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On Finsler metrics with vanishing S-curvature

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Abstract: In this paper, we consider Finsler metrics defined by a Riemannian metric and a 1-form on a manifold. We study these metrics with vanishing S-curvature. We find some conditions under which such a Finsler metric is Berwaldian or locally Minkowskian.

Key words: (α, β)-metric, Berwald metric, S-curvature.

1. Introduction

In Finsler geometry, there are several important non-Riemannian quantities: the Cartan torsion $\mathbf{C}$, the Berwald curvature $\mathbf{B}$, the S-curvature $\mathbf{S}$, the new non-Riemannian curvature $\mathbf{H}$, etc. They all vanish for Riemannian metrics; hence they are said to be non-Riemannian [6, 7, 9].

Let (M, F) be a Finsler manifold. The Finsler metric F on M induced a spray $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$, which determines the geodesics, where $G^i = G^i(x, y)$ are called the spray coefficients of $\mathbf{G}$. A Finsler metric F is called a Berwald metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ are quadratic in $y \in T_xM$ for any $x \in M$. The Berwald curvature $\mathbf{B}$ of Finsler metrics is an important non-Riemannian quantity constructed by L. Berwald.

The S-curvature is constructed by Shen for given comparison theorems on Finsler manifolds [10]. A natural problem is to study and characterize Finsler metrics of vanishing S-curvature. It is known that some Randers metrics are of vanishing S-curvature [8, 13]. This is one of our motivations to consider Finsler metrics with vanishing S-curvature. Shen proved that every Berwald metric satisfies $\mathbf{S} = 0$ [10]. In [2], Bao and Shen find a class of non-Berwaldian Randers metrics with vanishing S-curvature. Thus the converse of Shen's theorem is not true, generally. A natural question arises: *"Under which conditions does the converse of Shen's Theorem hold?"*

There are 2 basic tensors on Finsler manifolds: fundamental metric tensor $\mathbf{g}_y$ and the Cartan torsion $\mathbf{C}_y$, which are second and third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$, respectively. The rate of change of $\mathbf{C}$ along Finslerian geodesics is called Landsberg curvature $\mathbf{L}_y$. Taking a trace of $\mathbf{C}$ and $\mathbf{L}$ gives us mean Cartan torsion $\mathbf{I}$ and mean Landsberg curvature $\mathbf{J}$, respectively. $\mathbf{J}/\mathbf{I}$ is regarded as the relative rate of change of $\mathbf{I}$ along Finslerian geodesics. Then F is said to be an isotropic mean Landsberg metric if $\mathbf{J} + cF\mathbf{I} = 0$, where $c = c(x)$ is a scalar function on M.

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Theorem 1 *Let* $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ *be a non-Riemannian* (α, β)*-metric on manifold* M *with vanishing S-curvature and* $\phi \neq c_1\sqrt{1 + c_2 s^2} + c_3 s$ *for any constant* $c_1 > 0$, c_2, c_3. *Suppose that* $\mathbf{J}/\mathbf{I}$ *is isotropic,*

$$\mathbf{J} + c(x)F\mathbf{I} = 0,$$

where $c = c(x)$ *is a scalar function on* M. *Then* F *reduces to a Berwald metric.*

There is a weaker notion of Berwald metrics, namely R-quadratic metrics. For a Finsler space (M, F), the Riemann curvature is a family of linear transformations $\mathbf{R}_y : T_xM \to T_xM$, where $y \in T_xM$, with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\forall\lambda > 0$ (the definition will be given in §2). If F is Riemannian, i.e. $F(y) = \sqrt{g(y, y)}$ for some Riemannian metric g, then $\mathbf{R}_y := \mathrm{R}(\cdot, y)y$, where $\mathrm{R}(u, v)z$ denotes the Riemannian curvature tensor of g. In this case, $\mathbf{R}_y$ is quadratic in $y \in T_xM$. A Finsler metric is said to be R-quadratic if its Riemann curvature $\mathbf{R}_y$ is quadratic in $y \in T_xM$ [11]. There are many non-Riemannian R-quadratic Finsler metrics. For example, all Berwald metrics are R-quadratic. Indeed a Finsler metric is R-quadratic if and only if the h-curvature of Berwald connection depends on position only in the sense of Bácsó-Matsumoto [1]. The notion of R-quadratic Finsler metrics was introduced by Shen, and can be considered a generalization of R-flat metrics.

