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A characterization of certain geodesic hyperspheres in complex projective space

Juan de Dios PÉREZ[1,*], **Young Jin SUH**[2]
[1]Department of Geometry and Topology, Faculty of Sciences, University of Granada, Granada, Spain
[2]Department of Mathematics, Kyungpook National University, Taegu, Korea

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Abstract: We characterize geodesic hyperspheres of radius r such that $cot^2(r) = \frac{1}{2}$ as the unique real hypersurfaces in complex projective space whose structure Jacobi operator satisfies a pair of conditions.

Key words: Complex projective space, real hypersurface, structure Jacobi operator, Lie derivative

1. Introduction

We will consider connected orientable real hypersurfaces M in complex projective space $\mathbb{C}P^m$, $m \geq 3$, endowed with the Fubini–Study metric g of constant holomorphic sectional curvature equal to 4. Let J be the Kaehlerian structure of $\mathbb{C}P^m$ and N a unit normal vector field on M. Let (ϕ, ξ, η, g) be the almost contact metric structure induced on M by (J, g) (see Section 2) and $\mathbb{D}$ the maximal holomorphic distribution on M; that is, at any point $\mathbb{D}$ contains any tangent vector orthogonal to ξ. We will say that M is Hopf if ξ is a principal vector field for the shape operator associated to N.

Homogeneous real hypersurfaces were classified by Takagi in [14] and [15] into 6 classes: Type A_1 are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$. Type A_2 are tubes of radius r, $0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^n$, $0 < n < m-1$. Type B are tubes of radius r, $0 < r < \frac{\pi}{4}$, over the complex quadric. Type C are tubes of radius r, $0 < r < \frac{\pi}{4}$, over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^n$, where $2n+1 = m$ and $m \geq 5$. Type D are tubes of radius r, $0 < r < \frac{\pi}{4}$, over the Plucker embedding of the complex Grassmannian manifold $G(2,5)$ in $\mathbb{C}P^9$. Type E are tubes of radius r, $0 < r < \frac{\pi}{4}$, over the canonical embedding of the Hermitian symmetric space $SO(10)/U(5)$ in $\mathbb{C}P^{15}$. See also the papers by Okumura [7], Maeda [5], Kimura [3], and Cecil and Ryan [2]. All these hypersurfaces are Hopf and have constant principal curvatures. In [6] there is a survey of the most important results on these real hypersurfaces.

On the other hand, we can consider ruled real hypersurfaces introduced by Kimura [4]. Except for these real hypersurfaces there are few known examples of real hypersurfaces in $\mathbb{C}P^m$. Therefore, it is interesting to know whether certain families of real hypersurfaces do or do not exist once we know the behavior of their shape operator associated to N, A. In Section 3 we present 2 theorems of nonexistence of 2 families of such real hypersurfaces. Namely, we will prove:

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*Correspondence: jdperez@ugr.es
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Theorem 1.1 . *There do not exist real hypersurfaces in* $\mathbb{C}P^m$, $m \geq 3$, *whose shape operator is given by* $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi + \frac{\beta^2-1}{\alpha}U$, $A\phi U = \delta\phi U$, *where* U *is a unit vector field in* $\mathbb{D}$, *and there exists* $Z \in \mathbb{D}_U = Span\{\xi, U, \phi U\}^\perp$ *such that* $AZ = 0$, $A\phi Z = -\frac{1}{\alpha}$, α, β *being nonvanishing functions on* M *and* δ *is either* 0 *or* $-\frac{\beta^2}{\alpha}$.

Theorem 1.2 . *There do not exist real hypersurfaces in* $\mathbb{C}P^m$, $m \geq 3$, *whose shape operator is given by* $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi + \frac{\beta^2-1}{\alpha}U$, $A\phi U = \delta\phi U$, *and for any* $Z \in \mathbb{D}_U$, $AZ = \lambda Z$, *where* α, β *are nonvanishing functions defined on* M, δ *is either* 0 *or* $-\frac{\beta^2}{\alpha}$ *and* λ *is either* 0 *or* $-\frac{1}{2\alpha}$.

The Jacobi operator R_X with respect to a unit vector field X is defined by $R_X = R(.,X)X$, where R is the curvature tensor field on M. Then we see that R_X is a self-adjoint endomorphism of the tangent space. It is related to Jacobi vector fields, which are solutions of the second-order differential equation (the Jacobi equation) $\nabla_{\dot\gamma}(\nabla_{\dot\gamma}Y) + R(Y,\dot\gamma)\dot\gamma = 0$ along a geodesic γ in M. The structure vector field given by $\xi = -JN$ is called the Reeb vector field on M. The corresponding Jacobi operator R_ξ is called the structure Jacobi operator on M.

In the line of characterizing real hypersurfaces of $\mathbb{C}P^m$ in terms of R_ξ it is natural to consider the problem about the parallelism and the invariance, or Lie parallelism. In [8] the nonexistence of real hypersurfaces in nonflat complex space forms with parallel structure Jacobi operator was proven. In [9], we also proved the nonexistence of real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, whose structure Jacobi operator is invariant, that is, its Lie derivative in any tangent direction vanishes.

