, 2n + 2)$ is
\[
p_k(G(2, 2n + 2)) = 1 + t^2 + \cdots + t^{2n-2} + 2t^{2n} + t^{2n+2} + \cdots + t^{4n}.
\]

By Theorem 5.2, we have

(1) For $k \neq n$, $e^k(E) \in H^{2k}(G(2, 2n+2))$, $G(2, k + 2) \in H_{2k}(G(2, 2n+2))$ are the generators respectively;

(2) $e^n(E), e(F) \in H^{2n}(G(2, 2n + 2))$ and $G(2, n+2), G(1, 2n + 1) \in H_{2n}(G(2, 2n + 2))$ are the generators.

The characteristic classes $e^k(E), e(F)$ and the submanifolds $G(2, k + 2), G(1, 2n + 1)$ are integral cohomology and homology classes, respectively. However, they need not be the generators of the integral cohomology and homology groups. For example, when $k \neq n$, from $\int_{G(2, k+2)} e^k(E) = 2$ we know that $[G(2, k + 2)] \in H_{2k}(G(2, 2n+2), \mathbb{Z})$, $e^k(E) \in H^{2k}(G(2, 2n+2), \mathbb{Z})$ cannot be generators simultaneously. Now we compute $\int_{\mathbb{C}P^k} e^k(E)$ and $\int_{\mathbb{C}P^n} e(F)$.

Let $J$ be a complex structure on $\mathbb{R}^{2k+2} \subset \mathbb{R}^{2n+2}$ and $\mathbb{C}P^k = \{e_1 e_1^* e_1 \mid e_1 \in S^{2k+1}\}$. Let $e_1, e_2 = Je_1, e_2 = Je_{2\alpha - 1}, e_{2\alpha} = Je_{2\alpha}, \alpha = 2, 3, \cdots, k + 1$, be local orthonormal frame fields on $\mathbb{R}^{2k+2}$. By $de_2 = Jde_1$ we have $\omega_1^{2\alpha - 1} = \omega_2^{2\alpha}, \omega_1^{2\alpha} = -\omega_2^{2\alpha - 1}$; then
\[
d(e_1 \wedge e_2) = \sum_{\alpha=2}^{k+1} \omega_1^{2\alpha - 1}(E_{12\alpha - 1} + E_{22\alpha}) + \sum_{\alpha=2}^{k+1} \omega_2^{2\alpha}(E_{12\alpha} - E_{22\alpha}).
\]

The oriented volume element of $\mathbb{C}P^k$ is $dV = 2^k \omega_1^3 \wedge \omega_1^4 \wedge \cdots \wedge \omega_1^{2k+2}$.

Let $i: \mathbb{C}P^k \to G(2, 2n + 2)$ be inclusion, we have
\[
i^* e^k(E) = (-1)^k \frac{k!}{\pi^k} \omega_1^3 \wedge \omega_1^4 \wedge \cdots \wedge \omega_1^{2k+2}.
\]
By Proposition 2.3 (3),
\[ \int_{\mathbb{C}P^k} i^* e^k(E) = (-1)^k. \]

By \( p_k(E) = e^{2k}(E) \), we have \( \int_{\mathbb{C}P^k} i^* p_k(E) = 1. \)

For \( n = k \), \( J \) induces a complex structure on the induced bundle \( i^* F \to \mathbb{C}P^n \). Let \( F_C \) be the complex vector bundle formed by the \((1,0)\)-vectors of \( i^* F \otimes \mathbb{C} \). By \( i^* e(F) = c_n(F_C) \) (see [19]), we can show
\[ \int_{\mathbb{C}P^n} i^* e(F) = \int_{\mathbb{C}P^n} c_n(F_C) = 1. \]

See also Chern [3].

Let \( J \) be a complex structure on \( \mathbb{R}^{2k+2} \), and the orientation given by \( J \) is opposite to that of \( J \).

Let \( \overline{\mathbb{C}P}^k = \{ v \wedge J v \mid v \in S^{2k+1} \} \) be the complex projective space. The orientation on the vector bundle \( E(2, 2n+2)|_{\overline{\mathbb{C}P^n}} \) is given by \( v, J v \), and we have
\[ \int_{\overline{\mathbb{C}P^n}} i^* e^k(E) = (-1)^k. \]

Let \( \overline{F}_C \) be the complex vector bundle formed by the \((1,0)\)-vectors of \( F \otimes \mathbb{C}|_{\overline{\mathbb{C}P^n}} \). The orientation on realization vector bundle of \( \overline{F}_C \) given by \( J \) is opposite to that of \( F|_{\overline{\mathbb{C}P^n}} \). Hence \( e(F|_{\overline{\mathbb{C}P^n}}) = -c_n(\overline{F}_C) \) and we have
\[ \int_{\overline{\mathbb{C}P^n}} e(F) = -\int_{\overline{\mathbb{C}P^n}} c_n(\overline{F}_C) = -1. \]

These prove

**Proposition 5.3** (1) When \( k < n \), we have

\[ [G(2, k + 2)] = 2(-1)^k[^{\mathbb{C}P^k} \in H_{2k}(G(2, 2n + 2))]; \]

(2) In the homology group \( H_{2n}(G(2, 2n+2)) \), we have

\[ [G(2, n + 2)] = (-1)^n([\mathbb{C}P^n] + [\overline{\mathbb{C}P^n}]), \]

\[ [G(1, 2n + 1)] = [\mathbb{C}P^n] - [\overline{\mathbb{C}P^n}]. \]

For Grassmann manifold \( G(2, 2n+3) \), by the splitting principle of the characteristic classes, we can assume that there are oriented orthonormal sections \( s_1, s_2 \) and \( t_3, t_4, \cdots, t_{2n+2}, t_{2n+3} \) of vector bundle \( E = E(2, 2n+3) \) and \( F = F(2, 2n + 3) \) respectively, such that

\[ \frac{1}{2\pi} \nabla^2 \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \]

\[ \frac{1}{2\pi} \nabla^2 \begin{pmatrix} t_3 \\ t_4 \\ \vdots \\ t_{2n+1} \\ t_{2n+2} \\ t_{2n+3} \end{pmatrix} = \begin{pmatrix} 0 & -y_2 & \cdots & 0 \\ y_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ y_{n+1} & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} t_3 \\ t_4 \\ \vdots \\ t_{2n+1} \\ t_{2n+2} \\ t_{2n+3} \end{pmatrix}. \]
The total Pontrjagin classes of $F$ are $p(F) = \prod_{\alpha=2}^{n+1} (1 + y_\alpha^2)$.

$s_1 \otimes t_{2\alpha-1}, s_2 \otimes t_{2\alpha-1}, s_1 \otimes t_{2\alpha}, s_2 \otimes t_{2\alpha}$ and $s_1 \otimes t_{2n+3}, s_2 \otimes t_{2n+3}$ are orthonormal sections of $E \otimes F \cong TG(2, 2n + 3)$; they also give an orientation on $E \otimes F$. The curvature of $E \otimes F$ is

$$
\frac{1}{2\pi} \nabla^2 \begin{pmatrix}
  s_1 \otimes t_{2\alpha-1} \\
  s_2 \otimes t_{2\alpha-1} \\
  s_1 \otimes t_{2\alpha} \\
  s_2 \otimes t_{2\alpha}
\end{pmatrix} = \begin{pmatrix}
  0 & -x & -y_\alpha & 0 \\
  x & 0 & 0 & -y_\alpha \\
  y_\alpha & 0 & 0 & -x \\
  0 & y_\alpha & x & 0
\end{pmatrix} \begin{pmatrix}
  s_1 \otimes t_{2\alpha-1} \\
  s_2 \otimes t_{2\alpha-1} \\
  s_1 \otimes t_{2\alpha} \\
  s_2 \otimes t_{2\alpha}
\end{pmatrix},
$$

$$
\frac{1}{2\pi} \nabla^2 \begin{pmatrix}
  s_1 \otimes t_{2n+3} \\
  s_2 \otimes t_{2n+3}
\end{pmatrix} = \begin{pmatrix}
  0 & -x \\
  x & 0
\end{pmatrix} \begin{pmatrix}
  s_1 \otimes t_{2n+3} \\
  s_2 \otimes t_{2n+3}
\end{pmatrix},
$$

Hence the Euler class of $G(2, 2n + 3)$ is

$$e(TG(2, 2n + 3)) = e(E \otimes F) = x \prod_{\alpha=2}^{n+1} (x^2 - y_\alpha^2) = (n + 1)e^{2n+1}(E).$$

The odd dimensional homology groups of $G(2, 2n + 3)$ are trivial, and the even dimensional homology groups are one dimensional. The Euler-Poincaré number is $\chi(G(2, 2n + 3)) = 2n + 2$.

Similar to the case of $G(2, 2n + 2)$, we have

**Theorem 5.4** (1) The Pontrjagin classes of $F(2, 2n + 3)$ and $TG(2, 2n + 3)$ can all be represented by the Euler class $e(E(2, 2n + 3))$;

(2) $e(TG(2, 2n + 3)) = (n + 1)e^{2n+1}(E(2, 2n + 3))$;

(3) $\int_{G(2,k+2)} e^k(E(2, 2n + 3)) = 2, \ k = 1, \ldots, 2n + 1$;

(4) $\int_{CP^k} e^k(E(2, 2n + 3)) = (-1)^k, \ k = 1, \ldots, n + 1$.

As is well known, the Chern, Pontrjagin, and Euler classes are all integral cocycles. Let $D: H^k(G(2, N), \mathbb{Z}) \rightarrow H_{2N-k}(G(2, N), \mathbb{Z})$ be the Poincaré duality. The following theorem gives the structure of the integral homology and cohomology of $G(2, N)$.

