On Finsler metrics with vanishing S-curvature

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Abstract: In this paper, we consider Finsler metrics defined by a Riemannian metric and a 1-form on a manifold. We study these metrics with vanishing S-curvature. We find some conditions under which such a Finsler metric is Berwaldian or locally Minkowskian.

Key words: \((\alpha, \beta)\)-metric, Berwald metric, S-curvature.

1. Introduction

In Finsler geometry, there are several important non-Riemannian quantities: the Cartan torsion \(C\), the Berwald curvature \(B\), the S-curvature \(S\), the new non-Riemannian curvature \(H\), etc. They all vanish for Riemannian metrics; hence they are said to be non-Riemannian [6, 7, 9].

Let \((M, F)\) be a Finsler manifold. The Finsler metric \(F\) on \(M\) induced a spray \(G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}\), which determines the geodesics, where \(G^i = G^i(x, y)\) are called the spray coefficients of \(G\). A Finsler metric \(F\) is called a Berwald metric if \(G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k\) are quadratic in \(y\) for any \(x \in M\). The Berwald curvature \(B\) of Finsler metrics is an important non-Riemannian quantity constructed by L. Berwald.

The S-curvature is constructed by Shen for given comparison theorems on Finsler manifolds [10]. A natural problem is to study and characterize Finsler metrics of vanishing S-curvature. It is known that some Randers metrics are of vanishing S-curvature [8, 13]. This is one of our motivations to consider Finsler metrics with vanishing S-curvature. Shen proved that every Berwald metric satisfies \(S = 0\) [10]. In [2], Bao and Shen find a class of non-Berwaldian Randers metrics with vanishing S-curvature. Thus the converse of Shen’s theorem is not true, generally. A natural question arises: “Under which conditions does the converse of Shen’s Theorem hold?”

There are 2 basic tensors on Finsler manifolds: fundamental metric tensor \(g_y\) and the Cartan torsion \(C_y\), which are second and third order derivatives of \(\frac{1}{2}F^2\) at \(y \in T_zM\), respectively. The rate of change of \(C\) along Finslerian geodesics is called Landsberg curvature \(L_y\). Taking a trace of \(C\) and \(L\) gives us mean Cartan torsion \(I\) and mean Landsberg curvature \(J\), respectively. \(J/I\) is regarded as the relative rate of change of \(I\) along Finslerian geodesics. Then \(F\) is said to be an isotropic mean Landsberg metric if \(J + cFI = 0\), where \(c = c(x)\) is a scalar function on \(M\).
There are many non-Riemannian R-quadratic Finsler metrics. For example, all Berwald metrics are R-quadratic. Indeed a Finsler metric is R-quadratic if and only if the h-curvature of Berwald connection depends on position only in the sense of Bákcsó-Matsumoto [1]. The notion of R-quadratic Finsler metrics was introduced by Shen, and can be considered a generalization of R-flat metrics.

**Theorem 2** Let $F = \alpha \phi(s), \quad s = \frac{\beta}{\alpha}$ be a non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ with vanishing $S$-curvature and $\phi \neq c_1 \sqrt{1 + c_2 s^2} + c_3 s$ for any constant $c_1 > 0$, $c_2$, $c_3$. Suppose that $F$ is R-quadratic. Then $F$ reduces to a Berwald metric.

Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory, and multiterminal information theory. Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, and play a very important role in studying flat Finsler information structures. A Finsler metric $F = F(x, y)$ on a manifold $M$ is said to be locally dually flat if at any point there is a standard coordinate system $(x^i, y^i)$ in $TM$ satisfying $[F^2]_{x^i y^j} y^k = 2[F^2]_{x^i}$. It is easy to see that every locally Minkowskian metric satisfies in the above equation, hence is locally dually flat [14, 15]. Here, we find some conditions under which a locally dually flat non-Randers type $(\alpha, \beta)$-metric reduces to a locally Minkowskian metric. More precisely, we prove the following.

**Theorem 3** Let $F = \alpha \phi(s), \quad s = \frac{\beta}{\alpha}$ be a non-Randers type $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$ with vanishing $S$-curvature. Suppose that one of the following holds:

(a) $\phi'(0) \neq 0$ and $(k_2 - k_3 b^2)b^2 \neq -1$;

(b) $\phi'(0) = \phi''(0) = 0$ or $\phi$ is a polynomial that $\phi'(0) = 0$.