In [11] we obtained the following:

Theorem 1.3 . *Let* M *be a real hypersurface of* $\mathbb{C}P^m$, $m \geq 3$, *such that the structure Jacobi operator* R_ξ *is invariant under the structure vector field* ξ, *that is,* $\mathcal{L}_\xi R_\xi = 0$. *Then either* M *is locally congruent to a tube of radius* $\pi/4$ *over a complex submanifold in* $\mathbb{C}P^m$ *or to either a geodesic hypersphere or a tube over a totally geodesic* $\mathbb{C}P^k$, $0 < k < m-1$, *with radius* $r \neq \pi/4$.

On the other hand, we do not know the classification of real hypersurfaces in $\mathbb{C}P^m$ satisfying

$$AR_\xi = R_\xi A, \tag{1.1}$$

although we know that every Hopf real hypersurface satisfies it. The geometrical meaning of this condition is that any eigenspace of R_ξ is invariant by A.

In [12] we proved the nonexistence of real hypersurfaces in $\mathbb{C}P^m$ satisfying this condition if the structure Jacobi operator is $\mathbb{D}$-invariant, that is $\mathcal{L}_X R_\xi = 0$ for any $X \in \mathbb{D}$. In order to generalize this result we devote this paper to studying real hypersurfaces satisfying (1.1), and at the same time the Lie derivative of the structure Jacobi operator is always in the direction of the Reeb vector field ξ. Namely, we will suppose

$$(\mathcal{L}_X R_\xi)Y = -g(\phi AX, Y)\xi \tag{1.2}$$

for any $X, Y \in \mathbb{D}$. A geodesic hypersphere of radius r such that $cot^2(r) = \frac{1}{2}$ clearly satisfies (1.1). It is also easy to see that it satisfies (1.2) (see final section of this paper). Our purpose is to prove that the converse is also true by the

Main Theorem. *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then M satisfies (1.1) and (1.2) if and only if M is locally congruent to a geodesic hypersphere of radius r such that $cot^2(r) = \frac{1}{2}$.*

Remark If in addition we impose that the real hypersurface is complete, by Takagi [14], M should be congruent (globally) to a geodesic hypersphere of radius r such that $cot^2(r) = \frac{1}{2}$.

2. Preliminaries

Throughout this paper, all manifolds, vector fields, etc. will be considered of class C^∞ unless otherwise stated.

Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field of M. Let ∇ be the Levi–Civita connection on M and (J, g) the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$, where ϕX is the tangent component of JX. Then (ϕ, ξ, η, g) is an almost contact metric structure on M; see [1]. That is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.1}$$

for vector fields X, Y tangent to M. From (2.1) we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

From the parallelism of J we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{2.3}$$

and

$$\nabla_X \xi = \phi AX \tag{2.4}$$

for any vector fields X, Y tangent to M, where A denotes the shape operator associated to N. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given respectively by

$$\begin{aligned} R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ -2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \tag{2.6}$$

for any vector fields X, Y, and Z tangent to M, where R is the curvature tensor of M.

We will write, in general, $A\xi = \alpha\xi + \beta U$, U being a unit vector field in $\mathbb{D}$. M is Hopf (respectively, non-Hopf) if $\beta = 0$ (respectively, $\beta \neq 0$).

We will need the following results

Theorem 2.1 . *(See [5]). Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, such that $A\xi = \alpha\xi$. Then α is locally constant and if X is a tangent vector field on M such that $AX = \lambda X$ and X is orthogonal to ξ, then $A\phi X = \frac{\alpha\lambda+2}{2\lambda-\alpha}\phi X$.*

Theorem 2.2 . *(See [10]). There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, whose shape operator satisfies $A\xi = \alpha\xi + U$, $AU = \xi$, $A\phi U = -\frac{1}{\alpha}\phi U$, where U is a unit vector field in $\mathbb{D}$ and α a nonnull function defined on M.*

3. Proof of Theorems 1.1 and 1.2

We begin with the proof of Theorem 1.1.

By the Codazzi equation we get $(\nabla_Z A)\xi - (\nabla_\xi A)Z = -\phi Z$. This gives $Z(\alpha)\xi + Z(\beta)U + \beta\nabla_Z U + A\nabla_\xi Z = -\phi Z$. Taking its scalar product with ξ, we have

$$Z(\alpha) + \beta g(\nabla_\xi Z, U) = 0, \tag{3.1}$$

and its scalar product with U yields

$$Z(\beta) + \frac{\beta^2-1}{\alpha} g(\nabla_\xi Z, U) = 0. \tag{3.2}$$

From (3.1) and (3.2) we obtain

$$\alpha\beta Z(\beta) + (1-\beta^2)Z(\alpha) = 0. \tag{3.3}$$

The Codazzi equation also gives $(\nabla_Z A)U - (\nabla_U A)Z = 0$. That is, $Z(\beta)\xi + Z(\frac{\beta^2-1}{\alpha})U + \frac{\beta^2-1}{\alpha}\nabla_Z U - A\nabla_Z U + A\nabla_U Z = 0$. Its scalar product with ξ yields

$$Z(\beta) + \beta g(\nabla_U Z, U) = 0 \tag{3.4}$$

and its scalar product with U gives

$$Z(\frac{\beta^2-1}{\alpha}) + \frac{\beta^2-1}{\alpha} g(\nabla_U Z, U) = 0. \tag{3.5}$$

From (3.4) and (3.5) we obtain

$$Z(\frac{\beta^2-1}{\alpha}) + \frac{1-\beta^2}{\alpha} Z(\beta) = 0. \tag{3.6}$$

From (3.3) and (3.6) we get $\alpha Z(\beta) = 0$. Thus, $Z(\beta) = 0$. From (3.6) we have $\frac{1-\beta^2}{\alpha} Z(\alpha) = 0$.