Theorem 2 *Let* $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ *be a non-Riemannian* (α, β)*-metric on a manifold* M *with vanishing S-curvature and* $\phi \neq c_1\sqrt{1 + c_2 s^2} + c_3 s$ *for any constant* $c_1 > 0$, c_2, c_3. *Suppose that* F *is R-quadratic. Then* F *reduces to a Berwald metric.*

Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory, and multiterminal information theory. Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, and play a very important role in studying flat Finsler information structures. A Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat if at any point there is a standard coordinate system (x^i, y^i) in TM satisfying $[F^2]_{x^k y^l} y^k = 2[F^2]_{x^l}$. It is easy to see that every locally Minkowskian metric satisfies in the above equation, hence is locally dually flat [14, 15]. Here, we find some conditions under which a locally dually flat non-Randers type (α, β)-metric reduces to a locally Minkowskian metric. More precisely, we prove the following.

Theorem 3 *Let* $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ *be a non-Randers type* (α, β)*-metric on a manifold* M *of dimension* $n \geq 3$ *with vanishing S-curvature. Suppose that one of the following holds:*

(a) $\phi'(0) \neq 0$ *and* $(k_2 - k_3 b^2)b^2 \neq -1$;

(b) $\phi'(0) = \phi''(0) = 0$ *or* ϕ *is a polynomial that* $\phi'(0) = 0$.

If F *is locally dually flat then it reduces to a locally Minkowskian metric.*

In this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively [12].

2. Preliminary

A Finsler metric on a manifold M is a nonnegative function F on TM having the following properties:

(a) F is C^∞ on $TM_0 := TM \setminus \{0\}$;

(b) $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0$, $y \in TM$;

(c) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u,v) := \frac{1}{2}\Big[F^2(y+su+tv)\Big]\Big|_{s,t=0}, \qquad u,v \in T_xM.$$

At each point $x \in M$, $F_x := F|_{T_xM}$ is an Euclidean norm if and only if $\mathbf{g}_y$ is independent of $y \in T_xM_0$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]\Big|_{t=0}, \qquad u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion.

Given a Finsler manifold (M,F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i(x,y)$ are local functions on TM_0 satisfying

$$G^i(x,\lambda y) = \lambda^2 G^i(x,y) \quad \lambda > 0.$$

$\mathbf{G}$ is called the associated spray to (M,F). The projection of an integral curve of G is called a geodesic in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$. A Finsler metric F is called a Berwald metric if G^i are quadratic in $y \in T_xM$ for any $x \in M$ or equivalently the Berwald curvature

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$$

is vanishing.

A Finsler metric $F = F(x,y)$ on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the following form:

$$G^i = -\frac{1}{2} g^{ij} H_{y^j},$$

where $H = H(x,y)$ is a C^∞ scalar function on TM_0 satisfying $H(x,\lambda y) = \lambda^3 H(x,y)$ for all $\lambda > 0$. Such a coordinate system is called an adapted coordinate system [4]. In [8], Shen proved that the Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is dually flat if and only if it satisfies

$$(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}.$$

In this case, $H = -\frac{1}{6}[F^2]_{x^m}y^m$.

Let $U(t)$ be a vector field along a curve $c(t)$. The canonical covariant derivative $\mathrm{D}_{\dot c}U(t)$ is defined by

$$\mathrm{D}_{\dot c}U(t) := \Big\{\frac{dU^i}{dt}(t) + U^j(t)\frac{\partial G^i}{\partial y^j}(\dot c(t))\Big\}\frac{\partial}{\partial x^i}|_{c(t)}.$$

$U(t)$ is said to be parallel along c if $\mathrm{D}_{\dot c(t)}U(t) = 0$.

To measure the changes in the Cartan torsion $\mathbf{C}$ along geodesics, we define $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u,v,w) := \frac{d}{dt}\Big[\mathbf{C}_{\dot c(t)}(U(t),V(t),W(t))\Big]|_{t=0},$$

where $c(t)$ is a geodesic and $U(t), V(t), W(t)$ are parallel vector fields along $c(t)$ with $\dot c(0) = y, U(0) = u, V(0) = v, W(0) = w$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$. An important fact is that if F is Berwaldian, then it is Landsbergian. $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change in $\mathbf{C}$ along Finslerian geodesics. Then F is said to be an isotropic Landsberg metric if $\mathbf{L} = cF\mathbf{C}$, where $c = c(x)$ is a scalar function on M.