If $F$ is locally dually flat then it reduces to a locally Minkowskian metric.

In this paper, we use the Berwald connection and the $h$- and $v$-covariant derivatives of a Finsler tensor field are denoted by “$|$” and “$\prime$” respectively [12].
2. Preliminary

A Finsler metric on a manifold $M$ is a nonnegative function $F$ on $TM$ having the following properties:

(a) $F$ is $C^\infty$ on $TM_0 := TM \setminus \{0\}$;

(b) $F(\lambda y) = \lambda F(y)$, $\forall \lambda > 0$, $y \in TM$;

(c) for each $y \in T_xM$, the following quadratic form $g_y$ on $T_xM$ is positive definite,

$$g_y(u,v) := \frac{1}{2} \left[ F^2(y + su + tv) \right]_{s,t=0}, \quad u,v \in T_xM.$$

At each point $x \in M$, $F_x := F|_{T_xM}$ is an Euclidean norm if and only if $g_y$ is independent of $y \in T_xM_0$. To measure the non-Euclidean feature of $F_x$, define $C_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$C_y(u,v,w) := \frac{1}{2} \frac{d}{dt} \left[ g_y + tw(u,v) \right]_{t=0}, \quad u,v,w \in T_xM.$$

The family $C := \{C_y\}_{y \in TM_0}$ is called the Cartan torsion.

Given a Finsler manifold $(M,F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i,y^i)$ for $TM_0$ is given by

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y) \frac{\partial}{\partial y^i},$$

where $G^i(x,y)$ are local functions on $TM_0$ satisfying

$$G^i(x,\lambda y) = \lambda^2 G^i(x,y) \quad \lambda > 0.$$

$G$ is called the associated spray to $(M,F)$. The projection of an integral curve of $G$ is called a geodesic in $M$. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$. A Finsler metric $F$ is called a Berwald metric if $G^i$ are quadratic in $y \in T_xM$ for any $x \in M$ or equivalently the Berwald curvature

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$$

is vanishing.

A Finsler metric $F = F(x,y)$ on a manifold $M$ is said to be locally dually flat if at any point there is a coordinate system $(x^i)$ in which the spray coefficients are in the following form:

$$G^i = -\frac{1}{2} y^{ij} H_{y^j},$$

where $H = H(x,y)$ is a $C^\infty$ scalar function on $TM_0$ satisfying $H(x,\lambda y) = \lambda^3 H(x,y)$ for all $\lambda > 0$. Such a coordinate system is called an adapted coordinate system [4]. In [8], Shen proved that the Finsler metric $F$ on an open subset $U \subset \mathbb{R}^n$ is dually flat if and only if it satisfies

$$(F^2)_x y^i y^k = 2(F^2)_x.$$
In this case, \( H = -\frac{1}{6} |F|^2 x^m y^n \).

Let \( U(t) \) be a vector field along a curve \( c(t) \). The canonical covariant derivative \( D_c U(t) \) is defined by

\[
D_c U(t) := \left\{ \frac{dU^i}{dt}(t) + U^j(t) \frac{\partial G^i}{\partial y^j}(\dot{c}(t)) \right\} \frac{\partial}{\partial x^i}|_{c(t)}.
\]

If \( U(t) \) is said to be parallel along \( c \) if \( D_c U(t) = 0 \).

To measure the changes in the Cartan torsion \( \mathbf{C} \) along geodesics, we define \( L_y : T_x M \otimes T_y M \otimes T_z M \rightarrow \mathbb{R} \) by

\[
L_y(u, v, w) := \frac{d}{dt} \left[ \mathbf{C}_c(t)(U(t), V(t), W(t)) \right]|_{t=0},
\]

where \( c(t) \) is a geodesic and \( U(t), V(t), W(t) \) are parallel vector fields along \( c(t) \) with \( \dot{c}(0) = y, U(0) = u, V(0) = v, W(0) = w \). The family \( L := \{L_y\}_{y \in TM} \) is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if \( L = 0 \). An important fact is that if \( F \) is Berwaldian, then it is Landsbergian. \( L/C \) is regarded as the relative rate of change in \( C \) along Finslerian geodesics. Then \( F \) is said to be an isotropic Landsberg metric if \( L = cFC \), where \( c = c(x) \) is a scalar function on \( M \).

