Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2014) 38: 25 – 28
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doi:10.3906/mat-1203-22

On Betti series of the universal modules of second order derivations of $\frac{k[x_1,x_2,...,x_s]}{(f)}$

Ali ERDOĞAN, Halise Melis TEKİN AKÇİN*
Department of Mathematics, Hacettepe University, Ankara, Turkey

Received: 09.03.2012 • **Accepted:** 14.03.2013 • **Published Online:** 09.12.2013 • **Printed:** 20.01.2014

Abstract: Let R be a coordinate ring of an affine irreducible curve represented by $\frac{k[x_1,x_2,...,x_s]}{(f)}$ and m be a maximal ideal of R. In this article, the Betti series of $\Omega_2(R_m)$ is studied. We proved that the Betti series of $\Omega_2(R_m)$, where $\Omega_2(R_m)$ denotes the universal module of second order derivations of R_m, is a rational function under some conditions.

Key words: Universal module, universal differential operators, Betti series, minimal resolution

1. Preliminaries and notation

Derivations and their universal modules have been studied by many mathematicians. Erdoğan [2] has studied when the Betti series of a universal module of second order derivations is a rational function. In this work, the analogue of this question for the Betti series of $\Omega_2(R_m)$, where $R = \frac{k[x_1,x_2,...,x_s]}{(f)}$ and m is a maximal ideal of R, has been studied. At the end, we give an example to illustrate our result.

All rings we will study in this work will be commutative with identity. Now, we recall some important properties.

Let R be a commutative $k-$algebra, where k is a field of characteristic zero. We have the following exact sequence:

$$0 \to I \to R \otimes_k R \overset{\varphi}{\to} R \to 0,$$

where φ is defined as $\varphi(\sum_{i=1}^{n} a_i \otimes b_i) = \sum_{i=1}^{n} a_i b_i$ for $a_i, b_i \in R$ and I is the kernel of φ. Note that $\ker\varphi$ is generated by the set $\{1 \otimes r - r \otimes 1 : r \in R\}$.

Let $d_n : R \to (R \otimes_k R)/I^{n+1}$ be a $k-$linear map defined as

$$d_n(r) = 1 \otimes r - r \otimes 1 + I^{n+1} \text{ and } d_n(1) = 0.$$

Here d_n is called the universal derivation of order n. The $R-$module $\frac{I}{I^{n+1}}$ is called the universal module of nth order derivations and is denoted by $\Omega_n(R)$. If R is a finitely generated $k-$algebra, then $\Omega_n(R)$ is a finitely generated $R-$module. Let $R = k[x_1, ..., x_s]$ be a polynomial algebra; then $\Omega_n(R)$ is a free $R-$module of rank $\binom{n+s}{s} - 1$ with basis

$$\{d_n(x_1^{\alpha_1} x_2^{\alpha_2} ... x_s^{\alpha_s}) : 1 \leq \alpha_1 + \alpha_2 + ... + \alpha_s \leq n\}$$

*Correspondence: hmtekin@hacettepe.edu.tr

2000 *AMS Mathematics Subject Classification:* Primary 13N05.

(see [5]). It is known that $\Omega_n(R)\otimes_R R_S \cong \Omega_n(R_S)$, where S is a multiplicatively closed subset of R.

Suppose that R is a local $k-$algebra with maximal ideal m. A projective resolution of $\Omega_n(R)$ is called a minimal resolution if the following are satisfied:

$$...F_2 \overset{\partial_2}{\to} F_1 \overset{\partial_1}{\to} F_0 \overset{\varepsilon}{\to} \Omega_n(R) \to 0$$

F_i is a free module of finite rank for all i and $\partial_n(F_n) \subseteq mF_{n-1}$ for all $n \geq 1$(*see* [4] for definition). The Betti series of $\Omega_n(R)$ is defined to be the series

$$B(\Omega_n(R),t) = \sum_{i\geq 0} \dim_{R/m} Ext^i(\Omega_n(R), \frac{R}{m})t^i \text{ for all } n \geq 1.$$

We now give some well-known results.