If $Z(\alpha) \neq 0$, $\beta^2 = 1$. Maybe after changing ξ by $-\xi$ we can suppose that $\beta = 1$. Therefore, either $\delta = 0$ or $\delta = -\frac{1}{\alpha}$.

From the Codazzi equation we have $(\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi = -2\xi$. Its scalar product with U yields

$$\delta g(\nabla_\xi \phi U, U) + 1 + \alpha\delta = -1. \tag{3.7}$$

If $\delta = 0$, from (3.7) we have a contradiction. Thus $\delta = -\frac{1}{\alpha}$.

From the Codazzi equation, $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -2\xi$. Its scalar product with ξ gives

$$g(\nabla_U \phi U, U) = 1 \tag{3.8}$$

and taking its scalar product with U, bearing in mind (3.8), it follows that $-\frac{2}{\alpha} + g(A\phi U, \phi U) = 0$. That is, $-\frac{3}{\alpha} = 0$, which is impossible. Therefore, $Z(\alpha) = Z(\beta) = 0$.

Once more from the Codazzi equation we obtain $(\nabla_{\phi Z} A)\xi - (\nabla_\xi A)\phi Z = Z$. This yields $(\phi Z)(\alpha)\xi + \alpha\phi A\phi Z + (\phi Z)(\beta)U + \beta\nabla_{\phi Z}U - A\phi A\phi Z + \xi(\frac{1}{\alpha})\phi Z + \frac{1}{\alpha}\nabla_\xi \phi Z + A\nabla_\xi \phi Z = Z$. Taking its scalar product with ξ we get

$$g(\nabla_\xi \phi Z, U) = -\frac{1}{\beta}(\phi Z)(\alpha), \tag{3.9}$$

and if we take its scalar product with U,

$$(\phi Z)(\beta) + \frac{\beta^2}{\alpha} g(\nabla_\xi \phi Z, U) = 0. \tag{3.10}$$

From (3.9) and (3.10) we have

$$\alpha(\phi Z)(\beta) - \beta(\phi Z)(\alpha) = 0. \tag{3.11}$$

As $(\nabla_{\phi Z} A)U - (\nabla_U A)\phi Z = 0$, its scalar product with ξ yields

$$g(\nabla_U \phi Z, U) = -\frac{1}{\beta}(\phi Z)(\beta) \tag{3.12}$$

and its scalar product with U gives $(\phi Z)(\frac{\beta^2 - 1}{\alpha}) + \frac{\beta^2}{\alpha} g(\nabla_U \phi Z, U) = 0$. Bearing in mind (3.12), we get

$$(\phi Z)(\frac{\beta^2-1}{\alpha}) - \frac{\beta}{\alpha}(\phi Z)(\beta) = 0. \tag{3.13}$$

From (3.13) we have $\alpha\beta(\phi Z)(\beta) + (1-\beta^2)(\phi Z)(\alpha) = 0$. This equation and (3.11) yield $(\phi Z)(\alpha) = (\phi Z)(\beta) = 0$.

From the Codazzi equation, $(\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z = -2\xi$. Taking its scalar product with ξ we obtain

$$g([\phi Z, Z], U) = -\frac{1}{\beta}. \tag{3.14}$$

Taking its scalar product with U and bearing in mind (3.14), we have

$$g(\nabla_Z \phi Z, U) = \frac{1}{\beta} - 2\beta. \tag{3.15}$$

From the Codazzi equation we get $(\nabla_Z A)\phi U - (\nabla_{\phi U} A)Z = 0$. As $Z(\alpha) = Z(\beta) = 0$, $Z(\delta) = 0$. Therefore, we obtain $\delta\nabla_Z\phi U - A\nabla_Z\phi U + A\nabla_{\phi U}Z = 0$. Taking its scalar product with Z we have $\delta g(\nabla_Z\phi U, Z) = 0$. If $\delta \neq 0$, $g(\nabla_Z\phi U, Z) = 0$. From (2.3) this yields $g(\nabla_Z\phi Z, U) = 0$. Now from (3.15) we have

$$2\beta^2 = 1. \tag{3.16}$$

Thus, β is constant. If we develop $(\nabla_{\phi U} A)\xi - (\nabla_\xi A)\phi U = U$ and take its scalar product with U we obtain

$$-\alpha\delta + \delta(\tfrac{\beta^2-1}{\alpha}) - \beta^2 + (\tfrac{2\beta^2-1}{\alpha})g(\nabla_\xi\phi U, U) = 1. \tag{3.17}$$

As $2\beta^2 = 1$ and $\delta = -\frac{\beta^2}{\alpha}$, from (3.17) we conclude

$$4\alpha^2 = 1. \tag{3.18}$$

From the Codazzi equation, $(\nabla_{\phi U} A)U - (\nabla_U A)\phi U = -2\xi$. Taking its scalar product with U, from (3.16) and (3.17) we get $\beta^2 + 1 = 0$, which is impossible. Therefore, we must suppose $\delta = 0$.