**Theorem 5.5** (1) When $2k + 2 < N$, $[CP^k]$ and $e^k(E(2, N))$ are the generators of $H_{2k}(G(2, N), \mathbb{Z})$ and $H^{2k}(G(2, N), \mathbb{Z})$, respectively;

(2) When $2k + 2 > N$, $[G(2, k + 2)]$ and $\frac{1}{2}e^k(E(2, N))$ are the generators of $H_{2k}(G(2, N), \mathbb{Z})$ and $H^{2k}(G(2, N), \mathbb{Z})$, respectively;

(3) When $2k + 2 < N$, $D(e^k(E(2, N))) = [G(2, N - k)]$; when $2k + 2 > N$, $D(\frac{1}{2}e^k(E(2, N))) = (-1)^{n-k}[CP^{n-k-2}]$;

(4) $[CP^n], [\overline{CP^n}]$ and $\frac{1}{2}(e^n(E(2, 2n + 2)) + \frac{1}{2}e(F(2, 2n + 2)))$ are generators of $H_{2n}(G(2, 2n + 2), \mathbb{Z})$ and $H^{2n}(G(2, 2n + 2), \mathbb{Z})$, respectively. Furthermore,

$$D(\frac{1}{2}(-1)^ne^n(E(2, 2n + 2)) + \frac{1}{2}e(F(2, 2n + 2))) = [CP^n],$$

506


\[ D(\frac{1}{2}(-1)^n e^n(E(2, 2n + 2)) - \frac{1}{2} e(F(2, 2n + 2))) = [\mathbb{CP}^n]. \]

**Proof**  As is well known, the Euler classes and Pontrjagin classes are harmonic forms and are integral cocycles, and their products are also harmonic forms; see [7, 15]. When $2k + 2 < N$, from $\int_{\mathbb{CP}^k} e^k(E(2, N)) = (-1)^k$ we know $\mathbb{CP}^k \in H_{2k}(G(2, N), \mathbb{Z})$ and $e^k(E(2, N)) \in H^{2k}(G(2, N), \mathbb{Z})$ are generators, respectively.

By simple computation, we have

\[ e^k(E(2, N)) = \frac{k!}{(2\pi)^k} \sum_{\alpha_1 < \cdots < \alpha_k} \omega_1^{\alpha_1} \omega_2^{\alpha_2} \cdots \omega_1^{\alpha_k} \omega_2^{\alpha_k}, \]

\[ a = (e^k(E(2, N)), e^k(E(2, N))) = \frac{(k!)^2}{(2\pi)^{2k}} C_{N-2}^k V(G(2, N)), \]

\[ \frac{1}{a} * e^k(E(2, N)) = \frac{(N - k - 2)!}{2(2\pi)^{N-k-2}} \sum_{\beta_1 < \cdots < \beta_{N-k-2}} \omega_1^{\beta_1} \omega_2^{\beta_2} \cdots \omega_1^{\beta_{N-k-2}} \omega_2^{\beta_{N-k-2}} \]

\[ = \frac{1}{2} e^{N-k-2}(E(2, N)). \]

By Theorem 3.1, $\frac{1}{2} e^{N-k-2}(E(2, N))$ is a generator of $H^{2N-2k-4}(G(2, N), \mathbb{Z})$. By $\int_{G(2, N-k)} \frac{1}{2} e^{N-k-2}(E(2, N)) = 1$ we know that $G(2, N - k) \in H_{2N-2k-4}(G(2, N), \mathbb{Z})$ is a generator and $D(e^k(E(2, N))) = G(2, N - k)$. This proves (1), (2), (3) of the Theorem.

Let $[S_1], [S_2]$ be generators of $H_{2n}(G(2, 2n+2), \mathbb{Z})$ and harmonic forms $\xi_1, \xi_2$ be generators of $H_{2n}(G(2, 2n+2), \mathbb{Z})$; they satisfy $\int_{S_i} \xi_j = \delta_{ij}$. There are integers $a_{ij}, n_{ij}$ such that

\[ \left( \begin{array}{cc} e^n(E) \\ e(F) \end{array} \right) = \left( \begin{array}{cc} n_{11} & n_{12} \\ n_{21} & n_{22} \end{array} \right) \left( \begin{array}{cc} \xi_1 \\ \xi_2 \end{array} \right), \quad (\mathbb{CP}^n, \overline{\mathbb{CP}^n}) = (S_1, S_2) \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right). \]

Then

\[ \left( \begin{array}{cc} n_{11} & n_{12} \\ n_{21} & n_{22} \end{array} \right) \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) = \left( \begin{array}{cc} (-1)^n & (-1)^n \\ 1 & -1 \end{array} \right), \]

and we have $\det(a_{ij}) = \pm 1$ or $\det(n_{ij}) = \pm 1$.

If $\det(n_{ij}) = \pm 1$, $e^n(E), e(F)$ are also the generators of $H^{2n}(G(2, 2n + 2), \mathbb{Z})$, we can assume $\xi_1 = e^n(E), \xi_2 = e(F)$. It is easy to see

\[ (e^n(E), e(F)) = 0, \quad (e^n(E), e^n(E)) = (e(F), e(F)) = 2, \]

\[ *e^n(E) = e^n(E), \quad *e(F) = e(F). \]

By Theorem 3.1, $\frac{1}{2} e^n(E), \frac{1}{2} e(F)$ are also the generators of $H^{2n}(G(2, 2n + 2), \mathbb{Z})$. This contradicts the fact that $\int_{\overline{\mathbb{CP}^n}} e^n(E) = (-1)^n$.

Then we must have $\det(n_{ij}) = \pm 2$ and $\det(a_{ij}) = \pm 1$. This shows $\mathbb{CP}^n, \overline{\mathbb{CP}^n}$ are generators of $H_{2n}(G(2, 2n+2), \mathbb{Z})$, and $\frac{1}{2} \{(-1)^n e^n(E) + e(F)\}$, $\frac{1}{2} \{(-1)^n e^n(E) - e(F)\}$ are generators of $H^{2n}(G(2, 2n+2), \mathbb{Z})$. The Poincaré duals of these generators are easy to compute.

\[ \square \]
We give some applications to conclude this section.

Let \( f: M \to \mathbb{R}^N \) be an immersion of an oriented compact surface \( M \), \( g: M \to G(2, N) \) the induced Gauss map, \( g(p) = T_pM \). Then \( e(M) = g^*(e(E(2, N))) \) is the Euler class of \( M \). Let \([M] \in H_2(M)\) be the fundamental class of \( M \). When \( N \neq 4 \), we have

\[
g_*[M] = \frac{1}{2} \chi(M)[G(2, 3)] = \chi(M)[-(\mathbb{CP}^1)] \in H_2(G(2, N)).
\]

In [10, 17], we have shown there is a fibre bundle \( \tau: G(2, 8) = G(6, 8) \to S^6 \) with fibres \( \mathbb{CP}^3 \), where \( S^6 = \{ v \in S^7 \mid v \perp \bar{e}_1 = (1, 0, \cdots, 0) \} \), \( \tau^{-1}(\bar{e}_2) = \{ v \wedge Jv \mid v \in S^7 \} \), \( \bar{e}_2 = (0, 1, 0, \cdots, 0) \). On the other hand, the map \( f(v) = \bar{e}_1 \wedge v \) gives a section of \( \tau \). Let \( dV \) be the volume form on \( S^6 \) such that \( \int_{S^6} dV = 1 \). It is easy to see

\[
[\tau^*dV] = \frac{1}{2} e^3(E(2, 8)) + \frac{1}{2} e(F(2, 8)).
\]

Let \( \varphi: M \to \mathbb{R}^8 \) be an immersion of an oriented compact 6-dimensional manifold, and \( g: M \to G(6, 8) = G(2, 8) \) be the Gauss map. Then \( e(M) = g^*e(F(2, 8)) \) is the Euler class of tangent bundle of \( M \), and \( e(T^\perp M) = g^*e(E(2, 8)) \) is the Euler class of normal bundle of \( M \).

\[
\int_M (\tau \circ g)^*dV = \int_M g^*[\frac{1}{2} e^3(E(2, 8)) + \frac{1}{2} e(F(2, 8))] = \frac{1}{2} \int_M e^3(T^\perp M) + \frac{1}{2} \chi(M)
\]

is the degree of the map \( \tau \circ g: M \to S^6 \). If \( \varphi \) is an imbedding, \( e(T^\perp M) = 0 \); see Milnor, Stasheff [13], p.120.

Let \( J, \bar{J} \) be 2 complex structures on \( \mathbb{R}^4 \), with orthonormal basis \( e_1, e_2, e_3, e_4 \),

\[
J e_1 = e_2, \quad J e_2 = -e_1, \quad J e_3 = e_4, \quad J e_4 = -e_3;
\]

\[
\bar{J} e_1 = -e_2, \quad \bar{J} e_2 = e_1, \quad \bar{J} e_3 = e_4, \quad \bar{J} e_4 = -e_3.
\]

For any unit vector \( v = \sum v_i e_i \), we have

\[
vJv + *vJv = e_1 e_2 + e_3 e_4;
\]

\[
vJv - *vJv = (v_1^2 + v_2^2 - v_3^2 - v_4^2)(e_1 e_2 - e_3 e_4) + 2(v_1 v_4 + v_2 v_3)(e_1 e_4 - e_2 e_3) + 2(v_2 v_3 - v_1 v_4)(e_1 e_3 + e_2 e_4);
\]

\[
v\bar{J}v - *v\bar{J}v = -e_1 e_2 + e_3 e_4,
\]

\[
v\bar{J}v + *v\bar{J}v = (-v_1^2 - v_2^2 + v_3^2 + v_4^2)(e_1 e_2 + e_3 e_4) + 2(v_1 v_3 - v_2 v_4)(e_1 e_4 + e_2 e_3) - 2(v_1 v_4 + v_2 v_3)(e_1 e_3 - e_2 e_4).
\]

This shows \( \mathbb{CP}^1, \overline{\mathbb{CP}^1} \) are 2 spheres in \( G(2, 4) = S^2(\sqrt{2}) \times S^2(\sqrt{2}) \) where the decomposition is given by star operator \( *: G(2, 4) \to G(2, 4) \).

Let \( f: M \to \mathbb{R}^4 \) be an immersion of an oriented surface, and \( g: M \to G(2, 4) \) the Gauss map. Then we have \( g_*[M] = a[G(2, 3)] + b[G(1, 3)] \), where \( a = \frac{1}{2} \chi(M) \), \( b = \frac{1}{2} \int_M e(T^\perp M) \). If \( f \) is an imbedding,

\[
g_*[M] = \frac{1}{2} \chi(M)[G(2, 3)] = -\frac{1}{2} \chi(M)[\mathbb{CP}^1] - \frac{1}{2} \chi(M)[\overline{\mathbb{CP}^1}].
\]

See also the work by Chern and Spanier [4].

