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Conformally parallel Spin(7) structures on solvmanifolds

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Abstract: In this paper we review the $Spin(7)$ geometry in relation to solvmanifolds. Starting from a 7-dimensional nilpotent Lie group N endowed with an invariant G_2 structure, we present an example of a homogeneous conformally parallel $Spin(7)$ metric on an associated solvmanifold. It is thought that this paper could lead to very interesting and exciting areas of research and new results in the direction of (locally conformally) parallel $Spin(7)$ structures.

Key words: Holonomy, G_2 and $Spin(7)$ manifolds, conformally parallel structures, solvmanifolds

1. Introduction

The concept of the holonomy group for a Riemannian manifold was first defined by Cartan in 1923 and is known to be an efficient tool in the study of Riemannian manifolds [1]. The list of possible holonomy groups of irreducible, simply-connected, nonsymmetric Riemannian manifolds was given by Berger in 1955 [7]. The refinement of Berger's list (as corrected later by Alekseevskii [3] and Gray-Brown [27]) includes the groups $SO(n)$ in n-dimensions, $U(n)$, $SU(n)$ in $2n$-dimensions, $Sp(n)$, $Sp(n)Sp(1)$ in $4n$-dimensions, and 2 special cases, G_2 holonomy in 7-dimensions and $Spin(7)$ holonomy in 8-dimensions. Manifolds with holonomy $SO(n)$ constitute the generic case, all others are denoted as manifolds with "special holonomy", and the last 2 cases are described as manifolds with "exceptional holonomy".

The existence problem of manifolds with exceptional holonomy was first solved by Bryant [11], complete examples were given by Bryant and Salamon [13], and the first compact examples were found by Joyce in 1996 [30]. The study of manifolds with exceptional holonomy and the construction of explicit examples is still an active research area in mathematics and related sciences (see also references in [18, 25, 36, 39, 38]).

In the many areas of geometries, parallel structures (e.g., locally conformal) have been studied extensively up to now (see [1, 8, 16, 19, 22, 30, 38]). Other geometries, such as locally conformal hyperkaehler and quaternion Kaehler, were also investigated in the literature [6, 8, 19, 25, 30]. Because of the significance of the holonomy group structure in Riemannian geometry, the choices of G_2 and $Spin(7)$ also deserve attention [10, 20, 22, 30, 35].

In physics, there exists a special interest in the construction of G_2 and $Spin(7)$ holonomy metrics due to their application in supergravity compactification (for more details, see [1, 18, 25, 30, 36, 35]). Since manifolds with special holonomy provide some geometrical structures for reducing the number of supersymmetries, they

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are natural candidates for the extra dimensions in string and M-theory [1, 23, 24, 25, 30, 36]. Incomplete Ricci-flat metrics of holonomy G_2 with a 2-step nilpotent isometry group N acting on orbits of codimension one were investigated by Gibbons et al. in [25]. One of these Ricci-flat metrics has also been considered in the study of special domain walls in string theory [1, 25]. These metrics (called the domain-wall metrics) are obtained as Heisenberg limits of higher-dimensional metrics of special holonomy (see details in Section 2). Solvmanifolds (solvable Lie groups endowed with a left-invariant metric) and solvable extensions of nilpotent Lie groups provide examples of homogeneous Einstein manifolds [8, 16], and these are special topics for our study. This paper is in line with efforts to better understand the geometry of various classes of manifolds whose structure group of the tangent bundle reduces to one of the exceptional groups, G_2 or $Spin(7)$.

An 8-dimensional Riemannian manifold (M,g) is called a $Spin(7)$ manifold if it admits a reduction of the structure group of the tangent bundle to the Lie group $Spin(7)$. The presence of a $Spin(7)$ structure is equivalent to the existence of globally defined 4-form Ω, called the *Bonan form* [9, 39] (or *Cayley form or fundamental* 4*-form*). Whenever this special 4-form is covariantly constant with respect to the Levi–Civita connection, the holonomy group is contained in $Spin(7)$, and the corresponding manifold is called a parallel $Spin(7)$ manifold. The development of the theory of explicit metrics with holonomy $Spin(7)$ (also G_2) follows the by now classical line of Bonan [10], Fernandez [21], Bryant [11], Salamon [13], and Joyce [30]. We shall review and present a few relevant facts in Sections 2 and 3.

The motivation for the present work was the paper by Chiossi and Fino [16] who, starting with a 6-dimensional nilpotent Lie group N endowed with an invariant $SU(3)$ structure, construct ed a homogeneous conformally parallel G_2 metric on an associated solvmanifold. Inspired by this work [16], moving up one dimension, we study $Spin(7)$ structures on a rank-one solvable extension of a metric 7-dimensional nilpotent Lie algebra $\mathfrak{n}$ endowed with an G_2 structure φ and a nonsingular self-adjoint derivation D, which is diagonalizable by a unitary basis in order to obtain the noncompact examples found in [25].

Our aim is to investigate conformally parallel $Spin(7)$ structures on Riemannian products and present an explicit example of a homogeneous conformally parallel $Spin(7)$ metric on an associated solvmanifold. The classification of these types of manifolds is an ongoing problem in 8-dimensional space.