For a vector \( y \in T_x M_0 \), the Riemann curvature \( R_y : T_x M \rightarrow T_x M \) is defined by \( R_y(u) := R_i^k(y)u^k \frac{\partial}{\partial x^i} \), where

\[
R_i^k(y) = 2 \frac{\partial G^i}{\partial x^j} - \frac{\partial^2 G^i}{\partial y^j \partial y^k} y^j + 2 G^i_{jl} \frac{\partial G^j}{\partial y^l} - \frac{\partial G^i}{\partial y^l} \frac{\partial G^j}{\partial y^k}.
\]

The family \( R := \{R_y\}_{y \in TM} \) is called the Riemann curvature. There are many Finsler metrics whose Riemann curvature in every direction is quadratic. A Finsler metric \( F \) is said to be R-quadratic if \( R_y \) is quadratic in \( y \in T_x M \) at each point \( x \in M \).

Put

\[
R^i_{jkl}(y) := \frac{1}{3} \frac{\partial}{\partial y^j} \left[ \frac{\partial R^i_k}{\partial y^l} - \frac{\partial R^i_l}{\partial y^k} \right].
\]

\( R^i_{jkl} \) are the coefficients of the h-curvature of the Berwald connection, which are also denoted by \( H^i_{jkl} \) in the literature. We have

\[
R^i_k(y) = y^j R^i_{jkl}(y)y^l.
\]

Thus \( R^i_k(y) \) is quadratic in \( y \in T_x M \) if and only if \( R^i_{jkl}(y) \) are functions of \( x \) only.

For a Finsler metric \( F \) on an \( n \)-dimensional manifold \( M \), the Busemann-Hausdorff volume form \( dV_F = \sigma_F(x)dx^1 \cdots dx^n \) is defined by

\[
\sigma_F(x) := \frac{\text{Vol}(\mathbb{R}^n(1))}{\text{Vol}\left\{(y^j) \in \mathbb{R}^n \mid F(y^j \frac{\partial}{\partial x^j} | x) < 1 \right\}}.
\]

In general, the local scalar function \( \sigma_F(x) \) cannot be expressed in terms of elementary functions, even if \( F \) is locally expressed by elementary functions.

Let \( G^i(x, y) \) denote the geodesic coefficients of \( F \) in the same local coordinate system. The S-curvature is defined by

\[
S(y) := \frac{\partial G^i}{\partial y^j}(x, y) - y^i \frac{\partial}{\partial x^i}\left[ \ln \sigma_F(x) \right].
\]
where \( y = y^i \frac{\partial}{\partial x^i} |_{x} \in T_x M \). It is proved that \( S = 0 \) if \( F \) is a Berwald metric \([8]\). There are many non-Berwald metrics satisfying \( S = 0 \) \([2]\).

Given a Riemannian metric \( \alpha \), a 1-form \( \beta \) on a manifold \( M \), and a \( C^\infty \) function \( \phi = \phi(s) \) on \([-b_o, b_o]\), where \( b_o := \sup_{x \in M} \|\beta\|_x \), one can define a function on \( TM \) by

\[
F := \alpha \phi(s), \quad s = \frac{\beta}{\alpha}.
\]

If \( \phi \) and \( b_o \) satisfy (2.1) and (2.2) below, then \( F \) is a Finsler metric on \( M \). Finsler metrics in this form are called \((\alpha, \beta)\)-metrics. Randers metrics are special \((\alpha, \beta)\)-metrics.