508
6. The case of $G(3,6)$

The Poincaré polynomial of Grassmann manifold $G(3,6)$ is $p_t(G(3,6)) = 1 + t^4 + t^5 + t^9$. To study the homology of $G(3,6)$ we need only consider the dimension 4, 5.

Let $i: G(2,4) \to G(3,6)$ be an inclusion defined naturally. It is easy to see that $i^*p_t(E(3,6)) = p_t(E(2,4)) = e^2(E(2,4))$; then
\[
\int_{G(2,4)} p_t(E(3,6)) = 2.
\]

As §2, let $SLAG = \{G(\tilde{e}_1 \tilde{e}_3 \tilde{e}_5) | G \in SU(3) \subset SO(6)\}$ be a subspace of $G(3,6)$, and $e_i = G(\tilde{e}_i), e_{i+1} = G(\tilde{e}_{i+1}) = J e_i$ be $SU(3)$-frame fields, $i = 1, 3, 5$. Restricting the coframes $\omega_i = (d e_i, e_{i+1})$ on $SLAG$ we have
\[
\omega_i = \omega_i^j + 1, \, \omega_i^j = -\omega_i^{j+1}, \, i, j = 1, 3, 5 \text{ and } \omega_1^2 + \omega_3^4 + \omega_5^6 = 0.
\]

By the proof of Proposition 2.4, we have $d V_{SLAG} = 2^{\frac{3}{2}} \sqrt{3} \omega_1^6 \omega_5^6 \omega_6^4 \omega_5^4$ and $V(SLAG) = \sqrt{\frac{3}{2}} \pi^3$. Let $G(3,6)$ be generated by $e_1 e_3 e_5$ locally, and the first Pontrjagin class of canonical vector bundle $E(3,6)$ is
\[
p_t(E(3,6)) = \frac{1}{4\pi^2} [r_1(\Omega_{13})^2 + (\Omega_{15})^2 + (\Omega_{35})^2],
\]
where $\Omega_{ij} = -\sum_{\alpha} \omega_i^\alpha \wedge \omega_j^\alpha$, $\alpha = 2, 4, 6$. By computation we have
\[
*p_t(E(3,6))|_{SLAG} = \frac{\sqrt{6}}{4\pi^2} d V_{SLAG},
\]
\[
a = (p_t(E(3,6)), p_t(E(3,6))) = \frac{3 \cdot 4 \cdot 3}{(2\pi)^4} V(G(3,6)) = \frac{3}{2} \pi.
\]

From $\int_{G(2,4)} p_t(E(3,6)) = 2$, we know that $p_t(E(3,6))$ or $\frac{1}{a} p_t(E(3,6))$ is a generator of $H^4(G(3,6), \mathbb{Z})$. If $p_t(E(3,6))$ is a generator, by Theorem 3.1, $\frac{1}{a} * p_t(E(3,6))$ is a generator of $H^5(G(3,6), \mathbb{Z})$, but
\[
\int_{SLAG} \frac{1}{a} * p_t(E(3,6)) = \int_{SLAG} \frac{1}{\sqrt{6\pi^3}} d V_{SLAG} = \frac{1}{2}.
\]
Then $\frac{1}{2} p_t(E(3,6))$ is a generator of $H^4(G(3,6), \mathbb{Z})$ and $\int_{SLAG} \frac{4}{a} * \frac{1}{2} p_t(E(3,6)) = 1$.

We have proved the following theorem

**Theorem 6.1** (1) $\frac{1}{2} p_t(E(3,6)) \in H^4(G(3,6), \mathbb{Z})$ is a generator and its Poincaré dual $[SLAG]$ is a generator of $H_5(G(3,6), \mathbb{Z})$;

(2) $\frac{4}{a} * p_t(E(3,6)) \in H^5(G(3,6), \mathbb{Z})$ is a generator and its Poincaré dual $[G(2,4)]$ is a generator of $H_4(G(3,6), \mathbb{Z})$.

Let $\tilde{e}_1, \cdots, \tilde{e}_6$ be a fixed orthonormal basis of $\mathbb{R}^6$, $G \in SO(3)$ acts on the subspace generated by $\tilde{e}_4, \tilde{e}_5, \tilde{e}_6$, and denote $e_4 = G(\tilde{e}_4), e_5 = G(\tilde{e}_5), e_6 = G(\tilde{e}_6)$. As [7], let $PONT$ be the set of elements
\[
(\cos t \tilde{e}_1 + \sin t \tilde{e}_4)(\cos t \tilde{e}_2 + \sin t \tilde{e}_5)(\cos t \tilde{e}_3 + \sin t \tilde{e}_6), \, t \in [0, \frac{\pi}{2}].
\]
**PONT** is a calibrated submanifold (except 2 points correspond to \( t = 0, \frac{\pi}{2} \)) of the first Pontrjagin form \( p_1(E(3,6)) \). By moving the frame we can show

\[
\int_{PONT} p_1(E(3,6)) = 2 \quad \text{and} \quad V(PONT) = \sqrt{\frac{2}{3}} V(G(2,4)) = \frac{4\sqrt{6}}{3} \pi^2.
\]

Then 4-cycle \( PONT \) is homologous to the 4-cycle \( G(2,4) \) inside \( G(3,6) \).

**7. The case of** \( G(3,7) \)

The Poincaré polynomial of \( G(3,7) \) is \( p_i(G(3,7)) = 1 + 2t^4 + 2t^8 + t^{12} \).

Let \( \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_8 \) be a fixed orthonormal basis of \( \mathbb{R}^8 \) and \( \mathbb{R}^7 \) be a subspace generated by \( \bar{e}_2, \ldots, \bar{e}_8 \). The oriented Grassmann manifold \( G(3,7) \) is the set of subspaces of \( \mathbb{R}^7 \).

Let \( E = E(3,7) \) and \( F = F(3,7) \). As \( \S 4 \), we can show

\[
p_1(F) = -p_1(E), \quad p_2(F) = e^2(F) = p_1^2(E), \quad e(E \otimes F) = e(TG(3,7)) = 3e^3(F).
\]

By \( \int_{G(3,7)} e(TG(3,7)) = \chi(G(3,7)) = 6 \) we have

\[
\int_{G(3,7)} e^3(F) = 2.
\]

By inclusion \( G(2,6) \subset G(3,7) \), \( \mathbb{C}P^2 \) and \( \overline{\mathbb{C}P^2} \) can be imbedded in \( G(3,7) \).

**Lemma 7.1** \( \int_{\mathbb{C}P^2} p_1(E) = \int_{\overline{\mathbb{C}P^2}} p_1(E) = 1, \quad \int_{\mathbb{C}P^2} e(F) = -\int_{\overline{\mathbb{C}P^2}} e(F) = 1. \)

**Proof** By \( p_1(E(3,7))|_{\mathbb{C}P^2} = p_1(E(2,6))|_{\mathbb{C}P^2} = e^2(E(2,6))|_{\mathbb{C}P^2} \) and the results of \( \S 5 \), we have \( \int_{\mathbb{C}P^2} p_1(E) = 1. \)

The other equalities can be proved similarly.

Then

\[
\int_{\mathbb{C}P^2} \frac{1}{2}(p_1(E) + e(F)) = \int_{\overline{\mathbb{C}P^2}} \frac{1}{2}(p_1(E) - e(F)) = 1,
\]

\[
\int_{\mathbb{C}P^2} \frac{1}{2}(p_1(E) - e(F)) = \int_{\overline{\mathbb{C}P^2}} \frac{1}{2}(p_1(E) + e(F)) = 0,
\]

hence \( \mathbb{C}P^2, \overline{\mathbb{C}P^2} \in H_4(G(3,7)) \) and \( \frac{1}{2}(p_1(E) + e(F)), \frac{1}{2}(p_1(E) - e(F)) \in H^4(G(3,7)) \) are generators.

Let \( e_2, e_3, e_4, \ldots, e_8 \) be oriented orthonormal frame fields on \( \mathbb{R}^7 \), and \( G(3,7) \) be generated by \( e_2 \wedge e_3 \wedge e_4 \) locally. Euler class of \( F \) and first Pontrjagin class of \( E \) can be represented by

\[
e(F) = \frac{1}{2(4\pi)^2} \sum \varepsilon(\alpha_1\alpha_2\alpha_4\alpha_4)\Omega_{\alpha_1\alpha_2} \wedge \Omega_{\alpha_3\alpha_4} = \frac{1}{4\pi^2} \sum_{i,j=2}^{4} (\omega_i^1\omega_i^2\omega_i^3\omega_i^4 - \omega_i^1\omega_i^2\omega_i^3\omega_i^4 + \omega_i^1\omega_i^2\omega_i^3\omega_i^4),
\]

\[
p_1(E) = \frac{1}{4\pi^2}[(\Omega_{23})^2 + (\Omega_{24})^2 + (\Omega_{34})^2].
\]

Then we have

\[
p_1(E)e^2(F) = p_3^1(E) = 0.
\]
Lemma 7.2  (1) \( p_1(E) = \frac{4}{3} \pi^2 p_1(E) e(F), \) \( *e(F) = \frac{1}{2} \pi^2 e^2(F); \)

(2) \((p_1(E), e(F)) = 0, \ a = (p_1(E), p_1(E)) = \frac{8}{3} \pi^2, \ b = (e(F), e(F)) = \pi^2.\)

Proof  \( p_1(E), \ p_1(E)e(F) \) and \( *e(F), \ e^2(F) \) are the harmonic forms on \( G(3, 7). \) \( p_1(E) = \frac{4}{3} \pi^2 p_1(E) e(F) \) follows from the equalities such as

\[
* \omega^5_2 \omega^5_3 \omega^6_2 \omega^6_3 = \omega^7_2 \omega^7_3 \omega^8_2 \omega^8_3 = \omega^4_4 \omega^4_4 = \omega^2_2 = \omega^3_2 \omega^3_3 \omega^4_3 \omega^4_3.
\]

The proof of (2) is a direct computation. \( \square \)

To study \( G(3, 7), G(3, 8), \) and \( G(4, 8), \) we shall use Clifford algebras.