From the Codazzi equation, $(\nabla_Z A)\phi U - (\nabla_{\phi U} A)Z = 0$. Its scalar product with ϕZ yields

$$g(\nabla_Z\phi U, \phi Z) - g(\nabla_{\phi U} Z, \phi Z) = 0. \tag{3.19}$$

If we take the scalar product of $(\nabla_{\phi Z} A)\phi U - (\nabla_{\phi U} A)\phi Z = 0$ with ϕZ and bear in mind (2.3), we get

$$\tfrac{1}{\alpha}g(\nabla_{\phi Z} U, Z) + (\phi U)(\tfrac{1}{\alpha}) = 0. \tag{3.20}$$

Developing $(\nabla_Z A)\xi - (\nabla_\xi A)Z = \phi Z$ and taking its scalar product with ϕZ, we have

$$\beta g(\nabla_Z U, \phi Z) - \tfrac{1}{\alpha}g(\nabla_\xi Z, \phi Z) = -1. \tag{3.21}$$

Taking the scalar product of $(\nabla_Z A)U - (\nabla_U A)Z = 0$ with ϕZ, we also obtain

$$\beta^2 g(\nabla_Z U, \phi Z) - g(\nabla_U Z, \phi Z) = 0. \tag{3.22}$$

The scalar product of $(\nabla_{\phi Z} A)\xi - (\nabla_\xi A)\phi Z = Z$ with Z yields

$$\beta g(\nabla_{\phi Z} U, Z) - \tfrac{1}{\alpha}g(\nabla_\xi Z, \phi Z) = 0, \tag{3.23}$$

and the scalar product of $(\nabla_{\phi Z} A)U - (\nabla_U A)\phi Z = 0$ with Z implies

$$\beta + (\beta^2 - 1)g(\nabla_{\phi Z} U, Z) + g(\nabla_U\phi Z, Z) = 0. \tag{3.24}$$

From (3.21) and (3.23) it follows that

$$\beta g(\nabla_{\phi Z} U, Z) - \beta g(\nabla_Z U, \phi Z) = 1, \tag{3.25}$$

and from (3.22) and (3.24) we get

$$\beta + (\beta^2 - 1)g(\nabla_{\phi Z}U, Z) - \beta^2 g(\nabla_Z U, \phi Z) = 0. \tag{3.26}$$

From (3.25) and (3.26) we have

$$\begin{gathered} g(\nabla_{\phi Z}U, Z) = 2\beta \\ g(\nabla_Z U, \phi Z) = 2\beta - \tfrac{1}{\beta}. \end{gathered} \tag{3.27}$$

From (3.20) and (3.27), $(\phi U)(\frac{1}{\alpha}) = -\frac{2\beta}{\alpha}$. Thus,

$$(\phi U)(\alpha) = 2\alpha\beta. \tag{3.28}$$

Now $(\nabla_{\phi U}A)\xi - (\nabla_\xi A)\phi U = U$. Taking its scalar product with ξ and bearing in mind (3.28), we obtain

$$g(\nabla_\xi \phi U, U) = -\alpha \tag{3.29}$$

and its scalar product with U yields

$$(\phi U)(\beta) = 2\beta^2. \tag{3.30}$$

As $(\nabla_{\phi U}A)U - (\nabla_U A)\phi U = 2\xi$, if we take its scalar product with ξ and bear in mind (3.30), we have

$$\beta g(\nabla_U \phi U, U) = 1 - \beta^2 \tag{3.31}$$

and its scalar product with U, bearing in mind (3.31), gives

$$(\phi U)(\tfrac{\beta^2-1}{\alpha}) - \tfrac{2\beta(\beta^2-1)}{\alpha} + \tfrac{\beta^2-1}{\alpha\beta} = 0. \tag{3.32}$$

From (3.28), (3.30), and (3.32) we get $5\beta^2 - 1 = 0$. Therefore, β is constant. Thus, $(\phi U)(\beta) = 2\beta^2 = 0$, which is impossible. This finishes the proof of Theorem 1.1.

Now we begin proving Theorem 1.2.

If $\lambda = 0$, from the Codazzi equation we get $(\nabla_Z A)\phi Z - (\nabla_{\phi Z}A)Z = -2\xi$. That is, $A\nabla_{\phi Z}Z - A\nabla_Z \phi Z = -2\xi$. Taking its scalar product with ξ we have

$$\beta g([\phi Z, Z], U) = -2, \tag{3.33}$$

and its scalar product with U yields

$$\tfrac{\beta^2-1}{\alpha} g([\phi Z, Z], U) = 0. \tag{3.34}$$

From (3.33) and (3.34) we obtain $\beta^2 = 1$. If $\delta = -\frac{1}{\alpha}$ from Theorem 2.1, these real hypersurfaces do not exist. Let us suppose that $A\phi U = 0$ and that, after a possible change of ξ by $-\xi$, $\beta = 1$.

From the Codazzi equation, $(\nabla_{\phi U}A)\xi - (\nabla_\xi A)\phi U = U$. Taking its scalar product with U we obtain $1 = -1$.