Now we consider \((\alpha, \beta)\)-metrics. Let \( \alpha = \sqrt{a_{ij} y^i y^j} \) be a Riemannian metric and \( \beta = b_i y^i \) a 1-form on a manifold \( M \). Let

\[
\|\beta\|_x := \sqrt{a^{ij}(x)b_i(x)b_j(x)}.
\]

For a \( C^\infty \) function \( \phi = \phi(s) \) on \([-b_o, b_o]\), where \( b_o = \sup_{x \in M} \|\beta\|_x \), define

\[
F := \alpha \phi(s), \quad s = \frac{\beta}{\alpha}.
\]

By a direct computation, we obtain

\[
g_{ij} = \rho a_{ij} + \rho_0 b_i b_j - \rho_1 (b_i \alpha_j + b_j \alpha_i) + s \rho_1 \alpha_i \alpha_j,
\]

where \( \alpha_i := a_{ij} y^j / \alpha \), and

\[
\rho := \phi(\phi - s \phi'), \quad \rho_0 := \phi \phi'' + \phi' \phi', \quad \rho_1 := s(\phi \phi'' + \phi' \phi') - \phi \phi'.
\]

By further computation, one obtains

\[
\det(g_{ij}) = \phi^{n+1} (\phi - s \phi')^{n-2} \left[ (\phi - s \phi') + (\|\beta\|_x^2 - s^2) \phi'' \right] \det(a_{ij}) .
\]

Using the continuity, one can easily show that

**Lemma 1** Let \( b_o > 0 \). \( F = \alpha \phi(\beta/\alpha) \) is a Finsler metric on \( M \) for any pair \( \{\alpha, \beta\} \) with \( \sup_{x \in M} \|\beta\|_x \leq b_o \) if and only if \( \phi = \phi(s) \) satisfies the following conditions:

\[
\phi(s) > 0, \quad (|s| \leq b_o) \tag{2.1}
\]

\[
\phi(s) - s \phi'(s) + (b^2 - s^2) \phi''(s) > 0, \quad (|s| \leq b \leq b_o). \tag{2.2}
\]

Let

\[
r_{ij} := \frac{1}{2} (b_{ij} + b_{ji}), \quad s_{ij} := \frac{1}{2} (b_{ij} - b_{ji}).
\]
\[ r_j := b^i r_{ij}, \quad s_j := b^i s_{ij}. \]
Let \( r_{i0} := r_{ij} y^j, \ s_{i0} := s_{ij} y^j, \ r_0 := r_j y^j \) and \( s_0 := s_j y^j \). Suppose that \( G^i = G^i(x, y) \) and \( \tilde{G}^i = \tilde{G}^i(x, y) \) denote the coefficients of \( F \) and \( \alpha \) respectively in the same coordinate system. By definition, we obtain the following identity:
\[ G^i = \tilde{G}^i + Py^i + Q^i, \quad (2.3) \]
where
\[ P = \alpha^{-1} \Theta \left[ r_{00} - 2Qs_0 \right], \]
\[ Q^i = \alpha Q s^i_0 + \Psi \left[ r_{00} - 2Qs_0 \right] b^i, \]
\[ \Theta = \frac{\phi' - s(\phi \phi'' + \phi' \phi')}{2\phi \left((\phi - s\phi') + (b^2 - s^2)\phi''\right)}, \]
\[ \Psi = \frac{\phi''}{2 \left((\phi - s\phi') + (b^2 - s^2)\phi''\right)}. \]
Clearly, if \( \beta \) is parallel with respect to \( \alpha \) (\( r_{ij} = 0 \) and \( s_{ij} = 0 \)), then \( P = 0 \) and \( Q^i = 0 \). In this case, \( G^i = \tilde{G}^i \) are quadratic in \( y \), and \( F \) is a Berwald metric.

Now, let \( \phi = \phi(s) \) be a positive \( C^\infty \) function on \((-b_0, b_0)\). For a number \( b \in [0, b_0) \), let
\[ \Phi := -(Q - sQ') \{ n\Delta + 1 + sQ \} - (b^2 - s^2)(1 + sQ)Q'' \quad (2.4) \]
where
\[ \Delta := 1 + sQ + (b^2 - s^2)Q'. \quad (2.5) \]