For a vector $y \in T_xM_0$, the Riemann curvature $R_y : T_xM \to T_xM$ is defined by $R_y(u) := R^i{}_k(y)u^k\frac{\partial}{\partial x^i}$, where

$$R^i{}_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $R := \{R_y\}_{y \in TM_0}$ is called the Riemann curvature. There are many Finsler metrics whose Riemann curvature in every direction is quadratic. A Finsler metric F is said to be R-quadratic if R_y is quadratic in $y \in T_xM$ at each point $x \in M$.

Put

$$R_j{}^i{}_{kl}(y) := \frac{1}{3}\frac{\partial}{\partial y^j}\Big[\frac{\partial R^i{}_k}{\partial y^l} - \frac{\partial R^i{}_l}{\partial y^k}\Big].$$

$R_j{}^i{}_{kl}$ are the coefficients of the h-curvature of the Berwald connection, which are also denoted by $H_j{}^i{}_{kl}$ in the literature. We have

$$R^i{}_k(y) = y^j R_j{}^i{}_{kl}(y)y^l.$$

Thus $R^i{}_k(y)$ is quadratic in $y \in T_xM$ if and only if $R_j{}^i{}_{kl}(y)$ are functions of x only.

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{\mathrm{Vol}(\mathbb{B}^n(1))}{\mathrm{Vol}\Big\{(y^i) \in \mathbb{R}^n \ \Big|\ F\Big(y^i\frac{\partial}{\partial x^i}|_x\Big) < 1\Big\}}.$$

In general, the local scalar function $\sigma_F(x)$ cannot be expressed in terms of elementary functions, even if F is locally expressed by elementary functions.

Let $G^i(x,y)$ denote the geodesic coefficients of F in the same local coordinate system. The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i\frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big],$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$. It is proved that $\mathbf{S} = 0$ if F is a Berwald metric [8]. There are many non-Berwald metrics satisfying $\mathbf{S} = 0$ [2].

Given a Riemannian metric α, a 1-form β on a manifold M, and a C^∞ function $\phi = \phi(s)$ on $[-b_o, b_o]$, where $b_o := \sup_{x\in M} \|\beta\|_x$, one can define a function on TM by

$$F := \alpha\phi(s), \qquad s = \frac{\beta}{\alpha}.$$

If ϕ and b_o satisfy (2.1) and (2.2) below, then F is a Finsler metric on M. Finsler metrics in this form are called (α, β)-metrics. Randers metrics are special (α, β)-metrics.

Now we consider (α, β)-metrics. Let $\alpha = \sqrt{a_{ij}y^iy^j}$ be a Riemannian metric and $\beta = b_iy^i$ a 1-form on a manifod M. Let

$$\|\beta\|_x := \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

For a C^∞ function $\phi = \phi(s)$ on $[-b_o, b_o]$, where $b_o = \sup_{x\in M} \|\beta\|_x$, define

$$F := \alpha\phi(s), \quad s = \frac{\beta}{\alpha}.$$

By a direct computation, we obtain

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j - \rho_1(b_i\alpha_j + b_j\alpha_i) + s\rho_1\alpha_i\alpha_j,$$

where $\alpha_i := a_{ij}y^j/\alpha$, and

$$\begin{aligned} \rho &:= \phi(\phi - s\phi'), \\ \rho_0 &:= \phi\phi'' + \phi'\phi', \\ \rho_1 &:= s(\phi\phi'' + \phi'\phi') - \phi\phi'. \end{aligned}$$

By further computation, one obtains

$$\det(g_{ij}) = \phi^{n+1}(\phi - s\phi')^{n-2}\Big[(\phi - s\phi') + (\|\beta\|_x^2 - s^2)\phi''\Big]\det(a_{ij}).$$

Using the continuity, one can easily show that

Lemma 1 *Let $b_o > 0$. $F = \alpha\phi(\beta/\alpha)$ is a Finsler metric on M for any pair $\{\alpha, \beta\}$ with $\sup_{x\in M} \|\beta\|_x \le b_o$ if and only if $\phi = \phi(s)$ satisfies the following conditions:*

$$\phi(s) > 0, \qquad (|s| \le b_o) \tag{2.1}$$

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad (|s| \le b \le b_o). \tag{2.2}$$