Let \( \mathfrak{Cl}_8 \) be the Clifford algebra associated with the Euclidean space \( \mathbb{R}^8. \) Let \( \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_8 \) be a fixed orthonormal basis of \( \mathbb{R}^8, \) and the Clifford product be determined by the relations: \( \bar{e}_B \cdot \bar{e}_C + \bar{e}_C \cdot \bar{e}_B = -2 \delta_{BC}, \ B, C = 1, 2, \ldots, 8. \) Define the subspace \( V = V^+ \oplus V^- \) of \( \mathfrak{Cl}_8 \) by \( V^+ = \mathfrak{Cl}_8^{\text{even}} \cdot A, V^- = \mathfrak{Cl}_8^{\text{odd}} \cdot A, \) where

\[
A = \frac{1}{16} \text{Re} [(\bar{e}_1 + \sqrt{-1} \bar{e}_2) \cdots (\bar{e}_7 + \sqrt{-1} \bar{e}_8)(1 + \bar{e}_1 \bar{e}_3 \bar{e}_5 \bar{e}_7)].
\]

The space \( V = V^+ \oplus V^- \) is an irreducible module over \( \mathfrak{Cl}_8. \) The spaces \( V^+ \) and \( V^- \) are generated by \( \bar{e}_1 \bar{e}_B A \) and \( \bar{e}_B A \) respectively, \( B = 1, \ldots, 8; \) see [16,17].

Let \( \text{Spin}_7 = \{ G \in SO(8) \mid G(A) = A \} \) be the isotropy group of \( SO(8) \) acting on \( A. \) The group \( \text{Spin}_7 \) acts on \( G(2, 8), G(3, 8) \) and \( S^7 \) transitively. \( G_2 = \{ G \in \text{Spin}_7 \mid G(\bar{e}_1) = \bar{e}_1 \} \) is a subgroup of \( \text{Spin}_7. \)

The Grassmann manifold \( G(k, 8) \) can be viewed as a subset of Clifford algebra \( \mathfrak{Cl}_8 \) naturally. Then, for any \( \pi \in G(k, 8), \) there is \( v \in \mathbb{R}^8 \) such that \( \pi A = \bar{e}_1 v A \) or \( \pi A = v A \) according to the number \( k \) being even or odd, \( |v| = 1. \) Thus we have maps \( G(k, 8) \to S^7, \ \pi \mapsto v. \) Since \( \text{Spin}_7 \) acts on \( G(3, 8) \) transitively, from \( \bar{e}_2 \bar{e}_3 \bar{e}_4 A = \bar{e}_3 A \) we have \( G(\bar{e}_2 \bar{e}_3 \bar{e}_4) A = G(\bar{e}_1) A \) for any \( G \in \text{Spin}_7. \) This shows the map \( \tau: G(3, 8) \to S^7, \ \tau(\pi) = v, \) is a fibre bundle and \( v \perp \pi; \) see [10,17]. Let

\[
\text{ASSOC} = \tau^{-1}(\bar{e}_1) = \{ \pi \in G(3, 8) \mid \tau(\pi) = \bar{e}_1 \}
\]

be the fibre over \( \bar{e}_1. \) The group \( G_2 \) acts on \( \text{ASSOC} \) transitively, and we have \( \text{ASSOC} = \{ G(\bar{e}_2 \bar{e}_3 \bar{e}_4) \mid G \in G_2 \}. \)

We can show the isotropy group \( \{ G(\bar{e}_2 \bar{e}_3 \bar{e}_4) = \bar{e}_2 \bar{e}_3 \bar{e}_4 \mid G \in G_2 \} \) is isomorphic to the group \( SO(4); \) then \( \text{ASSOC} \approx G_2/\text{SO}(4). \)

Change the orientation of \( \mathbb{R}^7, \) and let \( \hat{A} = \frac{1}{16} \text{Re} [(\bar{e}_1 - \sqrt{-1} \bar{e}_2)(\bar{e}_3 + \sqrt{-1} \bar{e}_4) \cdots (\bar{e}_7 + \sqrt{-1} \bar{e}_8)(1 + \bar{e}_1 \bar{e}_3 \bar{e}_5 \bar{e}_7)]. \)

Define submanifold \( \text{ASSOC} = \{ \pi \in G(3, 8) \mid \pi \hat{A} = \bar{e}_1 \hat{A} \}, \) which is diffeomorphic to \( \text{ASSOC}. \)

Lemma 7.3  \( V(\text{ASSOC}) = \frac{5}{6} \pi^4. \)

Proof  Let \( \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_8 \) be \( \text{Spin}_7 \) frame fields on \( \mathbb{R}^8, \) and the 1-forms \( \omega^C_B = (\bar{e}_B, \bar{e}_C) \) satisfy (for proof, see [10])

511
\[ \omega_1^2 + \omega_4^8 + \omega_8^8 = 0, \quad \omega_1^8 - \omega_4^8 + \omega_6^8 - \omega_7^8 = 0, \]
\[ \omega_1^4 + \omega_2^8 + \omega_5^8 + \omega_7^8 = 0, \quad \omega_1^5 - \omega_6^8 + \omega_3^7 - \omega_4^8 = 0, \]
\[ \omega_1^6 + \omega_2^5 - \omega_3^8 - \omega_4^8 = 0, \quad \omega_1^7 - \omega_8^5 + \omega_3^6 + \omega_4^6 = 0, \]
\[ \omega_1^8 + \omega_2^3 + \omega_3^9 + \omega_4^7 = 0. \]

Since Spin(7) acts on G(3,8) transitively, G(3,8) is locally generated by \( \hat{e}_2 \hat{e}_3 \hat{e}_4 \). The volume element of G(3,8) is \( dV_{G(3,8)} = \omega_1^2 \omega_3^1 \omega_4^2 \omega_5^3 \omega_6^3 \omega_7^8 \omega_8^8 \).

Note that \( A \) can be represented by Spin(7) frames, that is
\[
A = \frac{1}{16} \text{Re} [(\hat{e}_1 + \sqrt{-1} \hat{e}_2) \cdots (\hat{e}_7 + \sqrt{-1} \hat{e}_8)(1 + \hat{e}_3 \hat{e}_5 \hat{e}_7)].
\]

Let \( \hat{e}_1 = \hat{e}_1 \) be a fixed vector, and \( \hat{e}_1, \hat{e}_2, \ldots, \hat{e}_8 \) be \( G_2 \) frame fields on \( \mathbb{R}^8 \); ASSOC is locally generated by \( \hat{e}_2 \hat{e}_3 \hat{e}_4 \) and
\[
d(\hat{e}_2 \hat{e}_3 \hat{e}_4) = \sum_{i=2}^{3} \sum_{\alpha=5}^{8} \omega_{i\alpha} E_{i\alpha}.
\]
\[
= \omega_3^3 (E_{25} + E_{47}) + \omega_3^6 (E_{26} - E_{48}) + \omega_3^7 (E_{27} - E_{45}) + \omega_3^8 (E_{28} + E_{46})
\]
\[
+ \omega_3^5 (E_{35} + E_{46}) + \omega_3^6 (E_{36} - E_{45}) + \omega_3^7 (E_{37} + E_{48}) + \omega_3^8 (E_{38} - E_{47}).
\]
The metric on ASSOC is
\[
ds^2 = 2(\omega_3^5)^2 + 2(\omega_3^8)^2 - 2\omega_3^5 \omega_3^8 + 2(\omega_3^6)^2 + 2(\omega_3^7)^2 - 2\omega_3^5 \omega_3^7
\]
\[
+ 2(\omega_2^7)^2 + 2(\omega_3^6)^2 + 2\omega_2^7 \omega_3^6 + 2(\omega_2^8)^2 + 2(\omega_3^5)^2 + 2\omega_2^8 \omega_3^5,
\]
with the volume form
\[
dV_{ASSOC} = 9 \omega_2^5 \omega_3^8 \cdots \omega_2^8 \omega_3^8.
\]
The normal space of ASSOC in G(3,8) at \( \hat{e}_2 \hat{e}_3 \hat{e}_4 \) is generated by
\[
E_{21}, E_{31}, E_{41}, E_{25} - E_{47}, E_{26} + E_{48}, E_{27} + E_{45}, E_{36}, E_{28} - E_{46} + E_{35}.
\]
The sphere \( S^7 \) is generated by \( \hat{e}_1 \), and \( dV_S^7 = \omega_1^2 \omega_3^1 \cdots \omega_8^1 \) is the volume form. From
\[
(E_{27} + E_{45} + E_{36}) A = -3 \hat{e}_8 A, \quad E_{21} A = -\hat{e}_2 A, \quad \cdots ,
\]
we can compute the tangent map of \( \tau : G(3,8) \rightarrow S^7 \),
\[
\tau_*(E_{27} + E_{45} + E_{36}) = -3 \hat{e}_8, \quad \tau_*(E_{21}) = -\hat{e}_2, \quad \cdots .
\]
Then we can compute the cotangent map \( \tau^* \) and we have
\[
\tau^* dV_{S^7} = \omega_1^1 \omega_3^1 \omega_4^1 (\omega_2^6 - \omega_3^7 + \omega_4^8)(-\omega_2^5 + \omega_3^8 + \omega_4^7)(\omega_2^8 + \omega_3^5 - \omega_4^6)(\omega_2^7 + \omega_3^6 + \omega_4^5),
\]
512
and
\[ dV_{G(3,8)} = -\frac{1}{9} \tau^* dV_{S^7} \cdot dV_{-1}(e_1). \]

From \( V(G(3,8)) = \frac{2}{7\pi^8}, \ V(S^7) = \frac{1}{3} \pi^4 \), we have \( V(\text{ASSOC}) = \frac{6}{7} \pi^4 \). \( \square \)

It is easy to see that \( \text{ASSOC} \) and \( \text{ASSOC} \) are submanifolds of \( G(3,7) \). In the following lemma \( E = E(3,7), F = F(3,7) \).