**Lemma 2** ([3]) Let \( F = \alpha \phi(s) \), \( s = \frac{\beta}{\alpha} \) be a non-Riemannian \((\alpha, \beta)\)-metric on a manifold and \( b := \| \beta_x \|_\alpha \). Suppose that \( \phi \neq c_1 \sqrt{1 + c_2 s^2 + c_3 s} \) for any constant \( c_1 > 0 \), \( c_2 \) and \( c_3 \). Then \( F \) is of isotropic S-curvature, \( S = (n + 1)cF \), if and only if one of the following holds:
(a) \( \beta \) satisfies
\[ r_{ij} = \varepsilon(b^2 a_{ij} - b_i b_j), \quad s_j = 0, \quad (2.6) \]
where \( \varepsilon = \varepsilon(x) \) is a scalar function, and \( \phi = \phi(s) \) satisfies
\[ \Phi = -2(n + 1)k \frac{\phi \Delta^2}{b^2 - s^2}, \quad (2.7) \]
where \( k \) is a constant. In this case, \( S = (n + 1)cF \) with \( c = k\varepsilon \).
(b) \( \beta \) satisfies
\[ r_{ij} = 0, \quad s_j = 0 \quad (2.8) \]
In this case, \( S = 0 \), regardless of choices of a particular \( \phi \).
3. Proof of Theorem 1

We have the following formula for the spray coefficient $G^i$ of $F$

$$G^i = G^i_\alpha + \alpha Q s^i_0 + (-2Q \alpha s_0 + r_{00})(\Theta \alpha^{-1} y^i + \Psi b^i),$$

(3.9)

where $s^i_j := a^{ij} s_{b^j}$, $s^i_0 := s_i y^i$, $r_{00} = r_{ij} y^i y^j$ and

$$\Theta = \frac{Q - s Q'}{2\Delta}, \quad \Psi = \frac{Q'}{2\Delta}.$$

By a direct computation, we can obtain a formula for the mean Cartan torsion of $(\alpha, \beta)$-metrics as follows:

$$I_i = -\frac{\Phi(\phi - s \phi')}{2\Delta \phi \alpha^2} (ab_i - sy_i).$$

(3.10)

According to Deicke's theorem, a Finsler metric is Riemannian if and only if $I = 0$. Clearly, an $(\alpha, \beta)$-metric $F = \alpha \phi(s)$ is Riemannian if and only if $\Phi = 0$.

In [5], Li and Shen obtained the mean Landsberg curvature of an $(\alpha, \beta)$-metric $F = \alpha \phi(s)$, $s = \frac{\beta}{\alpha}$ as follows

$$J_i = -\frac{1}{\alpha^2 \Delta (b^2 - s^2)} \left[ \phi \phi' + (n + 1)(Q - sQ') \right] (r_0 + s_0) h_i$$

$$-\frac{h_i}{2\alpha^2 \Delta (b^2 - s^2)} (\Psi_1 + s \frac{\Phi}{\Delta}) (r_{00} - 2\alpha Q s_0)$$

$$-\frac{\Phi}{2\alpha^2 \Delta^2} \left[ -\alpha Q s_0 h_i + \alpha Q (\alpha^2 s_i - y_i s_0) + \alpha^2 \Delta s_{i0} 

+ \alpha^2 (r_{i0} - 2\alpha Q s_0) - (r_{00} - 2\alpha Q s_0) y_i \right].$$

(3.11)

where $h_i := ab_i - sy_i$ and

$$\Psi_1 := \sqrt{b^2 - s^2} \Delta^\frac{1}{2} \left[ \frac{\sqrt{b^2 - s^2}^2}{\Delta^2} \right]' .$$

They also obtained

$$\bar{J} := J_i b^i = -\frac{\Delta}{2\alpha^2} \left[ \Psi_1 (r_{00} - 2\alpha Q s_0) + \alpha \Psi_2 (r_0 + s_0) \right],$$

(3.12)

where

$$\Psi_2 := 2(n + 1)(Q - sQ') + 3 \frac{\Phi}{\Delta}.$$

**Lemma 3** Let $F = \alpha \phi(s)$, $s = \frac{\beta}{\alpha}$ be a non-Riemannian $(\alpha, \beta)$-metric on manifold $M$. Suppose that $\phi \neq c_1 \sqrt{1 + c_2 s^2} + c_3 s$ for any constant $c_1 > 0$, $c_2$. If $F$ has vanishing S-curvature and a weakly Landsberg metric then $F$ is a Berwald metric.
Proof By (2.8) and (3.11) we have

\[ J_i = \frac{\Phi s_{i0}}{2\alpha \Delta}. \] (3.13)