Therefore, we must suppose that for any $Z \in \mathbb{D}_U$, $AZ = -\frac{1}{2\alpha}Z$. The same computations as above give us

$$\beta g([\phi Z, Z], U) = \tfrac{1}{2\alpha^2} - 1 \tag{3.35}$$

and

$$(2\beta^2 - 1)g([\phi Z, Z], U) = 2\beta. \tag{3.36}$$

From (3.35) and (3.36) we obtain

$$(2\beta^2 - 1)(1 - 2\alpha^2) = 4\alpha^2\beta^2. \tag{3.37}$$

From (3.37) for any $X \in TM$ we get

$$\tfrac{1-4\alpha^2}{\alpha}X(\beta) = \tfrac{4\beta^2-1}{\beta}X(\alpha). \tag{3.38}$$

Taking the scalar product of the Codazzi equation that we have used to obtain (3.35) and (3.36) with Z, and respectively with ϕZ, we have $Z(\alpha) = (\phi Z)(\alpha) = 0$. From (3.38) we have $\frac{1-4\alpha^2}{\alpha}Z(\beta) = \frac{1-4\alpha^2}{\alpha}(\phi Z)(\beta) = 0$. Thus, either $4\alpha^2 = 1$ or $Z(\beta) = (\phi Z)(\beta) = 0$. If $4\alpha^2 = 1$, from (3.37) we should have $-\frac{1}{2} = 0$, which is impossible. Therefore, $Z(\alpha) = (\phi Z)(\alpha) = Z(\beta) = (\phi Z)(\beta) = 0$.

From the Codazzi equation, $(\nabla_Z A)\xi - (\nabla_\xi A)Z = -\phi Z$. Its scalar product with ϕZ yields

$$g(\nabla_Z U, \phi Z) = \beta + \tfrac{1}{4\alpha^2\beta} - \tfrac{1}{\beta}. \tag{3.39}$$

The scalar product of $(\nabla_Z A)U - (\nabla_U A)Z = 0$ with ϕZ, bearing in mind (3.39), gives

$$2\beta^2 + 4\alpha^2 - 12\alpha^2\beta^2 - 1 = 0. \tag{3.40}$$

From (3.37) and (3.40) we obtain $\alpha^2(1 - 2\beta^2) = 0$. Thus, $2\beta^2 = 1$. This and (3.37) yield $\alpha = 0$, which is a contradiction. This finishes the proof of Theorem 1.2.

4. Proof of Main Theorem

First, we suppose that M is non-Hopf and write $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in $\mathbb{D}$ and β is a nonvanishing function defined on M. Condition (1.2) gives

$$\begin{gathered} g(\nabla_X A\xi, \xi)AY + g(A\xi, \phi AX)AY + \alpha\nabla_X AY - g(\nabla_X AY, \xi)A\xi \\ -g(AY, \phi AX)A\xi - \eta(AY)\nabla_X A\xi - \alpha\nabla_{AY}X + \eta(AY)\nabla_{A\xi}X - \alpha A\nabla_X Y \\ +g(\nabla_X Y, A\xi)A\xi - g(\nabla_Y X, \xi)\xi + \alpha A\nabla_Y X - g(\nabla_Y X, A\xi)A\xi = 0 \end{gathered} \tag{4.1}$$

for any $X, Y \in \mathbb{D}$. If we take the scalar product of (4.1) and ξ we obtain

$$\begin{array}{c} \beta\eta(AY)g(\phi AX, U) - \alpha g(AY, \phi AX) + \alpha g(X, \phi A^2 Y) \\ -\eta(AY)g(X, \phi A^2\xi) + g(X, \phi AY) = 0 \end{array} \tag{4.2}$$

for any $X, Y \in \mathbb{D}$. If we apply Condition (1.1) to ξ we obtain $\alpha AU = \alpha\beta\xi + (\beta^2 - 1)U$. Let us suppose $\alpha = 0$. Then $\beta^2 = 1$ and changing, if necessary, ξ by $-\xi$, we can suppose $\beta = 1$.

If in (4.2) we take $X = U$, $Y = \phi U$, we obtain

$$g(A\phi U, \phi U) = 0. \tag{4.3}$$

Taking $Y \in \mathbb{D}_U = Span\{\xi, U, \phi U\}^{\perp}$, $X = \phi U$ in (4.2), we have

$$g(AU, Y) = 0. \tag{4.4}$$

From (4.4) we know that AU has no component in $\mathbb{D}_U$. If we take $Y \in \mathbb{D}_U$, $X = U$ in (4.2), it follows that

$$g(A\phi U, Y) = 0. \tag{4.5}$$

Thus, $A\phi U$ has no component in $\mathbb{D}_U$. Take now $X = Y = U$ in (4.2). We get

$$g(AU, \phi U) = 0. \tag{4.6}$$

From (4.3), (4.5), and (4.6), we have $A\phi U = 0$. From (4.4) we get $A\xi = U$, $AU = \xi + \gamma U$. Take $X, Y \in \mathbb{D}_U$. From (4.2) we obtain $g(X, \phi AY) = 0$. From this, (4.4), and (4.5), $AY = 0$ for any $Y \in \mathbb{D}_U$. Therefore, the type number of M, that is, the rank of its shape operator, is at most 2. By [13], M must be ruled and $\gamma = 0$.

