Some results on $\mathcal{T}$-noncosingular modules

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Abstract: The notion of $\mathcal{T}$-noncosingularity of a module has been introduced and studied recently. In this article, a number of new results of this property are provided. It is shown that over a commutative semilocal ring $R$ such that $\text{Jac}(R)$ is a nil ideal, every $\mathcal{T}$-noncosingular module is semisimple. We prove that for a perfect ring $R$, the class of $\mathcal{T}$-noncosingular modules is closed under direct sums if and only if $R$ is a primary decomposable ring. Finitely generated $\mathcal{T}$-noncosingular modules over commutative rings are shown to be precisely those having zero Jacobson radical. We also show that for a simple module $S$, $E(S) \oplus S$ is $\mathcal{T}$-noncosingular if and only if $S$ is injective. Connections of $\mathcal{T}$-noncosingular modules to their endomorphism rings are investigated.

Key words: Small submodules, $\mathcal{T}$-noncosingular modules, endomorphism rings

1. Introduction

The concept of $\mathcal{T}$-noncosingularity of a module was introduced and studied recently by Tütüncü and Tribak in 2009 [20] as a dual notion of the $\mathcal{K}$-nonsingularity that was introduced and studied by Rizvi-Roman [14, 15]. It was shown in [21] that every dual Baer module is $\mathcal{T}$-noncosingular and that every $\mathcal{T}$-noncosingular lifting module is dual Baer. We note also that dual Rickart modules were introduced and studied by Lee et al. in 2011 [12] and it is easy to see that every dual Rickart module is $\mathcal{T}$-noncosingular. These links of the $\mathcal{T}$-noncosingularity with the dual Rickart and dual Baer properties are the motivations for the investigations in this paper. We obtain some new useful properties of this kind of module.

Throughout, $R$ will denote an associative ring with unity, $\text{Jac}(R)$ will denote the Jacobson radical of $R$, and $Z(R)$ will stand for the right singular ideal of $R$. For an $R$-module $M$, we write $E(M)$ and $\text{Rad}(M)$ for the injective hull and the Jacobson radical of $M$, respectively. If $N$ is a submodule of an $R$-module $M$, then the notation $N \ll M$ means that $N$ is small in $M$.

In Section 2 we investigate general properties of $\mathcal{T}$-noncosingular modules. We provide conditions for a $\mathcal{T}$-noncosingular module to have zero Jacobson radical. Among other results, we show that every finitely generated $\mathcal{T}$-noncosingular module over a commutative ring has zero Jacobson radical. The class of commutative rings $R$ for which every cyclic $R$-module is $\mathcal{T}$-noncosingular is characterized as that of von Neumann regular rings, while the class of commutative rings $R$ for which every finitely generated $\mathcal{T}$-noncosingular $R$-module is semisimple is shown to be precisely that of semilocal rings. It is also shown that over a commutative semilocal ring $R$ such that $\text{Jac}(R)$ is a nil ideal, every $\mathcal{T}$-noncosingular $R$-module is semisimple.

Section 3 is devoted to some results on direct sums of $\mathcal{T}$-noncosingular modules. We show that for a
simple module $S$, $E(S) \oplus S$ is $\mathcal{T}$-noncosingular if and only if $S$ is injective. We prove that for a perfect ring $R$, the class of $\mathcal{T}$-noncosingular modules is closed under direct sums if and only if $R$ is a primary decomposable ring.

The focus of our investigations in Section 4 is on connections of a $\mathcal{T}$-noncosingular module to its endomorphism ring.

2. Some properties of $\mathcal{T}$-noncosingular modules

**Definition 2.1** Let $M$ and $N$ be 2 modules.

1. We say that $M$ is $\mathcal{T}$-noncosingular relative to $N$ if $\forall \varphi \in \text{Hom}_R(M, N)$, $\text{Im} \varphi \ll N$ implies $\varphi = 0$.

2. The module $M$ is called $\mathcal{T}$-noncosingular if $M$ is $\mathcal{T}$-noncosingular relative to $M$ (or, equivalently, $\forall \varphi \in \text{End}_R(M)$, $\text{Im} \varphi \ll M \Rightarrow \varphi = 0$).

Many examples of $\mathcal{T}$-noncosingular modules are exhibited in [21] and [20]. Before presenting another example, we recall that a module $M$ is called radical if $M$ has no maximal submodules, i.e. $\text{Rad}(M) = M$.

**Example 2.2** Let $M$ be a simple radical module. That is, $M$ is a nonzero radical module that has no nonzero radical submodules (e.g., we can consider the $\mathbb{Z}$-modules $\mathbb{Z}(p^\infty)$ and $\mathbb{Q}$, where $p$ is a prime number). Let $\varphi$ be a nonzero endomorphism of $M$. Then $\text{Im} \varphi \cong M/\text{Ker} \varphi$ and so $\text{Rad}(\text{Im} \varphi) = \text{Im} \varphi$. Therefore, $\text{Im} \varphi = M$. Hence, $M$ is $\mathcal{T}$-noncosingular.