Lemma 1 *Let* $R = \frac{k[x_1,x_2,...,x_s]}{(f)}$ *where* $f \in k[x_1,...,x_5]$. *Then we have an exact sequence*

$$0 \to \ker\alpha \to \frac{\Omega_n(k[x_1,x_2,...,x_s])}{f\Omega_n(k[x_1,x_2,...,x_s])} \overset{\alpha}{\to} \Omega_n(R) \to 0$$

of $R-$ *modules* (*see* [3]).

Lemma 2 *Let* R *be a local ring with maximal ideal* m. *Let* M *be a finitely generated* $R-$ *module. Suppose that*

$$0 \to F_1 \overset{\partial}{\to} F_0 \to M \to 0$$

is a minimal resolution of M. *Then* $Ext^1(M,R/m)$ *is not zero.*

2. Main results

Lemma 3 *Let* $k[x_1,x_2...,x_s]$ *be a polynomial algebra and* m *be a maximal ideal of* $k[x_1,x_2...,x_s]$ *containing* f. *Let*

$$d_2 : k[x_1,x_2,...,x_s] \to \Omega_2(k[x_1,x_2,...,x_s])$$

be the universal derivation of second order. Suppose that $d_2(f)$ *and* $d_2(x_if)$ *belong to* $m\Omega_2(k[x_1,x_2,...,x_s])$ *for all* $i = 1,...,s$. *Then a module generated by the set*

$$\{d_2(g) : g \in fk[x_1,x_2,...,x_s]\}$$

is a submodule of $m\Omega_2(k[x_1,x_2,...,x_s])$.

Proof It suffices to show that $d_2(x_1^{\alpha_1}x_2^{\alpha_2}...x_s^{\alpha_s}f) \in m\Omega_2(k[x_1,x_2,...,x_s])$. By the properties of d_2, we have

$$\begin{aligned} d_2(x_1^{\alpha_1}x_2^{\alpha_2}...x_s^{\alpha_s}f) &= a_1(x_1,...x_s)d_2(x_1f) + ... + a_s(x_1,...x_s)d_2(x_sf) \\ &+a_{s+1}(x_1,...x_s)d_2(f) + f(\sum_{\gamma,\beta}\gamma(x_1,...x_s)d_2(x_1^{\beta_1}x_2^{\beta_2}...x_s^{\beta_s})) \end{aligned}$$

where $0 < \beta = \beta_1 + \beta_2 + ... + \beta_s \leq 2$ and $\gamma, a_i \in k[x_1,x_2,...,x_s]$ for all $i = 1,...,s+1$. Since $d_2(x_if), d_2(f) \in m\Omega_2(k[x_1,x_2,...,x_s])$ for all $i = 1,...,s$ and $f \in m$, we get $d_2(x_1^{\alpha_1}x_2^{\alpha_2}...x_s^{\alpha_s}f) \in m\Omega_2(k[x_1,x_2,...,x_s])$. Hence the result follows. □

Proposition 4 *Let $k[x_1, x_2, ..., x_s]$ be a polynomial algebra and m be a maximal ideal of $k[x_1, x_2, ..., x_s]$ containing an irreducible element f. If $d_2(f)$ and $d_2(x_i f)$ are elements of $m\Omega_2(k[x_1, x_2, ..., x_s])$ for all $i = 1, ..., s$ then $\Omega_2((\frac{k[x_1,x_2,...,x_s]}{(f)})_{\bar{m}})$ admits a minimal resolution of $(\frac{k[x_1,x_2,...,x_s]}{(f)})_{\bar{m}}$ modules, where $\bar{m} = m/(f)$ is a maximal ideal of $\frac{k[x_1,x_2,...,x_s]}{(f)}$.*

Proof Let $R = \frac{k[x_1,x_2,...,x_s]}{(f)}$ and $\bar{m}$ be a maximal ideal of R. Then, by using Lemma 1, we have the following exact sequence of $R_{\bar{m}}$ modules:

$$0 \to \ker \alpha_{\bar{m}} \to (\frac{\Omega_2(k[x_1, x_2, ..., x_s])}{f\Omega_2(k[x_1, x_2, ..., x_s])})_{\bar{m}} \overset{\alpha_{\bar{m}}}{\to} \Omega_2(R_{\bar{m}}) \to 0 \tag{1}$$