The outline of the paper is as follows. In Section 2, we present some Ricci-flat metrics of special holonomy in dimensions $d = 4, 6, 7, 8$. The exceptional holonomy group structures in dimensions $d = 7$ and $d = 8$ are presented in Section 3. In Section 4 we study solvable extensions of nilpotent Lie algebras. Conformally parallel $Spin(7)$ structures on solvmanifolds, along with a classification and an example, are studied in Section 5. The conclusions and plans for future studies are presented in Section 6.

2. Ricci-flat metrics of special holonomy in dimensions $d = 4, 6, 7$ and $d = 8$

In mathematical and physical theories, manifolds that admit metrics for which the Ricci tensor vanishes play a special role [30, 35]. These types of metrics are called Ricci-flat in the literature [8]. Ricci-flat metrics also appear as the fixed-points of the dynamical system called the Ricci-flow. This area is actively studied in mathematics (e.g., finding explicit examples of Ricci-flat metrics in some special dimensions). The proof of the Poincaré conjecture by Perelman makes use of this Ricci-flow dynamical system. The holonomy group of M imposes algebraic constraints on the Riemannian curvature. In particular:

Theorem 2.1 *[10] If M is a Riemannian manifold and $Hol(M)$ is contained in $SU(n)$, $Sp(n)$, G_2, or $Spin(7)$, then M is Ricci-flat.*

In [25], incomplete Ricci-flat metrics of special holonomy in dimensions $d = 4, 6, 7$ and $d = 8$ with a nilpotent isometry group acting on orbits of codimension one were presented. Gibbons et al. [25] studied the cases of the domain-wall solutions in $d = 5, 6$ and $d = 7$. Their results gave rise to Ricci-flat Heisenberg metrics of dimensions $d = 6, 7$ and $d = 8$. Since each domain wall preserves a fraction of the supersymmetry in the sense of Gibbons et al., it follows that the associated Ricci-flat metrics admit certain numbers of covariantly constant spinors [25]. In other words, they are metrics with a special holonomy type, and so we see that from Ricci-flat metrics in dimensions 6, 7, and 8, the special holonomy groups $SU(3)$, G_2 and $Spin(7)$ arise.

In [25], it was shown that such Ricci-flat metrics are closely related to a complete homogeneous Einstein manifold with a solvable isometry group. These Ricci-flat metrics (called the domain-wall metrics in [25]) are obtained as Heisenberg limits of higher-dimensional metrics of special holonomy and are given by

$$\begin{aligned}
ds_4^2 &= \mathcal{H}(dy^2 + dz_2^2 + dz_3^2) + \mathcal{H}^{-1}(dz_1 + mz_3dz_2)^2, \\
ds_6^2 &= \mathcal{H}^{-2}[dz_1 + m(z_3dz_2 + z_5dz_4)]^2 + \mathcal{H}^2dy^2 + \mathcal{H}(dz_2^2 + \ldots + dz_5^2), \\
ds_6^2 &= \mathcal{H}^2dy^2 + \mathcal{H}^{-1}(dz_1 + mz_4dz_3)^2 + \mathcal{H}^{-1}(dz_2 + mz_5dz_3)^2 + \mathcal{H}^2dz_3^2 + \mathcal{H}(dz_4^2 + dz_5^2), \\
ds_7^2 &= \mathcal{H}^4dy^2 + \mathcal{H}^{-2}[dz_1 + m(z_4dz_3 + z_6dz_5)]^2 + \mathcal{H}^{-2}[dz_2 + m(z_5dz_3 - z_6dz_4)]^2 \\
&\quad + \mathcal{H}^2(dz_3^2 + dz_4^2 + dz_5^2 + dz_6^2), \\
ds_7^2 &= \mathcal{H}^3dy^2 + \mathcal{H}^{-1}(dz_1 + mz_6dz_5)^2 + \mathcal{H}^{-1}(dz_2 - mz_6dz_4)^2 \\
&\quad + \mathcal{H}^{-1}(dz_3 + mz_5dz_4)^2 + \mathcal{H}^2(dz_4^2 + dz_5^2 + dz_6^2), \\
ds_8^2 &= \mathcal{H}^6dy^2 + \mathcal{H}^{-2}[dz_1 + m(z_5dz_4 + z_7dz_6)]^2 + \mathcal{H}^{-2}[dz_2 + m(z_6dz_4 - z_7dz_5)]^2 \\
&\quad + \mathcal{H}^{-2}[dz_3 + m(z_7dz_4 + z_6dz_5)]^2 + \mathcal{H}^3(dz_4^2 + dz_5^2 + dz_6^2 + dz_7^2),
\end{aligned}$$

where $\mathcal{H} = my$ with a nonzero real number m (see equation (5.34)); y and z_i are the coordinates on the manifolds (see [25]). Note that the lower index term in metric ds_4^2 (e.g., 4) shows the dimension of the manifold. For more details on the structure of these metrics and methods of construction, refer to [25].