A ring $R$ is called a right $V$-ring if every simple right $R$-module is injective. This is equivalent to the condition that for any right $R$-module $M$, we have $\text{Rad}(M) = 0$. Recall that a module $M$ is called $\mathcal{K}$-nonsingular if, for every $0 \neq \varphi \in \text{End}_R(M)$, $\text{Ker} \varphi$ is not essential in $M$ (see [15]). The next example shows the existence of a $\mathcal{T}$-noncosingular module that is not $\mathcal{K}$-nonsingular and provides a $\mathcal{K}$-nonsingular module that is not $\mathcal{T}$-noncosingular.

**Example 2.3** (1) Let $R$ be a right $V$-ring that is not semisimple (e.g., we can take $R = \prod_{i=1}^{\infty} F_i$, with $F_i = F$ is a field for all $i \geq 1$). By [20, Proposition 2.13], every $R$-module is $\mathcal{T}$-noncosingular. On the other hand, from [15, Corollary 2.21] it follows that $R$ has a module $M$ that is not $\mathcal{K}$-nonsingular.

(2) Let $F$ be a field and set $R = \begin{bmatrix} F & 0 \\ F & F \end{bmatrix}$. Then $\text{Jac}(R) = \begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix}$, and hence $R_R$ is not $\mathcal{T}$-noncosingular by [20, Corollary 2.7]. On the other hand, we have $Z(R_R) = 0$ by [4, Corollary 4.3]. Applying [15, Corollary 2.4], we conclude that $R_R$ is $\mathcal{K}$-nonsingular.

In [20, Proposition 2.3] it was showed that the $\mathcal{T}$-noncosingularity is inherited by direct summands. Next, we show that the $\mathcal{T}$-noncosingularity property does not always transfer from a module to each of its submodules and factor modules.

**Example 2.4** (1) Note that the $\mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z}$ is not $\mathcal{T}$-noncosingular, while $\mathbb{Z}$ is $\mathcal{T}$-noncosingular (see [20, Proposition 2.10]).

(2) Consider the $\mathbb{Z}$-module $M = \mathbb{Z}(2^\infty) \oplus \mathbb{Z}(2^\infty)$. Then $M$ has a submodule $N$, which is isomorphic to $\mathbb{Z}(2^\infty) \oplus \mathbb{Z}/2\mathbb{Z}$. Then $M$ is $\mathcal{T}$-noncosingular as every factor module of $M$ is injective, while $N$ is not $\mathcal{T}$-noncosingular by [20, Example 2.12].
Recall that a ring $R$ is said to be a right $H$-ring if, whenever $S_1$ and $S_2$ are simple $R$-modules such that $\text{Hom}_R(E(S_1), E(S_2)) \neq 0$, then $S_1 \cong S_2$. It is well known that every commutative Noetherian ring is an $H$-ring (see, e.g., [16]). Next, we deal with the $T$-noncosingularity of injective hulls of simple modules. First note that for any prime number $p$, the $\mathbb{Z}$-module $E(\mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}(p^\infty)$ is $T$-noncosingular.

**Proposition 2.5** Assume that $R$ is a ring that has a unique simple right $R$-module (up to isomorphism) or $R$ is a right $H$-ring. If $S$ is a simple $R$-module such that $E(S)$ is $T$-noncosingular, then $S$ is injective or $\text{Rad}(E(S)) = E(S)$.

**Proof** Suppose that $S$ is not injective and $\text{Rad}(E(S)) \neq E(S)$. Then $S \ll E(S)$ and $E(S)$ has a maximal submodule $N$. Let $S'$ denote the simple $R$-module $E(S)/N$. Taking the canonical projection $\pi : E(S) \rightarrow S'$ and the inclusion map $\alpha : S' \rightarrow E(S')$, the homomorphism $\alpha \pi : E(S) \rightarrow E(S')$ is nonzero and $\text{Im}(\alpha \pi) = S'$. By hypothesis, we get that $S' \cong S$. Hence, there exists a nonzero endomorphism $\varphi$ of $E(S)$ such that $\text{Im} \varphi = S \ll E(S)$. This contradicts the fact that $M$ is $T$-noncosingular. \hfill $\Box$

**Corollary 2.6** Let $m$ be a maximal ideal of a commutative Artinian ring $R$. Then $E(R/m)$ is $T$-noncosingular if and only if $R/m$ is injective.