From (3.13) we conclude if \( F \) is weakly Landsberg then \( s_{i0} = 0 \) and because of \( r_{00} = 0 \), \( F \) is a Berwald metric. \( \square \)

Proof of Theorem 1 Let \( F \) be a relatively isotropic mean Landsberg curvature metric with vanishing S-curvature. The following holds:

\[ J_k + cFI_k = 0. \] (3.14)

By (2.8) and (3.12) we have \( b_i J^i = 0 \). Multiplying (3.14) by \( b^k \) yields

\[ cF(b^k I_k) = 0. \] (3.15)

If \( c \neq 0 \) from (3.15) we have \( b^k I_k = 0 \) and so by (3.10) we conclude

\[ \Phi(\phi - s\phi') \frac{2\Delta \phi^3}{2\Delta \phi^3} (b^2 \alpha^2 - \beta^2) = 0 \] (3.16)

From (3.16) we conclude \( \Phi = 0 \) or \( \phi - s\phi' = 0 \). Then by (3.10) we have \( I = 0 \) and \( F \) is a Riemannian metric. By assumption \( F \) is a non-Riemannian metric and so \( c = 0 \). From (3.14), we conclude \( F \) is a weakly Landsberg metric. Then, by Lemma 3, \( F \) is a Berwald metric. The proof of Theorem 1 is complete.

4. Proof of Theorem 2

Lemma 4 ([9]) For the Berwald connection, the following Bianchi identities hold:

\[ R^i_{jkl|m} + R^i_{jlm|k} + R^i_{jmk|l} = B^i_{jku} R^u_{lm} + B^i_{jlu} R^u_{km} + B^i_{klu} R^u_{jm} \] (4.17)

\[ B^i_{jml} - B^i_{jkm} = R^i_{jkl,m} \] (4.18)

\[ B^i_{jkl,m} = B^i_{jkm,l}. \] (4.19)

Lemma 5 Let \( F = \alpha \phi(s) \), \( s = \frac{\beta}{\alpha} \) be a non-Riemannian \((\alpha, \beta)\)-metric on manifold \( M \). Suppose that \( \phi \neq c_1 \sqrt{1 + c_2 s^2} + c_3 s \) for any constant \( c_1 > 0 \), \( c_2 \). If \( F \) has vanishing S-curvature then we have

\[ b_m B^m_{jkl} = 0. \] (4.20)

Proof By (2.8), we have \( s_0 = 0 \). By assumption \( F \) has vanishing S curvature. By (2.8) and (3.9) we have

\[ G^i = G^i_{\alpha} + \alpha Q s^i_0. \] (4.21)

Multiplying (4.21) by \( b_i \) yields \( b_i G^i = b_i G^i_{\alpha} \). Thus \( b_m B^m_{jkl} = 0 \). \( \square \)

Proof of Theorem 2 According to Lemma 5, we have

\[ b_m B^m_{jkl} + b_m B^m_{jkl|s} = 0. \] (4.22)
By assumption $F$ is an $R$-quadratic metric. Thus (4.18) implies that
\[ B_{ijkl|m}^i - B_{jkl|m}^i = 0. \] (4.23)
Multiplying (4.23) by $b_i$ yields
\[ b_iB_{ijkl|m}^i = b_iB_{jkl|m}^i. \] (4.24)
From (4.22) and (4.24) we conclude
\[ b_{i|jm}B_{ijkl}^i = b_{i|lj}B_{ijkl}^j. \] (4.25)
Since $r_{ij} = 0$, then by multiplying (4.25) by $y^l$ we obtain
\[ s_{i0}B_{ijkl}^j = 0. \] (4.26)
By (4.21) we get $B_{ijkl}^i = [\alpha Qs_0^i]y^i + y^i$. From (4.26) we have
\[ [\alpha Q]_{y^i}y^{i'}s_{i0}s_0^i + [\alpha Q]_{y^i}y^{i}s_{i0}s_0^i + [\alpha Q]_{y^i}y^{i}s_{i0}s_0^i = 0. \] (4.27)
By (2.8), we have $s^i = s_i = 0$. Then multiplying (4.27) by $b^i b^k b^l$ yields
\[ \left[ b^i b^k b^l [\alpha Q]_{y^i}y^{i'} \right] s_{i0}s_0^i = 0. \] (4.28)
Then by (4.28), we conclude that $\beta$ is a closed 1-form and then $F$ reduces to a Berwald metric. The proof of Theorem 2 is complete. \[ \Box \]