Let

$$r_{ij} := \frac{1}{2}\Big(b_{i|j} + b_{j|i}\Big), \quad s_{ij} := \frac{1}{2}\Big(b_{i|j} - b_{j|i}\Big).$$

$$r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}.$$

Let $r_{i0} := r_{ij}y^j$, $s_{i0} := s_{ij}y^j$, $r_0 := r_jy^j$ and $s_0 := s_jy^j$. Suppose that $G^i = G^i(x,y)$ and $\bar{G}^i = \bar{G}^i(x,y)$ denote the coefficients of F and α respectively in the same coordinate system. By definition, we obtain the following identity:

$$G^i = \bar{G}^i + Py^i + Q^i, \tag{2.3}$$

where

$$\begin{aligned} P &= \alpha^{-1}\Theta\Big[r_{00} - 2Q\alpha s_0\Big] \\ Q^i &= \alpha Q s^i{}_0 + \Psi\Big[r_{00} - 2Q\alpha s_0\Big]b^i, \\ Q &= \frac{\phi'}{\phi - s\phi'} \\ \Theta &= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big((\phi - s\phi') + (b^2 - s^2)\phi''\Big)} \\ \Psi &= \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}. \end{aligned}$$

Clearly, if β is parallel with respect to α ($r_{ij} = 0$ and $s_{ij} = 0$), then $P = 0$ and $Q^i = 0$. In this case, $G^i = \bar{G}^i$ are quadratic in y, and F is a Berwald metric.

Now, let $\phi = \phi(s)$ be a positive C^∞ function on $(-b_0, b_0)$. For a number $b \in [0, b_0)$, let

$$\Phi := -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q'' \tag{2.4}$$

where

$$\Delta := 1 + sQ + (b^2 - s^2)Q' \tag{2.5}$$

Lemma 2 ([3]) Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be a non-Riemannian (α, β)-metric on a manifold and $b := \|\beta_x\|_\alpha$. Suppose that $\phi \neq c_1\sqrt{1 + c_2s^2} + c_3s$ for any constant $c_1 > 0$, c_2 and c_3. Then F is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$, if and only if one of the following holds:
(a) β satisfies

$$r_{ij} = \varepsilon(b^2 a_{ij} - b_ib_j), \quad s_j = 0, \tag{2.6}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{2.7}$$

where k is a constant. In this case, $\mathbf{S} = (n+1)cF$ with $c = k\varepsilon$.
(b) β satisfies

$$r_{ij} = 0, \quad s_j = 0 \tag{2.8}$$

In this case, $\mathbf{S} = 0$, regardless of choices of a particular ϕ.

3. Proof of Theorem 1

We have the following formula for the spray coefficient G^i of F

$$G^i = G^i_\alpha + \alpha Q s^i_0 + (-2Q\alpha s_0 + r_{00})(\Theta\alpha^{-1}y^i + \Psi b^i), \tag{3.9}$$

where $s^i_j := a^{ih}s_{hj}$, $s^i_0 := s_i y^i$, $r_{00} = r_{ij}y^i y^j$ and

$$\Theta = \frac{Q - sQ'}{2\Delta}, \quad \Psi = \frac{Q'}{2\Delta}.$$

By a direct computation, we can obtain a formula for the mean Cartan torsion of (α, β)- metrics as follows:

$$I_i = -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i - s y_i). \tag{3.10}$$

According to Deickeğs theorem, a Finsler metric is Riemannian if and only if $\mathbf{I} = 0$. Clearly, an (α, β)-metric $F = \alpha\phi(s)$ is Riemannian if and only if $\Phi = 0$.

In [5], Li and Shen obtained the mean Landsberg curvature of an (α, β)-metric $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ as follows

$$\begin{aligned} J_i = & -\frac{1}{\alpha^2\Delta(b^2 - s^2)}\big[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\big](r_0 + s_0)h_i \\ & - \frac{h_i}{2\alpha^3\Delta(b^2 - s^2)}(\Psi_1 + s\frac{\Phi}{\Delta})(r_{00} - 2\alpha Q s_0) \\ & - \frac{\Phi}{2\alpha^3\Delta^2}\Big[-\alpha Q' s_0 h_i + \alpha Q(\alpha^2 s_i - y_i s_0) + \alpha^2\Delta s_{i0} \\ & + \alpha^2(r_{i0} - 2\alpha Q s_0) - (r_{00} - 2\alpha Q s_0)y_i\Big]. \end{aligned} \tag{3.11}$$

where $h_i := \alpha b_i - s y_i$ and

$$\Psi_1 := \sqrt{b^2 - s^2}\,\Delta^{\frac{1}{2}}\Big[\frac{\sqrt{b^2 - s^2}}{\Delta^{\frac{3}{2}}}\Big]'.$$