In [6] it was shown that the scale-invariant Ricci-flat metric in dimension 4 is conformal to a complete homogeneous metric

$$ds^2 = z^{-4}(d\tau + xdy)^2 + z^{-2}(dx^2 + dy^2 + dz^2) \tag{2.1}$$

on a 4-dimensional solvable Lie group, and the metric (2.1) is the only nontrivial complete homogeneous hyper-Hermitian metric in dimension 4. The scale-invariant Ricci-flat metric in dimension 4 coincides also with the natural metric on the cotangent bundle $T^*\mathbb{H}$ of the upper half plane $\mathbb{H}$ induced from a special Kaehler metric on $\mathbb{H}$ [37].

In [16], it was proven that the previous scale-invariant Ricci-flat metrics in dimensions $d = 6$ and $d = 7$ are conformal to complete homogeneous metrics on a solvable Lie group and that the associated complete homogeneous Einstein manifold is exactly the same solvable Lie group with a homogeneous Einstein metric. The solvable Lie group is the isometry group of the Einstein metric, and it is a rank-one solvable extension of the nilpotent Lie group, which is the isometry group of the Ricci-flat metric.

All the examples of solvable Lie groups that we will consider will be standard solvmanifolds, i.e. a solvable Lie group S endowed with a left-invariant metric such that the orthogonal complement $\mathfrak{a} = (\mathfrak{s}^1)^\perp$ of the commutator $\mathfrak{s}^1 = [\mathfrak{s}, \mathfrak{s}]$ of the Lie algebra $\mathfrak{s}$ of S is abelian. We recall that, given a metric nilpotent Lie algebra $\mathfrak{n}$ with a inner product $<,>$, a metric solvable Lie algebra $(\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}, <,>')$ is called a metric solvable extension of $(\mathfrak{n}, <,>)$ if the restrictions of the Lie bracket of $\mathfrak{s}$ and of the inner product $<,>'$ to $\mathfrak{n}$ coincide with the Lie bracket and $<,>$ of $\mathfrak{n}$, respectively. The dimension of $\mathfrak{a}$ is called the algebraic rank of $\mathfrak{s}$ (see also Section 4).

In the present paper, we study conformally parallel $Spin(7)$ structures, which are characterized by the fact that the Riemannian metric g can be transformed to a metric with holonomy subgroup of $Spin(7)$ by a transformation

$$g \mapsto \tilde{g} = e^{2f} g, \tag{2.2}$$

for some function $f \in C^\infty(M)$. In light of some prior studies [16, 40], one can investigate some special $Spin(7)$ structures on a rank-one solvable extension of a metric 7-dimensional nilpotent Lie algebra $\mathfrak{n}$ endowed with a G_2 structure φ and a nonsingular self-adjoint derivation D, which is diagonalizable by a unitary basis, in order to obtain an incomplete Ricci-flat metric of exceptional holonomy as given in [25]. It is shown in Section 4 that an extension is given by a metric Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}H$ with bracket

$$[H, U] = DU, \quad [U, V] = [U, V]_{\mathfrak{n}\times\mathfrak{n}}, \tag{2.3}$$

where $U, V \in \mathfrak{n}$ and $H \perp \mathfrak{n}, \|H\| = 1$. The subscript denotes the Lie bracket on $\mathfrak{n}$, and the inner product extends that of $\mathfrak{n}$. There is a natural $Spin(7)$ structure on the manifold $M = Y \times \mathbb{R}$ and sometimes it is called a cone structure on Y [15, 35] (where the t-coordinate of $\mathbb{R}$ is considered to be the the time direction on M), corresponding to the 4-form

$$\Omega = \varphi \wedge dt + *\varphi \ \in \Lambda^4 T^* M \tag{2.4}$$

where the Hodge dual map of φ (i.e. $*\varphi$) is considered on 7-dimensional manifold Y (that is, φ and $*\varphi$ are the G_2 forms on Y [30]), and t is a coordinate on $\mathbb{R}$.

Detailed information related to these special dimensions (i.e. $d = 7$ and $d = 8$) will be given in the following section.

3. Exceptional structures in special dimensions $d = 7$ and $d = 8$

In this section we present the basics of G_2 and $Spin(7)$ geometries with their related properties [11, 15, 22, 29, 30, 35], and also the 4 classes of $Spin(7)$ manifolds with respect to the Lee 1-form [15, 17, 21, 29, 38]. Suppose that Y indicates a 7-dimensional nilmanifold with an invariant G_2 structure. Thus, Y is endowed with a nondegenerate 3-form φ that induces a Riemannian metric h. The fundamental material for the G_2 (also $Spin(7)$) geometry can be found in standard holonomy references books [30, 35]. Let us only recall that the Riemannian geometry of Y is completely determined by the following special form (called fundamental 3-form in 7-dimension):

$$\varphi = e^{125} - e^{345} + e^{567} + e^{136} + e^{246} - e^{237} + e^{147}. \tag{3.5}$$