**Proof** Note that $\text{Rad}(E(R/m)) \neq E(R/m)$ and $R$ is Noetherian by [2, Theorem 15.20 and Corollary 15.21]. Hence, $R$ is an $H$-ring, and the result follows from Proposition 2.5. \hfill $\Box$

We recall that a ring $R$ is called a right max ring if $\text{Rad}(M) \neq M$ for all nonzero right $R$-modules $M$.

**Corollary 2.7** Let $R$ be a right max local ring with maximal right ideal $m$. The following are equivalent:

1. $E(R/m)$ is $T$-noncosingular;
2. $R/m$ is injective;
3. $R$ is a division ring.

**Proof** (i) $\Rightarrow$ (ii) By Proposition 2.5 and the fact that $R$ is a right max ring.

(ii) $\Rightarrow$ (iii) By hypothesis, every simple $R$-module is injective. Thus, $R$ is a right $V$-ring and $m = \text{Rad}(R) = 0$. Therefore, $R$ is a division ring.

(iii) $\Rightarrow$ (i) This is obvious. \hfill $\Box$

**Proposition 2.8** Let $M$ be a module with $\text{Rad}(M) \neq 0$ and let $N$ be a nonzero small submodule of $M$. If $K$ is a module that is isomorphic to $N$, then the module $M \oplus K$ is not $T$-noncosingular.

**Proof** By hypothesis, there exists an isomorphism $\varphi : K \rightarrow N$. Let $\pi : M \oplus K \rightarrow K$ be the canonical projection and let $\mu : N \rightarrow M$ and $\rho : M \rightarrow M \oplus K$ be the inclusion maps. Then $\rho \mu \varphi \pi$ is a nonzero endomorphism of $M \oplus K$ such that $\text{Im}(\rho \mu \varphi \pi) = N \oplus 0 \ll M \oplus K$. \hfill $\Box$

It is easy to see that every module with zero Jacobson radical is $T$-noncosingular and that the converse is not true, in general (e.g., for any prime integer $p$, the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is $T$-noncosingular, but $\text{Rad}(\mathbb{Z}(p^\infty)) = \mathbb{Z}(p^\infty)$). In the next 3 results we present conditions under which the converse holds.

**Proposition 2.9** Let $M$ be a module such that every nonzero submodule contains a simple submodule. If $M \oplus S$ is $T$-noncosingular for every simple small submodule $S \leq M$, then $\text{Rad}(M) = 0$. 31
Proof Assume that $\text{Rad}(M) \neq 0$. Then $\text{Rad}(M)$ contains a simple submodule $S$. Thus, $S \preccurlyeq M$. From Proposition 2.8 it follows that $M \oplus S$ is not $T$-noncosingular. This completes the proof.

Definition 2.10 A module $M$ is said to be retractable if for every submodule $N \leq M$, $\text{Hom}(M,N) \neq 0$.

Retractable modules have been studied extensively by different authors (see, e.g., [6, 7, 8, 9, 17]).

Proposition 2.11 Let $M$ be a retractable module. If $M$ is $T$-noncosingular, then $\text{Rad}(M) = 0$.

Proof Suppose that $\text{Rad}(M) \neq 0$. Then $M$ contains a nonzero submodule $N$ such that $N \preccurlyeq M$. Since $M$ is retractable, there exists a nonzero endomorphism $f : M \rightarrow M$ with $\text{Im} f \subseteq N$. This contradicts the $T$-noncosingularity of $M$.

A ring $R$ is said to be right semi-Artinian if every nonzero right $R$-module contains a simple submodule. Recall that if $R$ is any ring, then a right $R$-module $M$ is nonsingular if $mE \neq 0$ for every nonzero element $m$ of $M$ and essential right ideal $E$ of $R$.

Corollary 2.12 If $M$ is a nonzero module that satisfies one of the following conditions:

(i) $M$ is a module over a commutative semi-Artinian ring,
(ii) $M$ is a projective module over a commutative Noetherian ring,
(iii) $M$ is a finitely generated module over a commutative ring,
(iv) $M$ is a nonsingular module over a right self-injective ring,

then $M$ is $T$-noncosingular if and only if $\text{Rad}(M) = 0$.

Proof By Proposition 2.11 [5, Theorems 2.7 and 2.8] and [17, Proposition 1.17 and Corollary 2.12].

Corollary 2.13 The following are equivalent for a commutative ring $R$:

(i) Every cyclic $R$-module is $T$-noncosingular;
(ii) $R$ is a von Neumann regular ring.

Proof (i) $\Rightarrow$ (ii) Let $I$ be an ideal of $R$. By hypothesis, the $R$-module $R/I$ is $T$-noncosingular. Then $\text{Rad}(R/I) = 0$ by Corollary 2.12. So $R$ is a $V$-ring (see [19, Theorem 22.1]). Thus, $R$ is von Neumann regular since $R$ is commutative (see [19, Theorem 22.4]).