We claim that this exact sequence is a minimal resolution of $\Omega_2(R_{\bar{m}})$. Since $\ker \alpha$ is of the form $\frac{N + f\Omega_2(k[x_1,x_2,...,x_s])}{f\Omega_2(k[x_1,x_2,...,x_s])}$, where N is a submodule of $\Omega_2(k[x_1, x_2, ..., x_s])$ generated by the elements $\{d_2(g) : g \in fk[x_1, x_2, ..., x_s]\}$, it is clear that $\ker \alpha_{\bar{m}} \subseteq \bar{m}(\frac{\Omega_2(k[x_1,x_2,...,x_n])}{f\Omega_2(k[x_1,x_2,...,x_n])})_{\bar{m}}$. Now we need to show that $\ker \alpha_{\bar{m}}$ is a free $R_{\bar{m}}$ module. $(\frac{\Omega_2(k[x_1,x_2,...,x_s])}{f\Omega_2(k[x_1,x_2,...,x_s])})_{\bar{m}}$ is free of rank $\binom{s+2}{s} - 1$. The Krull dimension of $R_{\bar{m}}$ is $s - 1$ and let K be the field of fractions of $R_{\bar{m}}$. Then $Tr \deg K = s - 1$. Note that $\dim_K \Omega_2(R_{\bar{m}}) \otimes_{R_{\bar{m}}} K = \dim_K \Omega_2(K) = \binom{s+1}{s-1} - 1$.
By tensoring the exact sequence given in (1) with K we have an exact sequence of $K-$vector spaces. Therefore,

$$\begin{aligned} \dim_K \ker \alpha_{\bar{m}} \otimes_{R_{\bar{m}}} K &= \dim_K (\frac{\Omega_2(k[x_1, x_2, ..., x_s])}{f\Omega_2(k[x_1, x_2, ..., x_s])})_{\bar{m}} \otimes_{R_{\bar{m}}} K - \dim_K \Omega_2(K) \\ &= \binom{s+2}{s} - \binom{s+1}{s-1} = s + 1. \end{aligned}$$

Since $\ker \alpha$ is generated by the elements $d_2(f), d_2(x_1 f), ..., d_2(x_s f)$ as an $R-$module, $\ker \alpha_{\bar{m}}$ is generated by the images of these elements in $R_{\bar{m}}$. Therefore, $\ker \alpha_{\bar{m}}$ is a free $R_{\bar{m}}$ module as required. □

Let R be a finitely generated regular algebra and m be a maximal ideal of R. Then $\Omega_2(R_m)$ is a free R_m-module. Thus, it is obvious that $B(\Omega_2(R_m), t)$ is rational.

Theorem 5 *Let $k[x_1, x_2, ..., x_s]$ be a polynomial ring and m be a maximal ideal containing an irreducible polynomial f. Suppose that $R = \frac{k[x_1,x_2,...,x_s]}{(f)}$ is not a regular ring at $\overline{m} = \frac{m}{(f)}$. Let $d_2(f)$ and $d_2(x_i f)$ be the elements of $m\Omega_2(k[x_1, x_2, ..., x_s])$ for all $i = 1, ..., s$. Then $B(\Omega_2(R_{\overline{m}}), t)$ is a rational function.*

Proof By proposition 4, we have that

$$0 \to \ker \alpha_{\bar{m}} \to (\frac{\Omega_2(k[x_1, x_2, ..., x_s])}{f\Omega_2(k[x_1, x_2, ..., x_s])})_{\bar{m}} \overset{\alpha_{\bar{m}}}{\to} \Omega_2(R_{\bar{m}}) \to 0$$

is a minimal resolution of $\Omega_2(R_{\bar{m}})$. By lemma 2, $Ext^1(\Omega_2(R_{\overline{m}}), \frac{R_{\overline{m}}}{\overline{m}R_{\overline{m}}}) \neq 0$. Therefore, we get the result. □

Example 6 *Let R be a k-algebra represented by $\frac{k[x,y,z]}{(y^2-x^3)}$. Then R is not regular at the origin and it is known that $pd(\Omega_2(\frac{k[x,y,z]}{(y^2-x^3)})) \leq 1$ (see [1]). Therefore, we see $B(\Omega_2(\frac{k[x,y,z]}{(y^2-x^3)}), t)$ is a rational function.*

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