It has become customary to suppress wedge signs when writing differential forms, so $e^{ij\cdots}$ indicates $e^i \wedge e^j \wedge \ldots$ from now on. The results of Fernandez and Gray [22] give allow one to describe G_2 geometry exclusively in algebraic terms, by looking at the various components of $d\varphi, d{*\varphi}$ in the irreducible summands $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$, and $\mathcal{X}_4$ of the space $T^*Y \otimes \mathfrak{g}_2^\perp$. Many authors have studied special classes of G_2 structures; see, for instance [5, 15, 23]. Before concentrating on a particular situation, recall that in general the exterior derivatives can be expressed as

$$\begin{cases} d{*\varphi} &= 4\tau_4 \wedge *\varphi + \tau_2 \wedge \varphi \\ d\varphi &= \tau_1 \wedge *\varphi + 3\tau_4 \wedge \varphi + *\tau_3 \end{cases} \tag{3.6}$$

where the various τ_is represent the differential forms corresponding to the representations $\mathcal{X}_i$, as in [12]. For example, τ_4 is the 1-form encoding the conformal data of the structure. With the convention of dropping all unnecessary wedge signs, the torsion 3-form of the unique G_2 connection [23] is given by

$$\Phi = \tfrac{7}{6}\tau_1\varphi - *d\varphi + *(4\tau_4\,\varphi) \tag{3.7}$$

where

$$\tau_1 = \tfrac{1}{7}g(d\varphi, *\varphi) \tag{3.8}$$

and

$$\tau_4 = -\tfrac{3}{4}*(*d\varphi \wedge \varphi). \tag{3.9}$$

τ_4 is known as the Lee 1-form of the 7-manifold (see details in [15, 20]). Moving up one dimension, we consider a product M of Y with $\mathbb{R}$, endowed with metric g. Indicating by e^8 the unit 1-form on the real line, one obtains a basis for the cotangent spaces T_p^*M. The manifold M inherits a nondegenerate 4-form

$$\Omega = \varphi \wedge e^8 + *\varphi, \tag{3.10}$$

which defines a reduction to the Lie group [29]. In equation (3.15), a special note for the Hodge dual map of φ (i.e. $*\varphi$) is considered on 7-dimensional manifold Y, as mentioned before in equation (2.4) [30].

3.1. The classes of $Spin(7)$ manifolds and Lee 1-form

In $Spin(7)$ geometry, we recall that the special 4-form Ω is self-dual $*\Omega = \Omega$, where $*$ is the Hodge operator and the 8-form $*\Omega\wedge\Omega$ coincides with the volume form. It is well known that the subgroup of $GL(8,R)$, which fixes Ω, is isomorphic to the double covering $Spin(7)$ of $SO(7)$ [35]. Moreover, $Spin(7)$ is a compact simply connected Lie group of dimension 21 [1].

The 4-form Ω corresponds to a real spinor ϕ and, therefore, $Spin(7)$ can be identified as the isotropy group of a nontrivial real spinor [14]. A 3-fold vector cross-product P on R^8 can be defined by

$$< P(x\wedge y\wedge z), t >= \Omega(x,y,z,t),\ \ for\ \ x,y,z,t \in R^8. \tag{3.11}$$

Then $Spin(7)$ is also characterized by

$$Spin(7) = \{a \in O(8) | P(ax\wedge ay\wedge az) = P(x\wedge y\wedge z), x,y,z \in R^8\}. \tag{3.12}$$

The inner product $<,>$ on R^8 can be reconstructed from Ω [14, 21], which corresponds with the fact that $Spin(7)$ is a subgroup of $SO(8)$. A $Spin(7)$ structure on an 8-dimensional manifold M is by definition a reduction of the structure group of the tangent bundle to $Spin(7)$, and we shall also say that M is a $Spin(7)$ manifold. This can be described geometrically by saying that there is a 3-fold vector cross-product P [33] defined on M, or equivalently there exists a nowhere vanishing differential 4-form Ω on M that can be locally written as

$$\begin{aligned}\Omega \;=\;& e^{1258} + e^{3458} + e^{1368} - e^{2468} + e^{1478} + e^{2378} - e^{5678}\\ & - e^{1267} - e^{3467} + e^{1357} - e^{2457} - e^{1456} - e^{2356} + e^{1234}.\end{aligned} \tag{3.13}$$

This special 4-form Ω is called the Bonan (Cayley or fundamental) form of the $Spin(7)$ manifold M [10, 11, 39, 38]. We also recall that a $Spin(7)$ manifold (M, g, Ω) is said to be parallel if the holonomy of the metric $Hol(g)$ is a subgroup of $Spin(7)$. This is equivalent to saying that the fundamental form Ω is parallel with respect to the Levi–Civita connection ∇^{LC} of metric g. Moreover, $Hol(g) \subset Spin(7)$ if and only if $d\Omega = 0$ [11] and any parallel $Spin(7)$ manifold is Ricci-flat [10] (see also Theorem 2.1).