(ii) $\Rightarrow$ (i) This follows from [20, Proposition 2.13] and [19, Theorem 22.4].

Following [18], a module $M$ is called noncosingular if $\text{Z}(M) = \cap\{N \mid M/N$ is small in its injective hull$\} = M$. That is, for every nonzero module $N$ and every nonzero homomorphism $f : M \rightarrow N$, $\text{Im} f$ is not a small submodule of $N$. This is obviously equivalent to the condition that $M$ is $T$-noncosingular relative to $N$ for every module $N$. Clearly, every noncosingular module is $T$-noncosingular. It is easy to check that if $S$ is a simple module that is not injective, then $S$ is $T$-noncosingular but not noncosingular. In the next result, we give conditions under which the $T$-noncosingularity of a module implies its noncosingularity. The following condition was studied in [1] as a dual notion of the retractability.

Definition 2.14 A module $M$ is called coretractable if, for any proper submodule $K$ of $M$, there exists a nonzero homomorphism $f : M \rightarrow M$ with $f(K) = 0$, that is, $\text{Hom}_R(M/K, M) \neq 0$. 
Proposition 2.15 Let $M$ be a coretractable injective module. If $M$ is $\mathcal{T}$-noncosingular, then $M$ is noncosingular.

Proof Suppose that there exists a proper submodule $X$ of $M$ such that $M/X \ll E(M/X)$. Let $\pi : M \rightarrow M/X$ be the canonical projection. Since $M$ is coretractable, there exists a nonzero homomorphism $\varphi : M/X \rightarrow M$. Since $M$ is injective, $\varphi$ can be extended to a homomorphism $\psi : E(M/X) \rightarrow M$. Taking the inclusion map $\alpha : M/X \rightarrow E(M/X)$, $\varphi\alpha\pi$ is a nonzero endomorphism of $M$. Since $\alpha\pi(M) \ll E(M/X)$, $\varphi\alpha\pi(M) \ll M$ by [13, Lemma 4.2(3)]. This contradicts the $\mathcal{T}$-noncosingularity of $M$. Hence, $M$ is noncosingular. \(\square\)

The next proposition can be regarded as the dual of [15, Proposition 2.18]. First we prove the following elementary known result.

Lemma 2.16 Let $N$ be a small submodule of a module $M$. If $L$ is a submodule of $M$ such that $(L+N)/N \ll M/N$, then $L \ll M$.

Proof Let $X$ be a submodule of $M$ such that $L+X = M$. Then, $[(L+N)/N] + [(X+N)/N] = M/N$. By hypothesis, we have $(X+N)/N = M/N$. Therefore, $X = M$ as $N \ll M$. So, $L \ll M$. \(\square\)

Proposition 2.17 Let $M$ be a module that has a projective cover $f : P \rightarrow M$. If $P$ is $\mathcal{T}$-noncosingular, then so is $M$.

Proof By hypothesis, $f : P \rightarrow M$ is an epimorphism with $Q = \ker f \ll P$. Thus, $P/Q \cong M$. To prove the $\mathcal{T}$-noncosingularity of $M$, let $\varphi \in \text{End}(P/Q)$ such that $\text{Im} \varphi \ll P/Q$. Consider the natural epimorphism $\pi : P \rightarrow P/Q$. Since $P$ is projective, there exists a homomorphism $\psi : P \rightarrow P$ such that $\varphi = \pi\psi$. Therefore, $\pi\psi(P) = \varphi(P/Q) \ll P/Q$. So $\psi(P) \ll P$ by Lemma 2.16. But $P$ is $\mathcal{T}$-noncosingular. Then $\psi = 0$, and hence $\varphi = 0$. This implies that $\varphi = 0$. Thus, $M$ is $\mathcal{T}$-noncosingular. \(\square\)

Proposition 2.18 The following are equivalent for a ring $R$:

(i) $R_R$ is $\mathcal{T}$-noncosingular;

(ii) Every projective $R$-module is $\mathcal{T}$-noncosingular;

(iii) Every $R$-module having a projective cover is $\mathcal{T}$-noncosingular.

Proof (i) $\Rightarrow$ (ii) By [20, Corollary 2.7], $\text{Jac}(R) = 0$. Let $P$ be a projective $R$-module. Hence $\text{Rad}(P) = P(\text{Jac}(R)) = 0$ by [2, Proposition 17.10]. So, $P$ is $\mathcal{T}$-noncosingular.

(ii) $\Rightarrow$ (iii) This follows from Proposition 2.17.