From the Codazzi equation, $(\nabla_U A)\xi - (\nabla_\xi A)U = -\phi U$. If we take its scalar product with ϕU we get

$$g(\nabla_U U, \phi U) = 0. \tag{4.7}$$

The Codazzi equation also yields $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -2\xi$. Its scalar product with ξ gives

$$g(\nabla_U \phi U, U) = 2. \tag{4.8}$$

As (4.7) and (4.8) give a contradiction, we assure that $\alpha \neq 0$ and we can write

$$AU = \beta\xi + \tfrac{\beta^2 - 1}{\alpha} U. \tag{4.9}$$

From (4.9), taking $X \in D_U$, $Y = U$ in (4.2), we get $g(A\phi U, X) = 0$. So we write $A\phi U = \delta\phi U$. Taking $Y = \phi U$, $X = U$ in (4.2), we obtain $-\alpha g(A\phi U, \phi AU) + \alpha g(U, \phi A^2\phi U) + g(U, \phi A\phi U) = 0$. That is, $\delta(\beta^2 - 1) + \alpha\delta^2 + \delta = 0$. Thus, either $\delta = 0$ or $\delta = -\frac{\beta^2}{\alpha}$.

Take now a unit $Z \in \mathbb{D}_U$. We have just seen that $\mathbb{D}_U$ is A-invariant, so we suppose that $AZ = \lambda Z$. Let $Y \in \mathbb{D}_U$ such that is orthogonal to $Span\{Z, \phi Z\}$. Applying (4.2) to Z and ϕY we obtain $(1+\alpha\lambda)g(A\phi Z, \phi Y) + \alpha g(A^2\phi Z, \phi Y) = 0$. Therefore, $(1+\alpha\lambda)A\phi Z + \alpha A^2\phi Z$ is proportional to ϕZ. Thus, we can write $(1+\alpha\lambda)A\phi Z + \alpha A^2\phi Z + \mu\phi Z = 0$ for some function μ. If in (4.2) we take $X = Z$, $Y = \phi Z$ we get $(1+\alpha\lambda)g(A\phi Z, \phi Z) + \alpha g(A^2\phi Z, \phi Z) = 0$. This yields $\mu = 0$ and we have

$$(1+\alpha\lambda)A\phi Z + \alpha A^2\phi Z = 0. \tag{4.10}$$

Suppose that ω is another eigenvalue of A in $\mathbb{D}_U$, $\omega \neq \lambda$, and that W is a unit eigenvector corresponding to ω. As above, we find

$$(1+\alpha\omega)A\phi W + \alpha A^2\phi W = 0. \tag{4.11}$$

If we take the scalar product of (4.10) and W and the scalar product of (4.11) and Z, we have $((1+\alpha\lambda)\omega + \alpha\omega^2)g(\phi Z, W) = 0$ and $((1+\alpha\omega)\lambda + \alpha\lambda^2)g(\phi W, Z) = 0$. If we add both expressions, as $\omega \neq \lambda$ we obtain

$$(1+\alpha(\omega+\lambda))g(\phi W, Z) = 0. \tag{4.12}$$

From (4.12) we have:

1. If there exists W such that $g(\phi W, Z) \neq 0$, $\omega + \lambda = -\frac{1}{\alpha}$.
2. If for any W in the above conditions $g(\phi W, Z) = 0$, then $A\phi Z = \lambda\phi Z$.

In the first case we can write $\phi Z = a_1 Z_1 + a_2 Z_2$, where ${a_1}^2 + {a_2}^2 = 1$, $AZ_1 = \lambda Z_1$, $AZ_2 = \omega Z_2$ with $1 + \alpha(\lambda+\omega) = 0$. From (4.10) we have $((1+\alpha\lambda)\lambda a_1 + \alpha\lambda^2 a_1)Z_1 + ((1+\alpha\lambda)\omega a_2 + \alpha\omega^2 a_2)Z_2 = 0$. From this equation we obtain $a_1\lambda(1+2\alpha\lambda) = 0$. If $1 + 2\alpha\lambda = 0$, as $1+\alpha(\lambda+\omega) = 0$ we should have $\lambda = \omega$, which is impossible. Then $a_1\lambda = 0$. If $\lambda = 0$ and $a_1 = 1$, $A\phi Z = 0$ and we are in Case 2. So we suppose $a_2 \neq 0$ and $A\phi Z = a_2\omega Z_2$. From (4.10), $(1+\alpha\lambda)a_2\omega + \alpha a_2^2\omega^2 = 0$. If $\lambda = 0$ we obtain $\omega = -\frac{1}{\alpha}$. The above equation yields $-\frac{a_2}{\alpha} + \frac{a_2^2}{\alpha} = 0$. Thus, $a_2(a_2 - 1) = 0$ and $a_2 = 1$. Thus, $A\phi Z = -\frac{1}{\alpha}\phi Z$. From Theorem 1.1 this case does not occur. Therefore, we must suppose that $\lambda \neq 0$, $a_1 = 0$ and $A\phi Z = \omega\phi Z$, $\lambda \neq \omega$ and $1 + \alpha(\lambda+\omega) = 0$.