According to the classification given by Fernandez [21], there are 4 classes of $Spin(7)$ manifolds obtained as irreducible representations of $Spin(7)$ of the space $\nabla^{LC}\Omega$. By using the fact given by Cabrera et al. [15], it is considered as the 1-form of the 8-manifold defined by

$$7\Theta = - * (*d\Omega \wedge \Omega) = *(\delta\Omega \wedge \Omega). \tag{3.14}$$

It is called the Lee form (this 1-form is denoted by Θ) of a given $Spin(7)$ structure [29]. The 4 classes of $Spin(7)$ manifolds in the Fernandez classification can be described in terms of the Lee form, as below [29, 38]:

$$W_0 : d\Omega = 0; \quad W_1 : \Theta = 0; \quad W_2 : d\Omega = \Theta \wedge \Omega; \quad W_4 = W_1 \oplus W_2. \tag{3.15}$$

A $Spin(7)$ structure of class W_1, that is, a $Spin(7)$ structure with Lee form equal to zero, is called a balanced $Spin(7)$ structure. Cabrera [14] showed that the Lee form of a $Spin(7)$ structure in the class W_2 is closed; therefore, such a manifold is locally conformally equivalent to a parallel $Spin(7)$ manifold and it is called locally conformally parallel. If the Lee form is not exact (i.e. the structure is not globally conformally parallel), it is called strict locally conformally parallel. We summarize these facts in the Table.

Table. Fernandez classification table of $Spin(7)$ manifolds.

The classes of $Spin(7)$ manifolds	Conditions
Parallel case: W_0	$d\Omega = 0, \Theta = 0$
Balanced case: W_1	$\Theta = 0$
Locally conformally parallel case: W_2	$d\Omega = \Theta \wedge \Omega$
Mixed type: $W_4 = W_1 + W_2$	-

In the present paper, our main goal is to study conformally parallel $Spin(7)$ structures on Riemannian products. If this is the case, the class W_2 in (3.15) can be considered as

$$d\Omega = -4\Omega \wedge \Theta, \tag{3.16}$$

where Θ is a closed 1-form, which we can assume is proportional to e^8. So let us rewrite those relations as

$$d\Omega \quad = \quad 4me^8 \wedge \Omega \ , \tag{3.17}$$

which also serves as a definition for the real constant m. To prevent the holonomy of the metric g from reducing to $Spin(7)$, we implicitly assume that m does not vanish. Suppose from now that Y is a nilpotent Lie group. This is indeed no real restriction since [41] any Riemannian manifold Y admitting a transitive nilpotent Lie group of isometries is essentially a nilpotent Lie group N with an invariant metric.

4. The rank-one solvable extension of Lie algebra

Let us consider (N, h) as a 7-dimensional connected and simply connected nilpotent Lie group with a left-invariant Riemannian metric, and $\mathfrak{n}$ its Lie algebra. By using the Iwasawa decomposition theorem (presented in Definition 4.1), the simply connected spaces can be identified with solvable groups $\mathcal{S}$ having the characteristic properties [31, 32]. The orthonormal basis $\{e^1, \dots, e^7\}$ of the cotangent bundle T^*N is intended to be nilpotent, i.e. such that $de^i \in \Lambda^2 V_{i-1}$, where the spaces $V_j = \mathrm{span}_{\mathbb{R}}\{e^1, \dots, e^{j-1}\}$ nested the dual Lie algebra,

$$0 \subset V_1 \subset V_2 \subset \ldots \subset V_6 \subset V_7 = \mathfrak{n}^*. \tag{4.18}$$

The step length of $\mathfrak{n}$ is defined as the number p of nonzero subspaces appearing in the lower central series:

$$\mathfrak{n} \supseteq [\mathfrak{n}, \mathfrak{n}] \supseteq \big[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}\big] \supseteq \ldots \supseteq \{0\}. \tag{4.19}$$

Considering this, we note that the terms for abelian and 1-step Lie algebras are the same [26]. We need the fact [34] that a nilmanifold $\Gamma\backslash N$ and the Lie algebra of its universal cover have isomorphic cohomology theories, $H^*(\mathfrak{n}) \cong H^*_{\mathrm{dR}}(\Gamma\backslash N)$. Let us fix a unit element $H \notin \mathfrak{n}$, and suppose that there exists a nonsingular self-adjoint derivation D of $\mathfrak{n}$ endowing

$$\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}H \tag{4.20}$$

with the structure of a solvable Lie algebra. That is, we consider $\mathfrak{s}$ as an extension of the following kind, in light of [16].

Definition 4.1 *[8] A simply connected solvable group $\mathcal{S}$ is of Iwasawa type if its metric solvable Lie algebra $\big(\mathfrak{s}, \langle\, , \rangle\big)$ satisfies the following conditions:*

i. $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ where $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and $\mathfrak{a} = \mathfrak{n}^\perp$ is abelian;

ii. ad_H is self-adjoint with respect to the scalar product $\langle\, , \rangle$ and nonzero, for all $H \in \mathfrak{a}, H \neq 0$;

iii. for some (canonical) element $\tilde{H} \in \mathfrak{a}$, the restriction of $\mathrm{ad}_{\tilde{H}}$ to $\mathfrak{n}$ is a positive-definite (i.e. $\mathrm{ad}_{\tilde{H}}\,|_n$ has positive eigenvalues).