(iii) $\Rightarrow$ (i) This is obvious. \(\square\)

Definition 2.19 (1) A module $M$ has $D_1$ property (or is called lifting) if for every submodule $N \subseteq M$, there exists a direct summand $K$ of $M$ with $K \subseteq N$ and $N/K \ll M/K$. $M$ has $D_3$ property if for any direct summands $K$, $L$ of $M$ with $M = K + L$, $K \cap L$ is a direct summand of $M$.

(2) A module $M$ satisfying $D_1$ and $D_3$ is called quasi-discrete.
Definition 2.20 A module $M$ is said to have the strong summand sum property, SSSP, if the sum of any family of direct summands is a direct summand of $M$. $M$ is said to have the summand intersection property, SIP, if the intersection of any 2 direct summands is a direct summand of $M$.

In the next result, we provide an application of $\mathcal{T}$-noncosingularity to quasi-discrete modules. It can be regarded as the dual of [15, Proposition 4.1].

Proposition 2.21 Let $M$ be a quasi-discrete module. If $M$ is $\mathcal{T}$-noncosingular, then $M$ has SSSP and SIP.

Proof Since $M$ is $\mathcal{T}$-noncosingular lifting, $M$ has SSSP by [21, Theorems 2.1 and 2.14]. To prove SIP, let $K_1$ and $K_2$ be 2 direct summands of $M$. Then $K = K_1 + K_2$ is a direct summand of $M$. Since $M$ has $(D_3)$, $K$ has $(D_3)$ by [13, Lemma 4.7]. Therefore, $K_1 \cap K_2$ is a direct summand of $K$. Hence, $K_1 \cap K_2$ is a direct summand of $M$.

Recall that a ring $R$ is said to be semilocal if the factor ring $R/Jac(R)$ is semisimple.

We conclude this section by describing the structure of some classes of $\mathcal{T}$-noncosingular modules over commutative semilocal rings. First we prove the following lemma.

Lemma 2.22 Let $I$ be a nil ideal of a commutative ring $R$. If $M$ is $\mathcal{T}$-noncosingular, then $MI = 0$.

Proof Let $a \in I$ and consider the endomorphism $\varphi_a$ of $M$ defined by $\varphi_a(x) = xa$ for all $x \in M$. Clearly, we have $Im\varphi_a = Ma$. Let $X$ be a submodule of $M$ such that $M = Ma + X$. By induction, we have $M = Ma^n + X$ for every integer $n \geq 1$. Then $X = M$ since the ideal $I$ is nil. It follows that $Ma \ll M$. Thus, $Ma = 0$ as $M$ is $\mathcal{T}$-noncosingular.

Theorem 2.23 Let $R$ be a commutative semilocal ring such that $Jac(R)$ is a nil ideal of $R$. Then an $R$-module $M$ is $\mathcal{T}$-noncosingular if and only if $M$ is semisimple.

Proof Let $M$ be a $\mathcal{T}$-noncosingular module. Since $R$ is semilocal, we have $Rad(M) = MJac(R)$ and $M/Rad(M)$ is semisimple by [2, Corollary 15.18]. Therefore, $Rad(M) = 0$ by Lemma 2.22. Thus, $M$ is semisimple. The converse is immediate.

Corollary 2.24 Let $R$ be a commutative perfect ring. Then an $R$-module $M$ is $\mathcal{T}$-noncosingular if and only if $M$ is semisimple.

Proof By Theorem 2.23 and [2, Theorem 28.4].

Theorem 2.25 The following statements are equivalent for a commutative ring $R$:

(i) Every $\mathcal{T}$-noncosingular $R$-module $M$ with $Rad(M) \ll M$ is semisimple;

(ii) Every finitely generated $\mathcal{T}$-noncosingular module is semisimple;

(iii) $R$ is semilocal.

Proof (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (iii) Since $Rad(R/Jac(R)) = 0$, the $R$-module $R/Jac(R)$ is $\mathcal{T}$-noncosingular. The result follows by (ii).
(iii) $\Rightarrow$ (i) Let $M$ be a $T$-noncosingular $R$-module with $Rad(M) \ll M$. Since $R$ is semilocal, $M/Rad(M)$ is semisimple and $Rad(M) = MJac(R)$ by [2, Corollary 15.18]. If $a \in Jac(R)$ and $\varphi_a$ is the endomorphism of $M$ defined by $\varphi_a(x) = xa$ for all $x \in M$, then we have $\text{Im} \varphi_a = M \varphi_a \subseteq MJac(R) \ll M$. But by $T$-noncosingularity, $Ma = 0$. Thus, $Rad(M) = 0$. This implies that $M$ is semisimple.