**Lemma 7.4** \( \int_{\text{ASSOC}} p_1^2(E) = \int_{\text{ASSOC}} p_1(E)e(F) = 1 \), \( \int_{\text{ASSOC}} p_1^2(E) = -\int_{\text{ASSOC}} p_1(E)e(F) = 1 \). Then \( \text{ASSOC}, \text{ASSOC} \) and \( p_1^2(E), p_1(E)e(F) \) are generators of \( H_8(G(3,7)) \) and \( H^8(G(3,7)) \), respectively. Furthermore, we have
\[ [G(2,6)] = [\text{ASSOC}] + [\text{ASSOC}]. \]

**Proof** From
\[ *\omega^6_2 \omega^6_3 \omega^6_4 |_{\text{ASSOC}} = \frac{1}{9} dV_{\text{ASSOC}}, \]
\[ *\omega^6_2 \omega^6_3 \omega^6_4 |_{\text{ASSOC}} = \frac{1}{9} dV_{\text{ASSOC}}, \]
\[ *\omega^6_2 \omega^6_3 \omega^6_4 |_{\text{ASSOC}} = 0, \]
we have \( \sum_{\alpha, \beta} *\omega^6_2 \omega^6_3 \omega^6_4 |_{\text{ASSOC}} = 8 \cdot \frac{1}{9} dV_{\text{ASSOC}}, \)
\[ *p_1(E)|_{\text{ASSOC}} = \frac{1}{4\pi^2} \cdot 3 \cdot 8 \cdot \frac{1}{9} dV_{\text{ASSOC}}. \]

Then by \( *p_1(E) = \frac{4}{7} \pi^2 p_1(E)e(F) \), we have
\[ \int_{\text{ASSOC}} p_1(E)e(F) = \int_{\text{ASSOC}} \frac{5}{4\pi^2} * p_1(E) = 1. \]

The proof of \( \int_{\text{ASSOC}} p_1^2(E) = 1 \) is similar.

Change the orientation of \( \mathbb{R}^7 \), and we have Euclidean space \( \mathbb{R}^7 \). Let \( \vec{E}, \vec{F} \to G(3,7) \) be canonical vector bundles with respect to \( \mathbb{R}^7 \). It is easy to see that \( \vec{E} = E \), but the orientations of \( \vec{F} \) and \( F \) are different. This shows
\[ \int_{\text{ASSOC}} p_1^2(E) = 1, \ \int_{\text{ASSOC}} p_1(E)e(F) = -1. \]
\[ [G(2,6)] = [\text{ASSOC}] + [\text{ASSOC}] \]
follows from \( \int_{G(2,6)} p_1^2(E) = 2 \) and \( \int_{G(2,6)} p_1(E)e(F) = 0. \) \( \square \)
Theorem 7.5 (1) $\frac{1}{2}(p_1(E) + e(F)), \frac{1}{2}(p_1(E) - e(F))$ are 2 generators of $H^4(G(3, 8), \mathbb{Z})$. Their Poincaré duals are $[ASSOC]$ and $[ASSOC]$ respectively;

(2) $\frac{1}{2}(p_1(E) e(F) + e^2(F)), \frac{1}{2}(p_1(E) e(F) - e^2(F)) \in H^8(G(3, 8), \mathbb{Z})$ are generators and their Poincaré duals are $[\mathbb{CP}^2], [\overline{\mathbb{CP}}^2]$ respectively;

(3) $\frac{1}{2}(p_1(E) e(F) + e^2(F)), \frac{1}{2}(p_1(E) e(F) - e^2(F))$ and $[ASSOC], [\overline{ASSOC}]$ are dual basis with respect to the universal coefficients theorem.

The proof is similar to Theorem 5.5.

8. The case of $G(3, 8)$

The Poincaré polynomial of Grassmann manifold $G(3, 8)$ is

$$p_t(G(3, 8)) = (1 + t^4 + t^8)(1 + t^7) = 1 + t^4 + t^7 + t^8 + t^{11} + t^{15}.$$ 

Let $E = E(3, 8), F = F(3, 8)$. By $\int_{\mathbb{CP}^2} p_1(E) = 1, \int_{ASSOC} p^2(E) = 1$ we know that

$$[\mathbb{CP}^2] \in H_4(G(3, 8), \mathbb{Z}), [ASSOC] \in H_8(G(3, 8), \mathbb{Z}),$$

$$p_1(E) \in H^4(G(3, 8), \mathbb{Z}), p^2(E) \in H^8(G(3, 8), \mathbb{Z})$$

are all generators. By Theorem 3.1, to understand the structure of the homology groups of dimension 7, 11, we need to compute the Poincaré duals of $p_1(E), p^2(E)$.

It is not difficult to compute

$$a = (p_1(E), p_1(E)) = \frac{15}{2\pi^4} V(G(3, 8)) = \frac{1}{3} \pi^4.$$ 

Then $\frac{1}{2} * p_1(E) \in H^{11}(G(3, 8), \mathbb{Z})$ is a generator, and we look for a submanifold $M$ such that $\int_M \frac{1}{2} * p_1(E) = 1$.

In §7 we define fibre bundle $\tau: G(3, 8) \to S^7, \tau(\pi) = \nu$ defined by $\pi A = \nu A$. As $v \perp \pi, \nu\pi \in G(4, 8)$ and $\nu\pi A = -A$. Let $\text{CAY} = \{ \pi \in G(4, 8) | \pi A = -A \}$, called the Cayley submanifold of $G(4, 8)$. Then we have a fibre bundle $\mu: G(3, 8) \to \text{CAY}$, $\mu \mapsto \nu\pi$, with fibre $S^3 = G(3, 4)$. Let $ASSOC = \{ \tilde{e}_1 e_2 e_3 e_4 | e_2 e_3 e_4 A = \tilde{e}_1 A \}$, where $\tilde{e}_1 = (1, 0, \ldots, 0)$. Then $M = \mu^{-1}(ASSOC)$ is an 11-dimensional submanifold of $G(3, 8)$.

Let $e_1, e_2, \ldots, e_8$ be $G_2$ frame fields on $\mathbb{R}^8$, and $ASSOC$ be generated by $\tilde{e}_1 e_2 e_3 e_4$. Represent the elements of $\mu^{-1}(\tilde{e}_1 e_2 e_3 e_4)$ by $\tilde{e}_2 \tilde{e}_3 \tilde{e}_4, \tau(\tilde{e}_2 \tilde{e}_3 \tilde{e}_4) = \tilde{e}_1$; then

$$\tilde{e}_1 \tilde{e}_2 \tilde{e}_3 \tilde{e}_4 = \tilde{e}_1 e_2 e_3 e_4.$$ 

Let $d(\tilde{e}_2 \tilde{e}_3 \tilde{e}_4) = \sum_{i=2}^4 \omega_i^2 \tilde{E}_i + \sum_{i=2}^8 \omega_i^a \tilde{E}_i^a, \ d\omega_i^a = \sum_{i=2}^8 (\omega_i^a)^2$ be the metric on $M$.

Let $\tilde{e}_1 = \lambda_1 \tilde{e}_1 + \sum \lambda_i e_i, \ d\tilde{e}_1 = \sum_{i=2} \tilde{E}_i + 8 \tilde{E}_i^a, \ d(e_2 e_3 e_4) = \sum \omega_i^a E_i^a$. It is easy to see

$$\tilde{\omega}_1^a = \sum_{i=2}^4 \lambda_i \omega_i^a, \sum_{i=2}^8 (\tilde{\omega}_1^i)^2$$ is the metric on the fibres of $\mu: M \to ASSOC$ and $(d(e_2 e_3 e_4), d(e_2 e_3 e_4)) = \sum (\omega_i^a)^2$. 

514
is the metric on $ASSOC$. From \(\langle \bar{e}_1 E_{i\alpha}, e_\beta \bar{e}_2 \bar{e}_3 \bar{e}_4 \rangle = \lambda_i \delta_{\alpha\beta}\) we have
\[
\langle \bar{e}_1 (d e_2 e_3 e_4), (d \bar{e}_1) \bar{e}_2 \bar{e}_3 \bar{e}_4 \rangle = \sum \omega_i^\alpha e_1 E_{i\alpha} - \lambda_j \omega_i^\beta e_\beta \bar{e}_2 \bar{e}_3 \bar{e}_4 = \sum \lambda_i \omega_i^\alpha \sum_j \lambda_j \omega_j^\beta.
\]

Hence
\[
\sum (\bar{\omega}_i^\alpha)^2 = \langle \bar{e}_1 d(\bar{e}_2 \bar{e}_3 \bar{e}_4), \bar{e}_1 d(\bar{e}_2 \bar{e}_3 \bar{e}_4) \rangle
= \langle \bar{e}_1 d(e_2 e_3 e_4) - (d \bar{e}_1) \bar{e}_2 \bar{e}_3 \bar{e}_4, \bar{e}_1 d(e_2 e_3 e_4) - (d \bar{e}_1) \bar{e}_2 \bar{e}_3 \bar{e}_4 \rangle
= \sum_{i, \alpha} (\omega_i^\alpha)^2 + \sum_{i} (\bar{\omega}_i^\alpha)^2 - 2 \sum_{i} (\lambda_i \omega_i^\alpha)^2
= \sum_{i, \alpha} (\omega_i^\alpha)^2 - \sum_{i} (\lambda_i \omega_i^\alpha)^2.
\]

Then the metric on $M$ can be represented by
\[
d_{\mathcal{S}_M}^2 = \sum_{i, \alpha} (\omega_i^\alpha)^2 - \sum_{i} (\lambda_i \omega_i^\alpha)^2 + \sum_{i=2}^{4} (\bar{\omega}_i^\alpha)^2.
\]

For fixed $\bar{e}_2 \bar{e}_3 \bar{e}_4$ of $M$, we can choose $G_2$ frame fields $e_1, e_2, \ldots, e_8$ such that $\bar{e}_1 = \lambda_1 e_1 + \lambda_2 e_2$. By
\[
\omega_4^5 = -\omega_2 - \omega_3^6, \ \omega_4^6 = \omega_2 + \omega_3^5, \ \omega_4^7 = \omega_2^5 - \omega_3^8, \ \omega_4^8 = -\omega_2^6 + \omega_3^7,
\]
we have
\[
dV_M = \frac{1}{9} (1 + 2 \lambda_1^2)^2 \omega_4^1 \omega_4^2 \omega_4^3 \omega_4^4 \mu^* dV_{ASSOC}.
\]

We can show that $\int_{G^{3 \times}} (1 + 2 \lambda_1^2)^2 dV_{G^{3 \times}} = 5 \pi^2$; then $V(M) = \frac{2}{3} \pi^6$.