The Iwasawa decomposition of a semisimple group G as the product of compact, abelian, and nilpotent subgroups generalizes the way in which a square real matrix can be written as a product of an orthogonal matrix and an upper triangular matrix (a consequence of the Gram–Schmidt method) [31]. Iwasawa type extensions are instances of standard solvmanifolds in the sense of Heber [28], and in a way they represent the basic model of standard Einstein manifolds. The nilpotent Lie groups of concern (actually all, up to dimension 6) always admit Einstein solvable extensions [31, 32], and all known examples of noncompact homogeneous spaces with Einstein metrics are of this kind [32]. These extensions are completely solvable, i.e. the eigenvalues of any inner derivation are real. The curvature of these spaces must be nonpositive, because Ricci-flat homogeneous manifolds are flat [4].

Consider N with an invariant G_2 structure. One can suppose there exists a diagonalizable operator $D \in \mathit{Der}(\mathfrak{n})$ with respect to a basis that determines the rank-one extension as given in equation (4.20). This requires that there is indeed a unitary basis consisting of eigenvectors (it is usually denoted by $\{e_i\}, i = 1 \ldots 7$) for which the matrix associated to $D = ad_{\tilde{H}}$ is diagonal. Thus,

$$\mathrm{ad}_{\tilde{H}}(e_i) = c_i e_i \tag{4.21}$$

for some real constants c_i, which must be positive in order to satisfy Definition 4.1. The derivation D is chosen to be precisely ad_{e_8}, and since the Cartan subalgebra $\mathfrak{a}$ is now one-dimensional, the only inner automorphism acting on $\mathfrak{n}$ is the bracket with the vector $\tilde{H} = e_8$, which is self-adjoint for the inner product, and nondegenerate because $c_j \neq 0$, for all js. Therefore, the Maurer–Cartan equations of the rank-one solvable extension $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}e_8$ assume the following form:

$$\begin{cases} de^j = \hat{d}e^j + c_j e^{j8}, & 1 \leqslant j \leqslant 7 \\ de^8 = 0, \end{cases} \tag{4.22}$$

where the 'hat' indicates derivatives relative to the 7-dimensional world, i.e. $\hat{d} = d|_{\Lambda^*\mathbb{R}^7}$, and $\{e^j\}$ is the basis of $\mathfrak{s}$ dual to $\{e_i\}$. In general, the Lie structure of $\mathfrak{n}$ is defined by

$$\begin{cases} \hat{d}e^1 &= & a_1 e^{12} & + & \dots & + & a_{21}e^{67} \\ \hat{d}e^2 &= & a_{22} e^{12} & + & \dots & + & a_{42}e^{67} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hat{d}e^6 &= & a_{106} e^{12} & + & \dots & + & a_{126}e^{67} \\ \hat{d}e^7 &= & a_{127} e^{12} & + & \dots & + & a_{147}e^{67} \end{cases} \tag{4.23}$$

where all coefficients a_k are real numbers.

5. Conformally parallel $Spin(7)$-structures on solvmanifolds

Let M be an 8-dimensional manifold with a $Spin(7)$-structure. The Lie group $Spin(7)$ is isomorphic to the subgroup of $GL(8,\mathbb{R})$, leaving invariant the 4-form

$$\Omega = \varphi \wedge e^8 + *\varphi.$$

Thus, the existence of a $Spin(7)$-structure is equivalent to the existence of a nondegenerate 4-form Ω on the manifold. In [21] the covariant derivative of Ω was studied, and a classification of such manifolds into 4 classes was obtained. If Ω is parallel with respect to the Levi–Civita connection, then the holonomy group is contained in $Spin(7)$ and the $Spin(7)$-structure is called parallel (see details in Section 3). If

$$d\Omega = -4\Omega \wedge \Theta$$

with Θ a closed 1-form, then the $Spin(7)$-structure is locally conformal to a parallel structure. Detailed study on the Lee 1-form with some special type of metrics was done recently [38].

5.1. A partial classification

If we write

$$\Omega = \varphi \wedge e^8 + *\varphi,$$

then one immediately finds that

$$d\Omega = d\varphi \wedge e^8 + d * \varphi. \tag{5.24}$$

Reflecting the splitting of the fibers of the cotangent bundle $T_p^*M = \mathbb{R}^7 \oplus \mathbb{R}e^8$, the relations (4.22) give the exterior derivatives of the real forms:

$$\begin{aligned} d\varphi = \hat{d}\varphi &+ (c_1+c_2+c_5)e^{1258} - (c_3+c_4+c_5)e^{3458} + (c_5+c_6+c_7)e^{5678} \\ &+ (c_1+c_3+c_6)e^{1368} + (c_2+c_4+c_6)e^{2468} - (c_2+c_3+c_7)e^{2378} \\ &+ (c_1+c_4+c_7)e^{1478}, \end{aligned}$$

and

$$\begin{aligned} d*\varphi = \hat{d}*\varphi &+ (c_3+c_4+c_6+c_7)e^{34678} - (c_1+c_2+c_6+c_7)e^{12678} \\ &+ (c_1+c_2+c_3+c_4)e^{12348} + (c_2+c_4+c_5+c_7)e^{24578} \\ &+ (c_1+c_3+c_5+c_7)e^{13578} - (c_1+c_4+c_5+c_6)e^{14568} \\ &+ (c_2+c_3+c_5+c_6)e^{23568}. \end{aligned}$$

When, in general, $Spin(7)$-manifolds M are constructed starting from 7 dimensions, many of their features are determined by the underlying G_2 structure, and the following proposition (using equation (3.17)) allows us to discover a geometrical constraint.