3. Direct sums of $T$-noncosingular modules

It is shown in [20, Example 2.12] that, in general, a direct sum of 2 $T$-noncosingular modules is not $T$-noncosingular. In this section we prove that for a simple module $S$, $E(S) \oplus S$ is $T$-noncosingular if and only if $S$ is injective (Proposition 3.4). The class of perfect rings for which arbitrary direct sums of $T$-noncosingular modules are $T$-noncosingular is shown to be exactly that of the primary decomposable rings (Theorem 3.7).

We begin with the next proposition, which is a direct consequence of [20, Corollary 2.7 and Proposition 2.11].

**Proposition 3.1** (i) If $M$ is a $T$-noncosingular module, then every direct sum of copies of $M$ is a $T$-noncosingular module.

(ii) If $R$ is a ring with $Jac(R) = 0$, then every free $R$-module is $T$-noncosingular.

Next, we provide a characterization for an arbitrary direct sum of $T$-noncosingular modules to be $T$-noncosingular when each module is fully invariant in the direct sum.

**Proposition 3.2** Let $M = \oplus_{i \in I} M_i$ be the direct sum of fully invariant submodules $M_i$. Then $M$ is $T$-noncosingular if and only if $M_i$ is $T$-noncosingular for all $i \in I$.

**Proof** The necessity follows from [20, Proposition 2.3]. Conversely, we need only to show that $M_i$ is a $T$-noncosingular module relative to $M_j$ for all $i, j \in I$ with $i \neq j$ (see [20, Proposition 2.11]). Let $f : M_i \to M_j$ ($i \neq j$) be a homomorphism. Let $\pi_i : M \to M_i$ be the projection map and $\alpha_j : M_j \to M$ be the inclusion map. Then $g = \alpha_j f \pi_i \in \text{End}_R(M)$ and $g(M) \subseteq M_j$. Since $M_i$ is fully invariant in $M$, we have $g(M_i) \subseteq M_i$. So, $g(M_i) \subseteq M_i \cap M_j = 0$. Hence, $f = 0$. Consequently, $M$ is $T$-noncosingular.

**Proposition 3.3** Let $M = N \oplus (\oplus_{i \in I} S_i)$ such that $S_i (i \in I)$ are simple modules. The following are equivalent:

(i) $M$ is $T$-noncosingular;

(ii) (a) $N$ is $T$-noncosingular, and

(b) For every simple small submodule $S$ of $N$, $S \not\cong S_i$ for all $i \in I$.

**Proof** (i) $\Rightarrow$ (ii) By [20, Proposition 2.3], $N$ and $N \oplus S_i$ are $T$-noncosingular modules for all $i \in I$. Proposition 2.8 now shows that condition (b) holds.

(ii) $\Rightarrow$ (i) By (b), each $S_i$ is $T$-noncosingular relative to $N$. Applying [20, Proposition 2.11], we obtain that $M$ is $T$-noncosingular.

Let $R$ be a Dedekind domain that is not a field and $P$ be a nonzero prime ideal of $R$. Let $R(P^\infty)$ denote the $P$-primary component of the torsion $R$-module $K/R$, where $K$ is the quotient field of $R$. In [20, Example 2.12] it is proven that the $R$-module $R(P^\infty) \oplus R/P$ is not $T$-noncosingular. That is, $E(R/P) \oplus R/P$
is not $T$-noncogulsingular. In the next result we provide a necessary and sufficient condition for $E(S) \oplus S$ to be $T$-noncogulsingular, where $S$ is a simple module.

**Proposition 3.4** Let $S$ be a simple module. Then the module $M = E(S) \oplus S$ is $T$-noncogulsingular if and only if $S$ is injective.

**Proof** The sufficiency is obvious. Conversely, suppose that $S$ is not injective. Then $S \ll E(S)$. Thus, $M$ is not $T$-noncogulsingular by Proposition 2.8.

By combining [20, Proposition 2.13] and Proposition 3.4, we get the following result.

**Corollary 3.5** The following are equivalent for a ring $R$:

(i) Every $R$-module is $T$-noncogulsingular;

(ii) For every simple $R$-module $S$, the module $E(S) \oplus S$ is $T$-noncogulsingular;

(iii) The ring $R$ is a right $V$-ring.

Next, we present other examples that show that the property of $T$-noncogulsingularity does not go to direct sums of $T$-noncogulsingular modules.

**Example 3.6** (1) Let $R$ be a right hereditary ring that is not a right $V$-ring. Therefore, $R$ has a simple $R$-module $S$ that is not injective (e.g., we can take a Dedekind domain $R$ that is not a field and $S$ any simple $R$-module). Then $E(S)$ and $S$ are both $T$-noncogulsingular $R$-modules by [20, Example 2.1]. However, the $R$-module $M = E(S) \oplus S$ is not $T$-noncogulsingular by Proposition 3.4.