5. Proof of Theorem 3
In this section, we are going to prove Theorem 3. First, we remark the following.

Lemma 6 ([16]) Let $F = \alpha \phi(s)$, $s = \frac{\beta}{\alpha}$, be a non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$, where $\phi'(0) \neq 0$ and $\beta \neq 0$. Then $F$ is locally dually flat if and only if $\alpha, \beta,$ and $\phi$ satisfy
\[ s_{i0} = \frac{1}{3}(\beta \theta - \theta b_l), \]
\[ r_{00} = \frac{2}{3} \theta \beta + \left[ \tau + \frac{2}{3}(\theta^2 \tau - \theta b_l) \right] \alpha^2 + \frac{1}{3}(3k_2 - 2 - 3k_3 b^2) \tau \beta^2, \]
\[ G_a^i = \frac{1}{3}[2\theta + (3k_1 - 2)\tau \beta]y^i + \frac{1}{3}(\theta^l - \tau b_l) \alpha^2 + \frac{1}{2}k_3 \tau \beta^2 b^l, \]
\[ \tau [s(k_2 - k_3 s^2)(\phi' - s \phi'') - s \phi'''] - (\phi'^2 + \phi''') - k_1 \phi (\phi - s \phi') = 0, \]
where $\tau := \tau(x)$ is a scalar function, $\theta := \theta_1(x)y^i$ is a 1-form on $M$, $\theta^l = \alpha^{lm} \theta_m$, and
\[ k_1 := \Pi(0), \quad k_2 := \frac{\Pi'(0)}{\Pi(0)}, \]
\[ k_3 := \frac{1}{6Q(0)^2} [3Q''(0)\Pi'(0) - 6\Pi'(0)^2 - Q(0)\Pi'''(0)], \]
\[ \Pi := \phi'^2 + \phi'''. \]
By Lemma 6, we can get the following.

**Corollary 1** ([16]) Let $F = \alpha \phi(s)$, $s = \frac{\beta}{\alpha}$ be an $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$ with the same assumption as Lemma 6. Let $\phi$ satisfy

$$s(k_2 - k_3 s^2)(\phi' - s \phi' - s \phi'' - (\phi'^2 + \phi''^2) + k_1 \phi - s \phi') \neq 0.$$ 

Then $F$ is locally dually flat on $M$ if and only if

\begin{align*}
s_{l0} &= \frac{1}{3} (\beta \theta_i - \theta b_i), \\
r_{00} &= \frac{2}{3} \left[ \theta \beta - (\theta b^l) \alpha^2 \right], \\
G^l_\alpha &= \frac{1}{3} \left[ 2 \theta y^l + \theta^l \alpha^2 \right].
\end{align*}  

where $k_i$ $(1 \leq i \leq 3)$ are the same as those of Theorem 6.

In [16], Xia proved the following.

**Lemma 7** ([16]) Let $F := \alpha \phi(s)$, $s = \frac{\beta}{\alpha}$ be a non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Suppose that $\phi(s)$ is an analytic function with $\phi'(0) = \phi''(0) = 0$ or $\phi(s)$ is a polynomial of $s$ with $\phi'(0) = 0$ and $\beta = b_i(x) y^i \neq 0$. Then $F$ is locally dually flat if and only if $\alpha$ and $\beta$ satisfy (5.29), (5.30) and (5.31), where $\theta = \theta_i(x) y^i$ is a 1-form on $M$ and $\theta^l := a^l m \theta_m$.