From the Codazzi equation, $(\nabla_Z A)\phi U - (\nabla_{\phi U} A)Z = 0$. If we take its scalar product with ϕZ we obtain

$$(\delta - \omega)g(\nabla_Z \phi U, \phi Z) + (\omega - \lambda)g(\nabla_{\phi U} Z, \phi Z) = 0 \tag{4.13}$$

and its scalar product with Z gives

$$(\phi U)(\lambda) = (\delta - \lambda)g(\nabla_Z \phi U, Z). \tag{4.14}$$

As $(\nabla_{\phi Z} A)\phi U - (\nabla_{\phi U} A)\phi Z = 0$, its scalar product with Z yields

$$(\delta - \lambda)g(\nabla_{\phi Z}\phi U, Z) + (\omega - \lambda)g(\nabla_{\phi U} Z, \phi Z) = 0. \tag{4.15}$$

If we take its scalar product with ϕZ we get

$$(\phi U)(\omega) = (\delta - \omega) g(\nabla_{\phi Z}\phi U, \phi Z). \tag{4.16}$$

From (4.14) and (4.16) we have

$$(\phi U)(-\tfrac{1}{\alpha}) = (\delta - \lambda) g(\nabla_Z \phi U, Z) + (\delta - \omega) g(\nabla_{\phi Z}\phi U, \phi Z). \tag{4.17}$$

Bearing in mind that $(\nabla_Z A)\xi - (\nabla_\xi A)Z = -\phi Z$ and taking its scalar product with ϕZ, we obtain

$$\lambda(\alpha - \omega) + \beta g(\nabla_Z U, \phi Z) + (\omega - \lambda) g(\nabla_\xi Z, \phi Z) = -1. \tag{4.18}$$

Taking the scalar product of $(\nabla_Z A)U - (\nabla_U A)Z = 0$ with ϕZ, it follows that

$$\lambda\beta + (\tfrac{\beta^2 - 1}{\alpha} - \omega) g(\nabla_Z U, \phi Z) + (\omega - \lambda) g(\nabla_U Z, \phi Z) = 0. \tag{4.19}$$

The scalar product of $(\nabla_{\phi Z} A)\xi - (\nabla_\xi A)\phi Z = Z$ with Z yields

$$\omega(\lambda - \alpha) + \beta g(\nabla_{\phi Z} U, Z) + (\lambda - \omega) g(\nabla_\xi \phi Z, Z) = 1. \tag{4.20}$$

The scalar product of $(\nabla_{\phi Z} A)U - (\nabla_U A)\phi Z = 0$ with Z gives

$$-\omega\beta + (\tfrac{\beta^2 - 1}{\alpha} - \lambda) g(\nabla_{\phi Z} U, Z) + (\omega - \lambda) g(\nabla_U Z, \phi Z) = 0. \tag{4.21}$$

From (4.18) and (4.20), bearing in mind that $\lambda + \omega = -\frac{1}{\alpha}$, we have

$$\beta g(\nabla_Z U, \phi Z) - \beta g(\nabla_{\phi Z} U, Z) = 2\lambda\omega - 1. \tag{4.22}$$

Similarly, (4.19) and (4.21) yield

$$(\beta^2 + \alpha\lambda) g(\nabla_Z U, \phi Z) - (\beta^2 + \alpha\omega) g(\nabla_{\phi Z} U, Z) = \beta. \tag{4.23}$$

The determinant of the matrix given by the coefficients of the system of equations (4.22) and (4.23) is $\alpha\beta(\lambda - \omega) \neq 0$. Thus, we have a unique solution given by

$$\begin{aligned} g(\nabla_Z U, \phi Z) &= \tfrac{\beta^2 - (\beta^2 + \alpha\omega)(2\lambda\omega - 1)}{\alpha\beta(\lambda - \omega)} \\ g(\nabla_{\phi Z} U, Z) &= \tfrac{\beta^2 - (\beta^2 + \alpha\lambda)(2\lambda\omega - 1)}{\alpha\beta(\lambda - \omega)} \end{aligned} \tag{4.24}$$

and from (2.3) $g(\nabla_Z \phi U, Z) = g(\phi \nabla_Z U, Z) = -g(\nabla_Z U, \phi Z)$. From (4.17) and (4.24) we have

$$-(\phi U)(\tfrac{1}{\alpha}) = \tfrac{\beta^2 + (1 - 2\lambda\omega)(\beta^2 + \alpha\delta)}{\alpha\beta}. \tag{4.25}$$

Thus, $(\phi U)(\alpha) = \alpha\beta + \frac{\alpha(1-2\lambda\omega)(\beta^2+\alpha\delta)}{\beta}$. Therefore,

$$\begin{array}{c} \text{If } \delta = -\frac{\beta^2}{\alpha}, \\ (\phi U)(\alpha) = \alpha\beta. \\ \text{If } \delta = 0, \\ (\phi U)(\alpha) = 2\alpha\beta(1-\lambda\omega). \end{array} \tag{4.26}$$

As $(\nabla_{\phi U}A)\xi - (\nabla_\xi A)\phi U = U$, if we take its scalar product with ξ, we obtain

$$(\phi U)(\alpha) + 3\delta\beta - \alpha\beta + \beta g(\nabla_\xi \phi U, U) = 0, \tag{4.27}$$

and its scalar product with U yields

$$(\phi U)(\beta) - \alpha\delta + \delta\frac{\beta^2-1}{\alpha} - \beta^2 + (\frac{\beta^2-1}{\alpha} - \delta)g(\nabla_\xi \phi U, U) = 1. \tag{4.28}$$

From (4.26), (4.27), and (4.28) we get

$$\begin{array}{c} \text{If } \delta = -\frac{\beta^2}{\alpha}, \\ g(\nabla_\xi \phi U, U) = \frac{3\beta^2}{\alpha}, \\ (\phi U)(\beta) = 1 + \frac{2\beta^2 - 5\beta^4}{\alpha^2}. \\ \text{If } \delta = 0, \\ g(\nabla_\xi \phi U, U) = \alpha - 2\alpha(1-\lambda\omega), \\ (\phi U)(\beta) = 2\beta^2 + 2(1-\beta^2)\lambda\omega. \end{array} \tag{4.29}$$