Proposition 5.1 *Let $M = Y \times \mathbb{R}$ be an 8-dimensional manifold. When (M, Ω) is conformal to a $Spin(7)$-holonomy manifold, Y has a cocalibrated G_2 structure.*

Proof That $\hat{d}*\varphi = 0$ is clear if one considers the terms in (5.24) that belong to $(e^8)^\perp$. On the other hand, the proof is easier to see when one notices the connection between equations (3.13) and (5.1) and the formulas for $d\varphi$ and $d*\varphi$ that immediately follow (5.1). □

Moreover, the components of (3.17) in the direction of e^8 read

$$\begin{aligned} \hat{d}\varphi \;=\; & -(4m+c_3+c_4+c_6+c_7)e^{3467} + (4m+c_1+c_2+c_6+c_7)e^{1267} \\ & -(4m+c_1+c_2+c_3+c_4)e^{1234} - (4m+c_2+c_4+c_5+c_7)e^{2457} \\ & -(4m+c_1+c_3+c_5+c_7)e^{1357} + (4m+c_1+c_4+c_5+c_6)e^{1456} \\ & -(4m+c_2+c_3+c_5+c_6)e^{2356}. \end{aligned} \tag{5.25}$$

Remark 5.2 *Note that a nearly parallel G_2 manifold is characterized by $d\varphi = \lambda * \varphi$ with constant λ [22].*

Proposition 5.3 *Let (M, Ω) be a conformal parallel $Spin(7)$-holonomy manifold as in Proposition 5.1. Then Y has a nearly parallel G_2 structure if and only if all c_i s are equal and different from $-m$.*

Proof If Y has a nearly parallel G_2 structure, then the coefficients of the exterior forms in equation (5.25) are all equal and different from zero. This implies the one-way proof that c_is are equal and different from $-m$. The other way is clear. □

In order to construct an explicit example of a conformally parallel $Spin(7)$ structure on a solvmanifold, the point is to find all possible coefficients $a_k, k = 1, \ldots, 147$ such that $d^2(e^j) = 0$ and (3.17) are satisfied for some nonvanishing m. The following section gives this construction explicitly.

5.2. An example of conformally parallel $Spin(7)$-structures on solvmanifolds

Consider the family of 3-step solvable Lie algebras $\mathfrak{s}_{c_1,c_2}$ with structure equations

$$\begin{aligned} de^i &= c_1 e^i \wedge e^8, i = 1, \ldots, 4, \\ de^5 &= 2c_1 e^5 \wedge e^8 + c_2 e^1 \wedge e^2 + c_2 e^3 \wedge e^4, \\ de^6 &= 2c_1 e^6 \wedge e^8 + c_2 e^1 \wedge e^3 - c_2 e^2 \wedge e^4, \\ de^7 &= 2c_1 e^7 \wedge e^8 + c_2 e^1 \wedge e^4 + c_2 e^2 \wedge e^3, \\ de^8 &= 0, \end{aligned} \tag{5.26}$$

with c_1, c_2 nonzero real numbers. For any $c_1, c_2 \neq 0$, the Lie algebras $\mathfrak{s}_{c_1,c_2}$ are isomorphic to the Lie algebra

$$\begin{aligned} dE^i &= E^i \wedge E^8, i = 1, \ldots, 4, \\ dE^5 &= 2E^5 \wedge E^8 + E^1 \wedge E^2 + E^3 \wedge E^4 \\ dE^6 &= 2E^6 \wedge E^8 + E^1 \wedge E^3 - E^2 \wedge E^4, \\ dE^7 &= 2E^7 \wedge E^8 + E^1 \wedge E^4 + E^2 \wedge E^3, \\ dE^8 &= 0, \end{aligned} \tag{5.27}$$

by the isomorphism

$$\begin{aligned} E^i &= c_2 e^i, i = 1, \ldots, 7, \\ E^8 &= c_1 e^8. \end{aligned} \tag{5.28}$$

The 4-form

$$\begin{aligned} \Omega = {} & e^{1258} + e^{3458} + e^{1368} - e^{2468} + e^{1478} + e^{2378} - e^{5678} - e^{1267} - e^{3467} \\ & + e^{1357} - e^{2457} - e^{1456} - e^{2356} + e^{1234} \end{aligned}$$

is such that

$$d\Omega = (6c_2 - 4c_1)e^{12348} + (c_2 - 6c_1)(e^{13578} - e^{24578} - e^{14568} - e^{12678} - e^{23568} - e^{34678}).$$

The metric

$$g = \sum_{i=1}^{8} (e^i)^2 \tag{5.29}$$

has Ricci tensor $\mathrm{Ric}(g)$ with respect to the orthonormal basis $(e_1, \ldots, e_8)$ given by

$$\begin{aligned} \mathrm{Ric}(e_i, e_i) &= -10c_1^2 - \frac{3}{2}c_2^2, i = 1, \ldots, 4, \\ \mathrm{Ric}(e_l, e_l) &= -20c_1^2 + c_2^2, l = 5, 6, 7, \\ \mathrm{Ric}(e_8, e_8) &= -16c_1^2. \end{aligned} \tag{5.30}$$