**Proof of Theorem 3** To prove Theorem 3, we consider some cases.

**Case (1):** $\phi'(0) = \phi''(0) = 0$ or $\phi(s)$ is a polynomial of $s$ with $\phi'(0) = 0$. In this case, by Lemma 7 we have

\begin{align*}
s_{l0} &= \frac{1}{3} (\beta \theta_i - \theta b_i), \\
r_{00} &= \frac{2}{3} \left[ \theta \beta - (\theta b^l) \alpha^2 \right], \\
G^l_\alpha &= \frac{1}{3} \left[ 2 \theta y^l + \theta^l \alpha^2 \right].
\end{align*}  

Since $s_0 = 0$ then (5.32) reduces to the following:

$$\theta = \frac{b^l \theta_i}{b^2} \beta.$$  

(5.35)

Plugging (5.35) into (5.33) implies that

$$r_{00} = \frac{(b \theta_i)}{b^2} \left[ \beta^2 - \beta^2 \alpha^2 \right].$$  

(5.36)
By (2.8), we have \( r_{00} = 0 \). By (5.36), we get \( b_l \theta_l = 0 \) and by (5.35) we have \( \theta = 0 \). Then \( G^l_\alpha = 0 \) and \( s^0_l = 0 \). So \( G^l = 0 \) and \( F \) is a locally Minkowski metric.

**Case (2):** \( \phi'(0) \neq 0 \) such that

\[
s(k_2 - k_3 s^2)(\phi \phi' - s \phi \phi'' - (\phi'^2 + \phi \phi'') + k_1 \phi (\phi - s \phi') \neq 0.
\]

According to Corollary 1, the proof of this case is similar to the proof of case (1).

**Case (3):** \( \phi'(0) \neq 0 \) such that

\[
s(k_2 - k_3 s^2)(\phi \phi' - s \phi \phi'' - (\phi'^2 + \phi \phi'') + k_1 \phi (\phi - s \phi') = 0.
\]

According to Lemma 6, we have

\[
s_{00} = \frac{1}{3} (\beta \theta_l - \theta b_l),
\]
\[
r_{00} = \frac{2}{3} \frac{\theta \beta}{b^2} + [\tau + \frac{2}{3} (b^2 \tau - \theta b_l)] \alpha^2 + \frac{1}{3} (3k_2 - 2 - 3k_3 b^2) \tau \beta^2,
\]
\[
G^l_\alpha = \frac{1}{3} [2 \theta + (3k_1 - 2) \tau \beta] y^l + \frac{1}{3} (\theta^2 - \tau b^l) \alpha^2 + \frac{1}{2} k_3 \tau \beta^2 b^l.
\]

By (2.8) we have \( s_0 \). Thus (5.37) implies that

\[
\theta = \frac{(b^l \theta_l)}{b^2} \beta.
\]

Plugging (5.40) into (5.38) we obtain

\[
r_{00} = \frac{2}{3} \frac{(b^l \theta_l)}{b^2} \beta^2 + [\tau + \frac{2}{3} (b^2 \tau - \theta b^l)] \alpha^2 + \frac{1}{3} (3k_2 - 2 - 3k_3 b^2) \tau \beta^2.
\]

By (2.8), since \( r_{00} = 0 \) then (5.41) reduces to the following:

\[
\frac{2}{3} \frac{(b^l \theta_l)}{b^2} \beta^2 + [\tau + \frac{2}{3} (b^2 \tau - \theta b^l)] \alpha^2 + \frac{1}{3} (3k_2 - 2 - 3k_3 b^2) \tau \beta^2 = 0.
\]

Differentiating (5.42) with respect to \( y^m \) yields

\[
\frac{4}{3} \frac{(b^l \theta_l)}{b^2} \beta b_m + 2 [\tau + \frac{2}{3} (b^2 \tau - \theta b^l)] y_m + \frac{2}{3} (3k_2 - 2 - 3k_3 b^2) \tau \beta b_m = 0.
\]

By multiplying (5.43) by \( b^m \) we get

\[
2 \left[ (k_2 - k_3 b^2) b^2 + 1 \right] \tau \beta = 0.
\]

By assumption, we have \((k_2 - k_3 b^2) b^2 + 1 \neq 0\). Then \( \tau = 0 \). Plugging \( \tau = 0 \) into (5.37), (5.38), and (5.39) yields (5.29), (5.30), and (5.31). Thus the proof of Theorem in this case is similar to the first case. This completes the proof.
References