They also obtained

$$\bar{J} := J_i b^i = -\frac{\Delta}{2\alpha^2}\Big[\Psi_1(r_{00} - 2\alpha Q s_0) + \alpha\Psi_2(r_0 + s_0)\Big], \tag{3.12}$$

where

$$\Psi_2 := 2(n+1)(Q - sQ') + 3\frac{\Phi}{\Delta}.$$

Lemma 3 *Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be n non-Riemannian (α, β)-metric on manifold M. Suppose that $\phi \neq c_1\sqrt{1 + c_2 s^2} + c_3 s$ for any constant $c_1 > 0$, c_2.If F has vanishing S-curvature and a weakly Landsberg metric then F is a Berwald metric.*

Proof By (2.8) and (3.11) we have

$$J_i = -\frac{\Phi s_{i0}}{2\alpha\Delta}. \tag{3.13}$$

From (3.13) we conclude if F is weakly Landsberg then $s^i_0 = 0$ and because of $r_{00} = 0$, F is a Berwald metric. □

Proof of Theorem 1 Let F be a relatively isotropic mean Landsberg curvature metric with vanishing S-curvature. The following holds:

$$J_k + cFI_k = 0. \tag{3.14}$$

By (2.8) and (3.12) we have $b_i J^i = 0$. Multiplying (3.14) by b^k yields

$$cF(b^k I_k) = 0. \tag{3.15}$$

If $c \neq 0$ from (3.15) we have $b^k I_k = 0$ and so by (3.10) we conclude

$$\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^3}(b^2\alpha^2 - \beta^2) = 0 \tag{3.16}$$

From (3.16) we conclude $\Phi = 0$ or $\phi - s\phi' = 0$. Then by (3.10) we have $I = 0$ and F is a Riemannian metric. By assumption F is a non-Riemannian metric and so $c = 0$. From (3.14), we conclude F is a weakly Landsberg metric. Then, by Lemma 3, F is a Berwald metric. The proof of Theorem 1 is complete.

4. Proof of Theorem 2

Lemma 4 ([9]) For the Berwald connection, the following Bianchi identities hold:

$$R^i{}_{jkl|m} + R^i{}_{jlm|k} + R^i{}_{jmk|l} = B^i{}_{jku}R^u{}_{lm} + B^i{}_{jlu}R^u{}_{km} + B^i{}_{klu}R^u{}_{jm} \tag{4.17}$$

$$B^i{}_{jml|k} - B^i{}_{jkm|l} = R^i{}_{jkl,m} \tag{4.18}$$

$$B^i{}_{jkl,m} = B^i{}_{jkm,l}. \tag{4.19}$$

Lemma 5 *Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ be a non-Riemannian (α, β)-metric on manifold M. Suppose that $\phi \neq c_1\sqrt{1 + c_2 s^2} + c_3 s$ for any constant $c_1 > 0$, c_2 If F has vanishing S-curvature then we have*

$$b_m B^m_{jkl} = 0 \tag{4.20}$$

Proof By (2.8), we have $s_0 = 0$. By assumption F has vanishing S curvature. By (2.8) and (3.9) we have

$$G^i = G^i_\alpha + \alpha Q s^i_0. \tag{4.21}$$

Multiplying (4.21) by b_i yields $b_i G^i = b_i G^i_\alpha$. Thus $b_m B^m_{jkl} = 0$. □

Proof of Theorem 2 According to Lemma 5, we have

$$b_{m|s}B^m_{jkl} + b_m B^m_{jkl|s} = 0. \tag{4.22}$$

By assumption F is an R-quadratic metric. Thus (4.18) implies that

$$B^i_{jkl|m} - B^i_{jkm|l} = 0. \tag{4.23}$$

Multiplying (4.23) by b_i yields

$$b_i B^i_{jkl|m} = b_i B^i_{jkm|l}. \tag{4.24}$$

From (4.22) and (4.24) we conclude

$$b_{i|m} B^i_{jkl} = b_{i|l} B^i_{jkm}. \tag{4.25}$$

Since $r_{ij} = 0$, then by multiplying (4.25) by y^l we obtain

$$s_{i0} B^i_{jkm} = 0. \tag{4.26}$$

By (4.21) we get $B^i_{jkl} = [\alpha Q s^i_0]_{y^j y^k y^l}$. From (4.26) we have

$$[\alpha Q]_{y^j y^k y^l} s_{i0} s^i_0 + [\alpha Q]_{y^j y^k} s_{i0} s^i_l + [\alpha Q]_{y^j y^l} s_{i0} s^i_k + [\alpha Q]_{y^k y^l} s_{i0} s^i_j = 0. \tag{4.27}$$