(2) Let $R$ be an almost DVR with maximal ideal $m$ and quotient field $Q$ (i.e. $R$ is a commutative local Noetherian domain of Krull dimension 1 and the integral closure $R'$ of $R$ in $Q$ is a finitely generated $R$-module and is a discrete valuation ring). Note that $E(R/m)$ is a simple radical $R$-module by [11, Proposition 4]. Therefore, $E(R/m)$ is a $T$-noncogulsingular $R$-module (see Example 2.2). Further, the $R$-module $R/m$ is $T$-noncogulsingular. On the other hand, the $R$-module $E(R/m) \oplus R/m$ is not $T$-noncogulsingular, since otherwise $R$ will be a $V$-ring and $m = 0$ by Corollary 3.5.

Recall that a ring $R$ is called left (resp. right) perfect if it is semilocal and every nonzero left $R$-module contains a maximal (resp. simple) submodule. A ring $R$ is said to be perfect if it is right and left perfect. A perfect ring is said to be primary if the ring $R/Jac(R)$ is simple Artinian. A perfect ring is called primary decomposable if it is isomorphic to a finite product of primary rings. A module $M$ is called supplemented if, for every submodule $N$ of $M$, there exists a submodule $K \leq M$ such that $M = N + K$ and $N \cap K \ll K$ (see, e.g., [3], [13], and [22]). It is easy to check that if $M$ is a module with zero Jacobson radical, then $M$ is supplemented if and only if $M$ is semisimple.

In the next result, we characterize the class of perfect rings $R$ for which arbitrary direct sums of $T$-noncogulsingular $R$-modules are $T$-noncogulsingular.

**Theorem 3.7** The following assertions are equivalent for a perfect ring $R$:

(i) Every $T$-noncogulsingular $R$-module is semisimple;

(ii) Every direct sum of $T$-noncogulsingular $R$-modules is $T$-noncogulsingular;

(iii) $R$ is primary decomposable.
**Proof** (i) ⇒ (ii) This is clear.

(ii) ⇒ (iii) Let \( M \) be a module such that \( S = \text{End}_R(M) \) is a division ring. Clearly, \( M \) is an indecomposable \( \mathcal{T} \)-noncosingular module. Since \( R \) is perfect, every \( R \)-module contains a simple submodule. Noting that \( M \oplus S \) is \( \mathcal{T} \)-noncosingular for every simple \( R \)-module \( S \), we conclude from Proposition 2.9 that \( \text{Rad}(M) = 0 \). Since \( R \) is perfect, \( M \) is supplemented by [13, Theorem 4.41]. Thus, \( M \) is semisimple, but \( M \) is indecomposable. Then \( M \) is simple. So \( R \) is primary decomposable by [10, Theorem 1.2].

(iii) ⇒ (i) By hypothesis, \( R = R_1 \oplus \cdots \oplus R_n \) is a direct sum of perfect primary rings \( R_i \) \((1 \leq i \leq n)\). We can write \( 1_R = e_1 + e_2 + \cdots + e_n \), where \( 1_R \) is the identity element of \( R \) and for each \( i \), \( e_i \in R_i \). Then for each \( i \), \( e_i \) is the identity element of the ring \( R_i \). Let \( M \) be an \( R \)-module. Then \( M = Me_1 + Me_2 + \cdots + Me_n \).

Also, \( Me_1 \) can be regarded as an \( R_i \)-module as well as an \( R \)-module, and its submodules are the same in both cases, because \( xe_i(r_1 + r_2 + \cdots + r_n) = xe_i r_i \), where \( x \in M \) and \( r_j \in R_j \) for each \( j \), \( 1 \leq j \leq n \). Now assume that \( R \) has a \( \mathcal{T} \)-noncosingular module \( M \) that is not semisimple. Without loss of generality we can assume that \( M_1 = Me_1 \) is not semisimple. Note that \( \text{End}_{R_i}(M_1) = \text{End}_R(M_1) \). So \( (M_1)_{R_i} \) is \( \mathcal{T} \)-noncosingular by [20, Proposition 2.3]. Since \( R_i \) is a perfect ring, \( (M_1)_{R_i} \) is supplemented by [13, Theorem 4.41]. Therefore, \( \text{Rad}_{R_i}(M_1) \neq 0 \). Hence, \( \text{Rad}_{R_i}(M_1) \) contains a simple submodule \( S_1 \). Moreover, \( (M_1)_{R_i} \) contains a maximal submodule \( K_i \) since \( R_1 \) is perfect. Consider the natural epimorphism \( \pi : M_1 \rightarrow M_1/K_1 \) and the inclusion map \( \alpha : S_1 \rightarrow M_1 \). Since \( R_1 \) is primary, \( R_1 \) has a unique isomorphism class of simple modules. So, there exists an isomorphism \( \theta : M_1/K_1 \rightarrow S_1 \). It follows that \( \varphi = \alpha \theta \pi \) is a nonzero endomorphism of \( M_1 \) such that \( \varphi(M_1) = S_1 \leq M_1 \). This shows that \( (M_1)_{R_1} \) is not \( \mathcal{T} \)-noncosingular, a contradiction. \( \square \)

**Corollary 3.8** If \( R \) is a finite product of local perfect rings (e.g., \( R \) is commutative perfect), then every \( \mathcal{T} \)-noncosingular module is semisimple.