**Lemma 8.1** $\int_M \frac{3}{4} \ast p_1(E) = 1$. Then $[M]$ and $\frac{3}{4} \ast p_1(E)$ are dual generators of $H_8(G(3, 8), \mathbb{Z})$, $H^8(G(3, 8), \mathbb{Z})$ respectively.

**Proof** By Theorem 3.1, the integration $\int_M \frac{1}{a} \ast p_1(E)$ is an integer. On the other hand, $2 \pi^2 p_1(E)$ is a calibration on $G(3, 8)$ with comass $\frac{1}{4}$; see [7]; then we have
\[
\left| \int_M \frac{1}{a} \ast p_1(E) \right| \leq \frac{2}{3a \pi^2} V(M) = \frac{4}{3}.
\]

We need to show that $\int_M \frac{1}{a} \ast p_1(E) \neq 0$.

Let $i$: $CAY \to G(4, 8)$ be the inclusion, $\mu$: $G(3, 8) \to CAY$ be a sphere bundle associated with the induced vector bundle $i^* E(4, 8) \to CAY$, and $e(i^* E(4, 8)) = i^* e(E(4, 8)) \in H^4(CAY, \mathbb{Z})$ be the Euler class. The induced bundle $(i \circ \mu)^* E(4, 8) \to G(3, 8)$ has a nonzero section, $(i \circ \mu)^* e(E(4, 8)) = 0$. By Gysin sequence for the sphere bundle $G(3, 8) \to CAY$, we can show $\mu_*(\frac{1}{a} \ast p_1(E)), (i^* e(E(4, 8)))^2$ are 2 generators of $H^8(CAY)$, where $\mu_*$: $H^{11}(G(3, 8)) \to H^8(CAY)$ is the integration along the fibre. From $e^2(E(4, 8)) = p_2(E(4, 8))$, we have $e^2(E(4, 8))|_{G(3, 7)} = 0$; then $\int_{ASSOC} (i^* e(E(4, 8)))^2 = 0$. If we also have $\int_{ASSOC} \mu_*(\frac{1}{a} \ast p_1(E)) = 0$, then
We have $[ASSOC] = 0$ in $H_8(CAY)$. This contradicts the fact that $i_*(ASSOC) \neq 0$. Then with a suitable choice of orientation on $M$, we have

$$\int_M \frac{1}{a} * p_1(E) = \int_{ASSOC} \mu_*(\frac{1}{a} * p_1(E)) = 1.$$ 

Finally, we study the $H_7(G(3,8))$ and $H_7^7(G(3,8))$. Let $I, J, K$ be the quaternion structures on $\mathbb{R}^8 = \mathbb{H}^2$ and $Sp(2)$ the symplectic group. As we know $Sp(2)$ is a subgroup of $SU(4) \subset SO(8)$; hence $Sp(2)$ is a subgroup of $Spin_7$. Let $f : S^7 \to G(3,8)$, and $f(v) = IvJuKv$. By $I\tilde{e}_1J\tilde{e}_1K\tilde{e}_1A = \tilde{e}_2\tilde{e}_3\tilde{e}_4A = \tilde{e}_1A$ and $Sp(2)$ acting on $S^7$ transitively, we have $IvJuKvA = vA$ for any $v \in S^7$, $\tau(f(v)) = v$.

**Lemma 8.2** $b = (p_1^2(E), p_1^2(E)) = \frac{5}{2}$.

**Proof** By computation, we have

$$p_1^2(E) = \frac{1}{2\pi^4} \left( \sum_{i<j} \sum_{a<\beta<\gamma<\tau} 3 \omega_i^a \omega_j^\beta \omega_j^\beta \omega_j^\gamma \omega_j^\gamma \right)$$

$$+ \sum_{j \neq k} \sum_{\alpha<\beta<\gamma<\tau} \omega_i^\alpha \omega_j^\beta \omega_j^\beta \omega_j^\gamma \omega_j^\gamma \omega_k^\gamma \omega_k^\gamma \omega_k^\gamma,$$

$i, j, k = 2, 3, 4, \alpha, \beta, \gamma, \tau = 1, 5, 6, 7, 8.$

Then

$$b = (p_1^2(E), p_1^2(E)) = \frac{1}{4\pi^8} (9 \cdot 3 \cdot C_5^4 + 3 \cdot C_5^2 \cdot C_3^2) V(G(3,8)) = \frac{5}{2}.$$

**Lemma 8.3** $\frac{1}{2} \int_{S^7} f^* | p_1^2(E) = 1$. Then $[f(S^7)]$ and $\frac{2}{5} * p_1^2(E)$ are dual generators of $H_7(G(3,8), \mathbb{Z})$ and $H_7^7(G(3,8), \mathbb{Z})$ respectively.

**Proof** Let $e_1, e_2 = Ie_1, e_3 = Je_1, e_4 = Ke_1, e_5, e_6 = Ie_5, e_7 = Je_5, e_8 = Ke_5$ be $Sp(2)$ frame fields on $\mathbb{R}^8$, $f(e_1) = e_2e_3e_4$. The 1 forms $\omega_i = (de_i, e_\alpha)$, $i = 1, 2, 3, 4, \alpha = 5, 6, 7, 8$, satisfy

$$\omega_1 = \omega_2 = \omega_3 = \omega_8, \quad \omega_1 = -\omega_2 = -\omega_3 = \omega_8,$$

$$\omega_1 = \omega_2 = -\omega_3 = -\omega_4, \quad \omega_1 = -\omega_2 = \omega_3 = -\omega_4.$$

We have

$$*\omega_1^5 \omega_2^5 \omega_3^5 \omega_4^5 \omega_5^5 \omega_6^5 \omega_7^5 \omega_8^5 |_{f(S^7)} = \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 |_{f(S^7)} = \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 |_{f(S^7)} = 0,$$

$$*\omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 |_{f(S^7)} = -\omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 |_{f(S^7)} = 0,$$

$$*\omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 |_{f(S^7)} = \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 |_{f(S^7)} = dV_{S^7},$$

$$*\omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 |_{f(S^7)} = -\omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 |_{f(S^7)} = 0,$$

$$*\omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 |_{f(S^7)} = \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 |_{f(S^7)} = dV_{S^7},$$

$$*\omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 |_{f(S^7)} = -\omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 \omega_1^5 |_{f(S^7)} = 0, \ldots$$
Then
\[ \frac{1}{5} f^* \rho_1^2(E) = \frac{2}{5} \cdot \frac{1}{2\pi^4} (3 \cdot 3 + 3 \cdot 2) dV_{S^7} = \frac{3}{\pi^4} dV_{S^7}, \]
\[ \int_{S^7} f^* \rho_1^2(E) = 1. \]

Obviously, we also have \( \int_{f(S^7)} \tau^* \frac{3}{\pi^4} dV_{S^7} = 1; \) then
\[ [\tau^* \frac{3}{\pi^4} dV_{S^7}] = \frac{2}{5} \rho_1^2(E) \in H^7(G(3,8)). \]

**Theorem 8.4** (1) The Poincaré dual of \( p_1(E) \) is \([M]\) and the Poincaré dual of \( \frac{3}{\pi^4} \rho_1(E) \) is \([CP^2]\); 
(2) The Poincaré dual of \( p_2(F) = \rho_1^2(E) \) is \([f(S^7)]\) and the Poincaré dual of \( \frac{2}{5} \rho_1^2(E) \) is \([ASSOC]\).

**9. The case of** \( G(4,8) \)

Let \( E = E(4,8), \ F = F(4,8) \) be canonical vector bundles on Grassmann manifold \( G(4,8) \). We have
\[ p_1(G(4,8)) = 1 + 3t^4 + 4t^8 + 3t^{12} + t^{16}, \]
\[ e(E)t(F) = 0, \ p_1(E) = -p_1(F), \ p_2(E) = e^2(E), \ p_2(F) = e^2(F), \]
\[ p_1^2(E) = p_2(E) + p_2(F), \ p_1(E)p_2(E) = p_1(E)p_2(F) = \frac{1}{2}p_1^3(E), \]
\[ p_1^2(E)e(E) = e^3(E), \ p_1^2(F)e(E) = e^3(F). \]

By the method used in §4, we can show \( e(E \otimes F) = 6e^4(E) = 6e^4(F) \). Then by \( \int_{G(4,8)} e(E \otimes F) = \chi(G(4,8)) = 12 \), we have
\[ \int_{G(4,8)} e^4(E) = \int_{G(4,8)} e^4(F) = 2. \]

We first study the cases of 4 and 12. Under the star operator \( *: G(4,8) \to G(4,8) \), \( *CP^2 \) is a submanifold of \( G(4,8) \). From \( CP^2 = C_{\mathbb{C}}(1,3) \subset G(4,8) \), we have \( *CP^2 = C_{\mathbb{C}}(2,3) \). The following table computes the integration of the characteristic classes on the submanifolds of \( G(4,8) \).