Thus, g is Einstein if and only if $c_2^2 = 4c_1^2$, with, in this case, the Ricci tensor given by

$$\mathrm{Ric}(g) = -16c_1^2 g. \tag{5.31}$$

The metric g is conformal to a metric with holonomy $Spin(7)$ if $c_2 = -\frac{2}{5}c_1$. One can find global coordinates $(x_1, \dots, x_7, t)$ on the simply connected Lie group S_{c_1,c_2} such that

$$\begin{aligned} e^i &= e^{-c_1 t}dx_i, \quad i = 1, \dots, 4, \\ e^5 &= e^{-2c_1 t}(c_2 dx_5 + c_2 x_1 dx_2 + c_2 x_3 dx_4), \\ e^6 &= e^{-2c_1 t}(c_2 dx_6 + c_2 x_1 dx_3 - c_2 x_2 dx_4), \\ e^7 &= e^{-2c_1 t}(c_2 dx_7 + c_2 x_1 dx_4 + c_2 x_2 dx_3), \\ e^8 &= dt. \end{aligned} \tag{5.32}$$

The Riemannian metric (with $c_2^2 = 4c_1^2$)

$$\begin{aligned} g_1 = {} & e^{-2c_1 t}\left((dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2\right) + 4c_1^2 e^{-4c_1 t}(dx_5 + x_1 dx_2 + x_3 dx_4)^2 \\ & + 4c_1^2 e^{-4c_1 t}(dx_6 + x_1 dx_3 - x_2 dx_4)^2 + 4c_1^2 e^{-4c_1 t}(dx_7 + x_1 dx_4 + x_2 dx_3)^2 + dt^2 \end{aligned}$$

is the Einstein metric considered in [25, Section 7.1] for

$$k = -c_1, z_1 = x_5, z_2 = -x_6, z_3 = x_7, z_4 = x_4, z_5 = x_3, z_6 = x_2, z_7 = x_1. \tag{5.33}$$

For $c_2 = -\frac{2}{5}c_1$, the metric g is conformal parallel and the conformal metric with holonomy $Spin(7)$ is given by

$$\begin{aligned} e^{\frac{16}{5}c_1 t} g = {} & e^{\frac{6}{5}c_1 t}\left((dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2\right) + e^{\frac{16}{5}c_1 t}dt^2 \\ & + \tfrac{4}{25}c_1^2 e^{-\frac{4}{5}c_1 t}(dx_5 + x_1 dx_2 + x_3 dx_4)^2 + \tfrac{4}{25}c_1^2 e^{-\frac{4}{5}c_1 t}(dx_6 + x_1 dx_3 - x_2 dx_4)^2 \\ & + \tfrac{4}{25}c_1^2 e^{-\frac{4}{5}c_1 t}(dx_7 + x_1 dx_4 + x_2 dx_3)^2. \end{aligned}$$

This metric is a nonhomogeneous metric on the 3-step solvable Lie group $S_{c_1, -\frac{2}{5}c_1}$ and is locally isometric to the scale-invariant metric with holonomy $Spin(7)$

$$\begin{aligned} ds_8^2 = {} & H^6 dy^2 + H^{-2}[dz_1 + m(z_5 dz_4 + z_7 dz_6)]^2 + H^{-2}[dz_2 + m(z_6 dz_4 - z_7 dz_5)]^2 \\ + {} & H^{-2}[dz_3 + m(z_7 dz_4 + z_6 dz_5)]^2 + H^3(dz_4^2 + dz_5^2 + dz_6^2 + dz_7^2) \end{aligned}$$

constructed in [25, Section 4.3.1] on the product $M = Y^7 \times \mathbb{R}$, where Y^7 is the total space of a principal T^4-bundle over T^3. Indeed, one has:

$$\begin{aligned} & z_1 = \tfrac{2}{5}x_5, && z_2 = -\frac{2}{5}x_6, \\ & z_3 = \tfrac{2}{5}x_7, && z_4 = x_4, \\ & z_5 = x_3, && z_6 = x_2, \\ & z_7 = x_1, && y = \frac{2c_1}{5}e^{\frac{2}{5}c_1 t}, \\ & m = \tfrac{2}{5}c_1, && H = my. \end{aligned} \tag{5.34}$$

6. Conclusions

In light of [16], we studied $Spin(7)$ structures on a rank-one solvable extension of a metric 7-dimensional nilpotent Lie algebra $\mathfrak{n}$ endowed with an G_2 structure φ and a nonsingular self-adjoint derivation D, which is diagonalizable by a unitary basis, in order to obtain the noncompact examples found in [25]. The classification of these types of manifolds is an ongoing problem, also being treated by the authors of [16]. Finally, we mentioned different directions on G_2 and $Spin(7)$ manifolds related to the geometric structures on these spaces. Starting from certain classes of G_2-manifolds Y, conformally parallel $Spin(7)$ metrics on Riemannian products associated to these manifolds with some special geometric properties should be studied [2, 5, 33]. All of them give also new research areas related to these exceptional geometries in dimensions $d = 7$ and $d = 8$.

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