**Proof** This is a direct consequence of Theorem 3.7. \( \square \)

4. The endomorphism ring of a \( \mathcal{T} \)-noncosingular module

We conclude this paper by investigating the connection of the \( \mathcal{T} \)-noncosingularity of a module to its endomorphism ring. Recall that a ring \( R \) is called reduced if it has no nonzero nilpotent elements.

**Proposition 4.1** Let \( M \) be a quasi-discrete module with \( S = \text{End}_R(M) \). If \( M \) is \( \mathcal{T} \)-noncosingular, then \( S = S_1 \times S_2 \) such that \( S_1 \) is von Neumann regular and \( S_2 \) is reduced.

**Proof** Let \( \nabla(M) = \{ \varphi \in S : \text{Im} \varphi \leq M \} \). By [13, Proposition 5.7], \( S/\nabla(M) = S_1 \times S_2 \) such that \( S_1 \) is von Neumann regular and \( S_2 \) is reduced. However, since \( M \) is \( \mathcal{T} \)-noncosingular, \( \nabla(M) = 0 \). \( \square \)

**Proposition 4.2** Let \( P \) be a quasi-projective module with \( S = \text{End}_R(P) \). The following are equivalent:

(i) \( P \) is \( \mathcal{T} \)-noncosingular;

(ii) \( \text{Jac}(S) = 0 \);

(iii) \( SS \) is \( \mathcal{T} \)-noncosingular.

**Proof** This follows from [20, Corollary 2.7] and the fact that \( \varphi \in \text{Jac}(S) \) if and only if \( \text{Im} \varphi \leq P \) (see, e.g., [22, 22.2]). \( \square \)
Proposition 4.2 is not true, in general, as the next example shows.

**Example 4.3** Consider the \( \mathbb{Z} \)-module \( M = \mathbb{Z}(p^\infty) \), where \( p \) is a prime number. It is well known that \( S = \text{End}_\mathbb{Z}(M) \) is a local ring that is not a division ring. Then \( \text{Jac}(S) \neq 0 \), while \( M \) is \( T \)-noncosingular.

**Definition 4.4** A module \( M \) has \( D_2 \) property (or is called direct projective) if, for any direct summand \( K \) of \( M \) and submodule \( N \) of \( M \) with \( M/N \cong K \), \( N \) is a direct summand of \( M \).

**Proposition 4.5** Let \( M \) be a direct projective module with \( S = \text{End}_R(M) \). If \( \text{Jac}(S) = 0 \), then \( M \) is \( T \)-noncosingular.

**Proof** By [22, 41.19(1)]. \( \square \)

**Proposition 4.6** Let \( R \) be a commutative ring. If \( R \) is \( T \)-noncosingular, then \( R \) is \( K \)-nonsingular.

**Proof** By [15, Proposition 2.7], it suffices to show that \( R \) is nonsingular. Since \( R \) is \( T \)-noncosingular, we have \( \text{Jac}(R) = 0 \) by [20, Corollary 2.7]. So, \( R \) is a semiprime ring. Therefore, \( Z(R) = 0 \) by [4, Proposition 1.27(b)]. \( \square \)

The converse of Proposition 4.6 is not true, in general, as shown below.

**Example 4.7** Let \( R \) be a discrete valuation ring with maximal ideal \( m \). It is clear that \( Z(R) = 0 \), while \( \text{Jac}(R) = m \). So \( R \) is \( K \)-nonsingular, but \( R \) is not \( T \)-noncosingular by [20, Corollary 2.7] and [15, Proposition 2.7].

Following [22, p. 261], a module \( M \) is called semi-injective if for any monomorphism \( f : N \to M \), where \( N \) is a factor module of \( M \), and for any homomorphism \( g : N \to M \), there exists \( h : M \to M \) such that \( hf = g \). Note that every quasi-injective module is semi-injective.

In the next result, we provide a condition under which Proposition 4.6 holds true for modules.

**Proposition 4.8** Let \( M \) be a coretractable module and let \( S = \text{End}_R(M) \). If \( M \) is \( T \)-noncosingular, then \( SS \) is \( K \)-nonsingular. The converse holds when \( M \) is semi-injective.

**Proof** This follows from [1, Corollary 4.8] and [15, Proposition 2.7]. \( \square \)

**References**