<table>
<thead>
<tr>
<th></th>
<th>( CP^2 )</th>
<th>( *CP^2 )</th>
<th>( G(2,4) )</th>
<th>( G(1,5) )</th>
<th>( G(4,5) )</th>
<th>( CP^2 )</th>
<th>( SP^1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e(E) )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>( e(F) )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( p_1(E) )</td>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Note that \( \det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} = -2 \), as proof of Theorem 5.5, we can show \( e(E), e(F), p_1(E) \in H^4(G(4,8), \mathbb{Z}) \) or \( CP^2, *CP^2, G(2,4) \in H_4(G(4,8), \mathbb{Z}) \) are the generators.
By Proposition 2.2, $V(G(4, 8)) = \frac{8}{135} \pi^8$ and we can compute

$$a = (e(E), e(E)) = (e(F), e(F)) = \frac{1}{2} (p_1(E), p_1(E)) = \frac{4}{15} \pi^4,$$

$$(e(E), e(F)) = (e(E), p_1(E)) = (e(F), p_1(E)) = 0.$$ 

In the last section we have Cayley submanifold $CAY = \{ \pi \in G(4, 8) \mid \pi A = -A \}$ of $G(4, 8)$. The Lie group $Spin_7$ acts on $CAY$ transitively. Let $e_1, e_2, \ldots, e_8$ be $Spin_7$ frame fields on $\mathbb{R}^8$ and then $CAY$ is generated by $e_1e_2e_3e_4$. By the equations listed in the proof of Lemma 7.3, we have

$$d(e_1e_2e_3e_4) = \sum \omega_i^a F_{i\alpha}$$

$$= \omega_1^5(E_{15} + E_{48}) + \omega_1^6(E_{16} + E_{47}) + \omega_1^7(E_{17} - E_{46}) + \omega_1^8(E_{18} - E_{45})$$

$$+ \omega_2^5(E_{25} + E_{47}) + \omega_2^6(E_{26} - E_{48}) + \omega_2^7(E_{27} - E_{45}) + \omega_2^8(E_{28} + E_{46})$$

$$+ \omega_3^5(E_{35} + E_{46}) + \omega_3^6(E_{36} - E_{45}) + \omega_3^7(E_{37} + E_{48}) + \omega_3^8(E_{38} - E_{47}).$$

Then the induced metric is

$$ds_{CAY}^2 = (\omega_1^5 - \omega_2^5)^2 + (\omega_1^6 + \omega_3^7)^2 + (\omega_2^6 - \omega_3^7)^2$$

$$+ (\omega_1^7 - \omega_2^7)^2 + (\omega_1^6 - \omega_2^8)^2 + (\omega_2^5 - \omega_3^8)^2$$

$$+ (\omega_1^8 - \omega_2^8)^2 + (\omega_1^7 - \omega_3^8)^2 + (\omega_2^6 + \omega_3^7)^2$$

$$+ (\omega_1^8 + \omega_2^7)^2 + (\omega_1^8 - \omega_3^8)^2 + (\omega_2^8 + \omega_3^7)^2.$$

By $(\omega_1^5 - \omega_2^5)(\omega_1^6 + \omega_3^7)(\omega_2^6 - \omega_3^7) = 2\omega_1^5\omega_2^5\omega_3^7$, \ldots, we get

$$dV_{CAY} = 16\omega_1^5\omega_2^5\omega_3^7 \cdots \omega_1^8\omega_2^8\omega_3^8.$$

As shown in [7], $2\pi^2 p_1(E)$ is a calibration on $G(4, 8)$ with comass $\frac{3}{2}$ and $CAY$ is a calibrated submanifold of $2\pi^2 * p_1(E)$; then

$$2\pi^2 * p_1(E)|_{CAY} = \frac{3}{2} dV_{CAY}.$$

By triality transformation, we can show that $CAY$ is isometric to $G(3, 7)$; then $V(CAY) = V(G(3, 7)) = \frac{16\pi^6}{45}$; see [14]. This shows

$$\int_{CAY} \frac{1}{2\pi} * p_1(E) = \int_{CAY} \frac{45}{32\pi^6} dV_{CAY} = \frac{1}{2}.$$ 

By Theorem 3.1, $e(E), e(F), p_1(E)$ cannot be the generators of $H^4(G(4, 8), \mathbb{Z})$. We have proved

**Lemma 9.1** $CP^2, *CP^2, G(2, 4)$ are the generators of $H_4(G(4, 8), \mathbb{Z})$, and the dual generators of $H^4(G(4, 8), \mathbb{Z})$ are $e(F), e(E), \frac{1}{2} (p_1(E) + e(E) - e(F))$, respectively.

The inner product of $e(E), e(F), \frac{1}{2} (p_1(E) + e(E) - e(F))$ forms a matrix

$$A = \frac{4\pi^4}{15} \begin{pmatrix}
0 & 1 & \frac{1}{2} \\
1 & 0 & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & 1
\end{pmatrix}, \quad A^{-1} = \frac{15}{4\pi^4} \begin{pmatrix}
\frac{3}{2} & -\frac{1}{2} & 1 \\
-\frac{1}{2} & \frac{3}{2} & 1 \\
1 & 1 & 2
\end{pmatrix}.$$ 

518
Lemma 9.2 (1) $e(E) = \frac{2\pi^4}{15} e^3(E), e(F) = \frac{2\pi^4}{15} e^3(F), p_1(E) = \frac{2\pi^4}{15} p_1^3(E)$; 
(2) $\frac{1}{2} e^3(E) - \frac{1}{4} p_1^3(E), \frac{1}{2} e^3(F) + \frac{1}{4} p_1^3(E), \frac{1}{2} p_1^3(E) \in H^{12}(G(4,8), \mathbb{Z})$ are the generators.

**Proof** $e(E), e(F), p_1(E)$ and $e^3(F), p_1^3(E)$ are 2 generators of the cohomology group $H^{12}(G(4,8))$, and they are all the harmonic forms on $G(4,8)$. Then $e^3(E), e^3(F), p_1^3(E)$ can be represented by $e(E), e(F), p_1(E)$. Assuming $e^3(E) = \lambda e(E) + \mu e(F) + \nu p_1(E)$, by $e(F) \wedge e(E) = 0$, $e(F) \wedge p_1(E) = 0$, we have $\mu = \nu = 0$.

$$2 = \int_{G(4,8)} e(E) \wedge e^3(E) = \lambda \int_{G(4,8)} e(E) \wedge e(F) = \lambda \frac{4\pi^4}{15}.$$ 

Then $e(E) = \frac{2\pi^4}{15} e^3(E)$. The other 2 equalities can be proved similarly.

Then

$$(e(E), e(F), \frac{1}{2}(p_1(E) + e(E) - e(F)))A^{-1}$$

$$= (\frac{1}{2} e^3(E) - \frac{1}{2} p_1^3(E)), \frac{1}{2} (e^3(F) + \frac{1}{2} p_1^3(E)), \frac{1}{2} p_1^3(E)).$$

\[ \Box \]

**Lemma 9.3** The following table computes the integration of the characteristic classes on the submanifolds of $G(4,8)$ in dimension 12.

<table>
<thead>
<tr>
<th></th>
<th>$G(4,7)$</th>
<th>$G(3,7)$</th>
<th>CAY</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^3(E)$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$e^3(F)$</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$p_1^3(E)$</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

**Proof** The second column follows from $\int_{G(4,7)} e^3(E) = \int_{G(3,7)} e^3(F) = 2$ and $e(F)|_{G(4,7)} = 0, p_1^3(E) = 2p_1(E)p_2(F) = 2p_1(E)e^2(F)$. The third column can be proved similarly. For the fourth column, we have proved $\int_{CAY} \frac{1}{2} \ast p_1(E) = \frac{1}{2};$ then $\int_{CAY} p_1^3(E) = 2$. By computing $e(E)|_{CAY}, e(F)|_{CAY}$, we can show $\int_{CAY} e^3(E) = -1$ and $\int_{CAY} e^3(F) = 1$.

\[ \Box \]

**Theorem 9.4** (1) $e(E), e(F), \frac{1}{2}(p_1(E) + e(E) - e(F)) \in H^4(G(4,8), \mathbb{Z})$ are the generators, and their dual generators are $[CP^2], [\ast CP^2], [G(2,4)] \in H_4(G(4,8), \mathbb{Z});$

(2) $\frac{1}{2} e^3(E), \frac{1}{2} e^3(F), \frac{1}{2} p_1^3(E)$ and $[G(4,7)], [G(3,7)], [CAY]$ are the generators of $H^{12}(G(4,8), \mathbb{Z})$ and $H_{12}(G(4,8), \mathbb{Z})$, respectively;

(3) The Poincaré duals of $e(E), e(F), \frac{1}{2}(p_1(E) + e(E) - e(F))$ are

$[G(4,7)], [G(3,7)], [CAY] + [G(4,7)] - [G(3,7)]$

respectively.

519
By Lemma 9.2, $\frac{1}{2}e^3(E) - \frac{1}{4}p_1(E)$, $\frac{1}{2}e^3(F) + \frac{1}{4}p_1(E)$, $\frac{1}{2}p_1^3(E)$ are the generators of $H^{12}(G(4,8), \mathbb{Z})$. Then $\frac{1}{2}e^3(E), \frac{1}{2}e^3(F), \frac{1}{2}p_1^3(E)$ are also the generators of $H^{12}(G(4,8), \mathbb{Z})$.

By Theorem 3.1, we can compute the Poincaré duals of $e$ to verify this equation. By Theorem 9.4, $\frac{1}{2}(p_1(E) + e(E) - e(F))e(E) = \frac{1}{2}(p_1(E)e(E) + e^2(E))$ and $\frac{1}{2}(p_1(E)e(F) - e^2(E))$, $\frac{1}{2}(p_1(E)e(F) + e^2(F))$ are integral cocycles. The submanifolds $ASSOC, \sim ASSOC$ defined in §7 are also the submanifolds of $G(4,8)$; then $*ASSOC, \sim *ASSOC$ are submanifolds of $G(4,8)$. The following table can be proved by Lemma 7.4.

<table>
<thead>
<tr>
<th></th>
<th>$ASSOC$</th>
<th>$ASSOC$</th>
<th>$*ASSOC$</th>
<th>$*ASSOC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}e^3(F) + \frac{1}{2}p_1(E)e(F)$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{2}e^3(F) - \frac{1}{2}p_1(E)e(F)$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{2}e^3(F) + \frac{1}{2}p_1(E)e(E)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{2}e^3(F) - \frac{1}{2}p_1(E)e(E)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Theorem 9.5 The characteristic classes

$$\frac{1}{2}e^2(F) + \frac{1}{2}p_1(E)e(F), \quad \frac{1}{2}e^2(F) - \frac{1}{2}p_1(E)e(F),$$

$$\frac{1}{2}e^2(E) + \frac{1}{2}p_1(E)e(E), \quad \frac{1}{2}e^2(E) - \frac{1}{2}p_1(E)e(E)$$

are the generators of $H^8(G(4,8), \mathbb{Z})$. Their Poincaré duals are

$[ASSOC], \ [\sim ASSOC], \ [*ASSOC], \ [*\sim ASSOC]$ respectively.