By (2.8), we have $s^i = s_i = 0$. Then multiplying (4.27) by $b^j b^k b^l$ yields

$$\left[b^j b^k b^l [\alpha Q]_{y^j y^k y^l} \right] s_{i0} s^i_0 = 0. \tag{4.28}$$

Then by (4.28), we conclude that β is a closed 1-form and then F reduces to a Berwald metric. The proof of Theorem 2 is complete. □

5. Proof of Theorem 3

In this section, we are going to prove Theorem 3. First, we remark the following.

Lemma 6 ([16]) Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, be a non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$, where $\phi'(0) \neq 0$ and $\beta \neq 0$. Then F is locally dually flat if and only if α, β, and ϕ satisfy

$$\begin{aligned}
s_{l0} &= \frac{1}{3}(\beta\theta_l - \theta b_l), \\
r_{00} &= \frac{2}{3}\theta\beta + \left[\tau + \frac{2}{3}(b^2\tau - \theta_l b^l)\right]\alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3 b^2)\tau\beta^2, \\
G^l_\alpha &= \frac{1}{3}[2\theta + (3k_1 - 2)\tau\beta]y^l + \frac{1}{3}(\theta^l - \tau b^l)\alpha^2 + \frac{1}{2}k_3\tau\beta^2 b^l, \\
&\tau[s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi')] = 0,
\end{aligned}$$

where $\tau := \tau(x)$ is a scalar function, $\theta := \theta_i(x) y^i$ is a 1-form on M, $\theta^l = a^{lm}\theta_m$, and

$$\begin{aligned}
k_1 &:= \Pi(0), \qquad k_2 := \frac{\Pi'(0)}{Q(0)}, \\
k_3 &:= \frac{1}{6Q(0)^2}[3Q''(0)\Pi'(0) - 6\Pi'(0)^2 - Q(0)\Pi'''(0)], \\
\Pi &:= \frac{\phi'^2 + \phi\phi''}{\phi(\phi - s\phi')}.
\end{aligned}$$

By Lemma 6, we can get the following.

Corollary 1 ([16]) Let $F=\alpha\phi(s),\ s=\frac{\beta}{\alpha}$ be an (α,β)-metric on a manifold M of dimension $n\geq 3$ with the same assumption as Lemma 6. Let ϕ satisfy

$$s(k_2-k_3s^2)(\phi\phi'-s\phi'-s\phi\phi'')-(\phi'^2+\phi\phi'')+k_1\phi(\phi-s\phi')\neq 0.$$

Then F is locally dually flat on M if and only if

$$s_{l0}=\frac{1}{3}(\beta\theta_l-\theta b_l), \tag{5.29}$$

$$r_{00}=\frac{2}{3}\left[\theta\beta-(\theta_l b^l)\alpha^2\right], \tag{5.30}$$

$$G^l_\alpha=\frac{1}{3}\left[2\theta y^l+\theta^l\alpha^2\right]. \tag{5.31}$$

where k_i $(1\leq i\leq 3)$ are the same as those of Theorem 6.

In [16], Xia proved the following.

Lemma 7 ([16]) Let $F:=\alpha\phi(s),\ s=\frac{\beta}{\alpha}$ be a non-Riemannian $(\alpha,\beta)-$metric on a manifold M of dimension $n\geq 3$. Suppose that $\phi(s)$ is an analytic function with $\phi'(0)=\phi''(0)=0$ or $\phi(s)$ is a polynomial of s with $\phi'(0)=0$ and $\beta=b_i(x)y^i\neq 0$. Then F is locally dually flat if and only if α and β satisfy (5.29), (5.30) and (5.31), where $\theta=\theta_i(x)y^i$ is a 1-form on M and $\theta^l:=a^{lm}\theta_m$.

Proof of Theorem 3 To prove Theorem 3, we consider some cases.

Case (1): $\phi'(0)=\phi''(0)=0$ or $\phi(s)$ is a polynomial of s with $\phi'(0)=0$. In this case, by Lemma 7 we have

$$s_{l0}=\frac{1}{3}(\beta\theta_l-\theta b_l), \tag{5.32}$$

$$r_{00}=\frac{2}{3}\left[\theta\beta-(\theta_l b^l)\alpha^2\right], \tag{5.33}$$

$$G^l_\alpha=\frac{1}{3}\left[2\theta y^l+\theta^l\alpha^2\right]. \tag{5.34}$$

Since $s_0=0$ then (5.32) reduces to the following:

$$\theta=\frac{b^l\theta_l}{b^2}\beta. \tag{5.35}$$

Plugging (5.35) into (5.33) implies that

$$r_{00}=\frac{(b^l\theta_l)}{b^2}[\beta^2-b^2\alpha^2]. \tag{5.36}$$

By (2.8), we have $r_{00}=0$. By (5.36), we get $b^l\theta_l=0$ and by (5.35) we have $\theta=0$. Then $G^l_\alpha=0$ and $s^l_0=0$. So $G^l=0$ and F is a locally Minkowski metric.

Case(2): $\phi'(0)\neq 0$ such that

$$s(k_2-k_3s^2)(\phi\phi'-s\phi'-s\phi\phi'')-(\phi'^2+\phi\phi'')+k_1\phi(\phi-s\phi')\neq 0.$$

According to Corollary 1, the proof of this case is similar to the proof of case (1).

Case(3): $\phi'(0)\neq 0$ such that

$$s(k_2-k_3s^2)(\phi\phi'-s\phi'-s\phi\phi'')-(\phi'^2+\phi\phi'')+k_1\phi(\phi-s\phi')=0.$$

According to Lemma 6, we have

$$s_{l0}=\frac{1}{3}(\beta\theta_l-\theta b_l), \tag{5.37}$$

$$r_{00}=\frac{2}{3}\theta\beta+\big[\tau+\frac{2}{3}(b^2\tau-\theta_l b^l)\big]\alpha^2+\frac{1}{3}(3k_2-2-3k_3b^2)\tau\beta^2, \tag{5.38}$$

$$G^l_\alpha=\frac{1}{3}[2\theta+(3k_1-2)\tau\beta]y^l+\frac{1}{3}(\theta^l-\tau b^l)\alpha^2+\frac{1}{2}k_3\tau\beta^2b^l. \tag{5.39}$$

By (2.8) we have $s_0=0$. Thus (5.37) implies that

$$\theta=\frac{(b^l\theta_l)}{b^2}\beta. \tag{5.40}$$

Plugging (5.40) into (5.38) we obtain

$$r_{00}=\frac{2}{3}\frac{(b^l\theta_l)}{b^2}\beta^2+\big[\tau+\frac{2}{3}(b^2\tau-\theta_l b^l)\big]\alpha^2+\frac{1}{3}(3k_2-2-3k_3b^2)\tau\beta^2. \tag{5.41}$$

By (2.8), since $r_{00}=0$ then (5.41) reduces to the following:

$$\frac{2}{3}\frac{(b^l\theta_l)}{b^2}\beta^2+\big[\tau+\frac{2}{3}(b^2\tau-\theta_l b^l)\big]\alpha^2+\frac{1}{3}(3k_2-2-3k_3b^2)\tau\beta^2=0. \tag{5.42}$$

Differentiating (5.42) with respect to y^m yields

$$\frac{4}{3}\frac{(b^l\theta_l)}{b^2}\beta b_m+2\big[\tau+\frac{2}{3}(b^2\tau-\theta_l b^l)\big]y_m+\frac{2}{3}(3k_2-2-3k_3b^2)\tau\beta b_m=0. \tag{5.43}$$

By multiplying (5.43) by b^m we get

$$2\Big[(k_2-k_3b^2)b^2+1\Big]\tau\beta=0. \tag{5.44}$$

By assumption, we have $(k_2-k_3b^2)b^2+1\neq 0$. Then $\tau=0$. Plugging $\tau=0$ into (5.37), (5.38), and (5.39) yields (5.29), (5.30), and (5.31). Thus the proof of Theorem in this case is similar to the first case. This completes the proof. □

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