Proof To see that the Poincaré dual of $\xi = \frac{1}{2}(e^2(F) + p_1(E)e(F))$ is $ASSOC$, we want to show that for any $\eta \in H^8(G(4,8))$ we have $\int_{G(4,8)} \xi \wedge \eta = \int_{ASSOC} \eta$. We can take $\eta = \frac{1}{2}(e^2(F) \pm p_1(E)e(F))$, $\frac{1}{2}(e^2(E) \pm p_1(E)e(E))$ to verify this equation.

By $\mathbb{R}^8 = \mathbb{R}^3 \oplus \mathbb{R}^5$, we see the product Grassmann $G(2,3) \times G(2,5), G(1,3) \times G(3,5)$ can imbedded in $G(4,8)$ and we have

<table>
<thead>
<tr>
<th></th>
<th>$G(4,6)$</th>
<th>$G(2,6)$</th>
<th>$G(2,3) \times G(2,5)$</th>
<th>$G(1,3) \times G(3,5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^2(E)$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$e^2(F)$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$p_1(E)e(E)$</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>$p_1(E)e(F)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

Then

$G(4,6), G(2,6), G(2,3) \times G(2,5), G(1,3) \times G(3,5) \in H_8(G(4,8), \mathbb{R})$
and
\[ e^2(E), \ e^2(F), \ p_1(E)e(E), \ p_1(E)e(F) \in H^8(G(4,8), \mathbb{R}) \]
are also the generators.

As an application, we consider the immersion \( f: M \to \mathbb{R}^8 \) of a compact oriented 4-dimensional manifold, with \( g: M \to G(4,8) \) as its Gauss map. We have
\[
g_*(M) = \frac{1}{2} \chi(M)(G(4,5)) + \lambda[G(1,5)] + \frac{3}{2} \tau(M)[G(2,4)],
\]
where \( \lambda = \frac{1}{2} \int_M g^* e(F(4,8)) \) and \( \tau(M) = \frac{1}{3} \int_M g^* p_1(E(4,8)) = \frac{1}{3} \int_M p_1(TM) \) is the signature of \( M \). \( \lambda = 0 \) if \( f \) is an imbedding.

If \( g \) is the Gauss map of immersion \( M \) in \( \mathbb{R}^7 \) or \( \mathbb{R}^6 \), we have
\[
g_*(M) = \frac{1}{2} \chi(M)(G(4,5)) + \frac{3}{2} \tau(M)[G(2,4)].
\]

10. The cohomology groups on \( ASSOC \)

The submanifold \( ASSOC \approx G_2/SO(4) \) of Grassmann manifold \( G(3,7) \) is important in the theory of calibrations; see [7,9]. In [6] Borel and Hirzebruch studied the characteristic classes on homogeneous spaces, and they computed the cohomology of \( ASSOC \). In what follows we use Gysin sequence to study the cohomology of \( ASSOC \).

As \( G(2,7) \) and \( G(3,7) \) be Grassmann manifolds on \( \mathbb{R}^7 \subset \mathbb{R}^8 \) generated by \( \bar{e}_2, \cdots, \bar{e}_8 \), and \( S^6 \subset \mathbb{R}^7 \) the unit sphere. There is a fibre bundle \( \tau_1: G(2,7) \to S^6 \) defined by \( \pi A = \bar{e}_1vA, \ \tau_1(\pi) = v \), where \( A \in \mathcal{C}^\mathcal{C} \) is defined in \( \S 7 \). For any \( G \in G_2 \), we have the following commutative diagram
\[
\begin{array}{ccc}
G(2,7) & \xrightarrow{G} & G(2,7) \\
\downarrow \tau_1 & & \downarrow \tau_1 \\
S^6 & \xrightarrow{G} & S^6 .
\end{array}
\]
The fibre \( \tau_1^{-1}(\bar{e}_2) = \{ v \wedge Jv \mid v \in S^6, \ v \perp \bar{e}_2 \} \approx \mathbb{C}P^2 \); see [10].

Then for any \( \pi \in G(2,7), \ v = \tau_1(\pi), \ v \wedge \pi \in ASSOC \). This defines the map
\[
\tau_2: G(2,7) \to ASSOC, \ \pi \mapsto v \wedge \pi.
\]
For any \( e_2e_3e_4 \in ASSOC, \ e_2e_3e_4A = \bar{e}_1A \), then \( \tau_2(e_3e_4) = e_2e_3e_4 \). This shows

Lemma 10.1 \( \tau_2: G(2,7) \to ASSOC \) is a fibre bundle with fibre \( G(2,3) = S^2 \).

Let \( i: ASSOC \to G(3,7) \) be an inclusion. It is easy to see \( G(2,7) \) is isomorphic to the sphere bundle \( S(\bar{E}) = \{ v \in \bar{E}, \ |v| = 1 \} \) of the induced bundle \( \bar{E} = i^*E(3,7) \). Let \( e(E(3,7)) \in H^3(G(3,7), \mathbb{Z}) \) be the Euler class of \( E(3,7), \ 2e(E(3,7)) = 0 \); see [13] p. 95–103. Then \( e(\bar{E}) = i^*e(E(3,7)) \in H^3(ASSOC, \mathbb{Z}) \) is the Euler class of the induced bundle \( \bar{E} \). There is a Gysin exact sequence for the sphere bundle \( G(2,7) \to ASSOC \),
\[
\to H^*(ASSOC) \xrightarrow{\tau_2^*} H^*(G(2,7)) \xrightarrow{\tau_2} H^{q-2}(ASSOC) \\
\xrightarrow{\wedge \bar{e}(\bar{E})} H^{q+1}(ASSOC) \xrightarrow{\tau_2^*} H^{q+1}(G(2,7)) \to ,
\]
521
where \( \tau_{2*} \) is the integration along the fibre. The coefficients of the cohomology groups can be \( \mathbb{R}, \mathbb{Z} \), or \( \mathbb{Z}_2 \).

**Lemma 10.2** \( \mathcal{e}(\mathcal{E}) = i^*\mathcal{e}(E(3,7)) \neq 0 \).

**Proof** The map \( \tau_{2*} : H^q(G(2,7), \mathbb{Z}) \to H^{q-2}(\text{ASSOC}, \mathbb{Z}) \) is the integration along the fibre. Let \( \bar{e}_1, e_2, e_3, \cdots, e_8 \) be \( G_2 \) frame fields, and \( G(2,7) \) generated by \( e_3e_4 \) and \( \tau_2(e_3e_4) = e_2e_3e_4 \). Then the Euler class of vector bundle \( E(2,7) \) can be represented by

\[
\mathcal{e}(E(2,7)) = \frac{1}{2\pi} \omega_3^g \wedge \omega_4^g + \frac{1}{2\pi} \sum_{i=5}^{8} \omega_i^g \wedge \omega_4^g
\]

and \( \omega_3^g \wedge \omega_4^g \) is the volume element of the fibre at \( e_4 \). Then \( \tau_{2*}(\mathcal{e}(E(2,7))) = 2 \).

By Gysin sequence, the map \( \tau_{2*} : H^2(G(2,7)) \to H^0(\text{ASSOC}) \) is surjective if \( \mathcal{e}(\mathcal{E}) = 0 \). This contradicts the fact that \( \tau_{2*}(\mathcal{e}(E(2,7))) = 2 \) and \( \mathcal{e}(E(2,7)) \in H^2(G(2,7), \mathbb{Z}) \) is a generator.

Then \( \mathcal{e}(E(3,7)) \in H^3(G(3,7), \mathbb{Z}) \) is a nonzero torsion.

**Theorem 10.3** The cohomology groups of \( \text{ASSOC} \) are

\[
H^q(\text{ASSOC}, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & q \neq 1, 7, \\ 0, & q = 1, 7; \end{cases}
\]

\[
H^q(\text{ASSOC}, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0, 4, 8, \\ \mathbb{Z}_2, & q = 3, 6, \\ 0, & q = 1, 2, 5, 7; \end{cases}
\]

\[
H^q(\text{ASSOC}, \mathbb{R}) = \begin{cases} \mathbb{R}, & q = 0, 4, 8, \\ 0, & q \neq 0, 4, 8. \end{cases}
\]

**Proof** \( G(2,7) \) is a Kähler manifold, and the cohomology of \( G(2,7) \) is generated by Euler class \( \mathcal{e}(E(2,7)) \). We prove the case of \( \mathbb{Z}_2 \) coefficients; the other cases are left to the reader. By Gysin sequence, we have

\[
0 = H^{-2}(\text{ASSOC}) \xrightarrow{\wedge \mathcal{e}(\mathcal{E})} H^1(\text{ASSOC}) \xrightarrow{\tau_{2*}} H^1(G(2,7)) = 0,
\]

\[
0 = H^{-1}(\text{ASSOC}) \xrightarrow{\wedge \mathcal{e}(\mathcal{E})} H^2(\text{ASSOC}) \xrightarrow{\tau_{2*}} H^2(G(2,7)) \xrightarrow{\tau_{2*} \circ \mathcal{e}(\mathcal{E})} H^0(\text{ASSOC})
\]

\[
\xrightarrow{\wedge \mathcal{e}(\mathcal{E})} H^3(\text{ASSOC}) \xrightarrow{\tau_{2*}} H^3(G(2,7)) = 0.
\]

This shows \( H^1(\text{ASSOC}) = 0 \) and \( H^2(\text{ASSOC}) \cong H^2(G(2,7)), \ H^0(\text{ASSOC}) \cong H^3(\text{ASSOC}) \). By

\[
0 = H^1(\text{ASSOC}) \xrightarrow{\wedge \mathcal{e}(\mathcal{E})} H^4(\text{ASSOC}) \xrightarrow{\tau_{2*}} H^4(G(2,7)) \xrightarrow{\tau_{2*} \circ \mathcal{e}(\mathcal{E})} H^2(\text{ASSOC})
\]

and \( \tau_{2*} = 0 : H^4(G(2,7), \mathbb{Z}_2) \to H^2(\text{ASSOC}, \mathbb{Z}_2) \), we have

\[ H^4(\text{ASSOC}) \cong H^4(G(2,7)).\]

The cases of \( q = 5, \cdots, 8 \) can be proved similarly. \( \Box \)
References