From the Codazzi equation, $(\nabla_{\phi U}A)U - (\nabla_U A)\phi U = -2\xi$. Its scalar product with ξ yields $(\phi U)(\beta) + 2\delta\frac{\beta^2-1}{\alpha} - \alpha\delta - (\beta^2-1) + \beta g(\nabla_U \phi U, U) = 2$. From (4.29), we obtain

$$\begin{array}{c} \text{If } \delta = -\frac{\beta^2}{\alpha}, \\ g(\nabla_U \phi U, U) = \frac{7\beta^3 - 4\beta}{\alpha^2}. \\ \text{If } \delta = 0, \\ \beta g(\nabla_U \phi U, U) = (1-\beta^2)(1-2\lambda\omega). \end{array} \tag{4.30}$$

Its scalar product with U gives $-2\beta\delta + (\phi U)(\frac{\beta^2-1}{\alpha}) - \beta\frac{\beta^2-1}{\alpha} + (\frac{\beta^2-1}{\alpha} - \delta)g(\nabla_U \phi U, U) = 0$. This and (4.30) imply

$$\begin{array}{c} \text{If } \delta = -\frac{\beta^2}{\alpha}, \\ 4\alpha^2 + 4\beta^4 - 11\beta^2 + 4 = 0. \\ \text{If } \delta = 0, \\ (5-2\lambda\omega)\beta^2 + 2\lambda\omega - 1 = 0. \end{array} \tag{4.31}$$

In the case of $\delta \neq 0$, from (4.31) we have $8\alpha(\phi U)(\alpha) = (22\beta - 16\beta^3)(\phi U)(\beta)$. Therefore, $4\alpha^4 = (-8\beta^2 + 11)(\alpha^2 + 2\beta^2 - 5\beta^4)$. From (4.31) and this equation we obtain that β must be a root of an equation with constant coefficients. Therefore, β is constant and α is also constant. As $(\phi U)(\alpha) = \alpha\beta$, we should have $\alpha\beta = 0$, which is impossible. Thus, $\delta = 0$.

From (4.31) we get $\lambda\omega = \frac{1}{2} - \frac{2\beta^2}{1-\beta^2}$ and from (4.29), $(\phi U)(\beta) = 1 - 3\beta^2$. As moreover $\lambda + \omega = -\frac{1}{\alpha}$, we find $\lambda = -\frac{1}{2\alpha} + \sqrt{\frac{1}{4\alpha^2} - \frac{1}{2} + \frac{2\beta^2}{1-\beta^2}}$ and $\omega = -\frac{1}{2\alpha} - \sqrt{\frac{1}{4\alpha^2} - \frac{1}{2} + \frac{2\beta^2}{1-\beta^2}}$. If we take $X = Z$, $Y = U$ in (4.1) and its scalar product with ϕZ, we have

$$g(\nabla_U Z, \phi Z) = g(\nabla_Z U, \phi Z). \tag{4.32}$$

Taking $X = \phi Z$, $Y = U$ in (4.1) and its scalar product with Z, we get

$$g(\nabla_{\phi Z} U, Z) = g(\nabla_U \phi Z, Z). \tag{4.33}$$

From (4.32), (4.33), (2.3), and (4.24) we obtain $-\beta^2 + (\beta^2 + \alpha\omega)(2\lambda\omega - 1) = \beta^2 - (\beta^2 + \alpha\lambda)(2\lambda\omega - 1)$. Bearing in mind the value of $\lambda\omega$, it follows that $3\beta^2 = 1$, and thus β is constant. Moreover $\lambda\omega = -\frac{1}{2}$, $(\phi U)(\alpha) = 3\alpha\beta$.

As $(\nabla_{\phi Z} A)\xi - (\nabla_\xi A)\phi Z = Z$, taking its scalar product with ξ we have

$$(\phi Z)(\alpha) + \beta g(\nabla_\xi \phi Z, U) = 0, \tag{4.34}$$

and its scalar product with U gives

$$(2 + 3\alpha\omega) g(\nabla_\xi \phi Z, U) = 0. \tag{4.35}$$

Looking at (4.35), if we suppose $2 + 3\alpha\omega = 0$, $\omega = -\frac{2}{3\alpha}$. Now $-\frac{2}{3\alpha} = \frac{1}{2\alpha}(-1 - \sqrt{1 + 2\alpha^2})$. This yields $9\alpha^4 + 4\alpha^2 = 0$. Thus, α should be constant and $(\phi U)(\alpha) = 3\alpha\beta = 0$ should give a contradiction. Thus, we must suppose $g(\nabla_\xi \phi Z, U) = 0$. From (4.34), $(\phi Z)(\alpha) = 0$. Analogously, we can obtain $Z(\alpha) = 0$.

If we develop $(\nabla_Z A)\phi Z - (\nabla_{\phi Z} A)Z = -2\xi$ and take its scalar product with ξ, we get $\beta g([\phi Z, Z], U) = -2$. If now we take its scalar product with U, considering (4.24) and the values of λ and ω, we arrive to $3\alpha^2 + 2\alpha + 2 = 0$. Thus, α is constant and we have a new contradiction. This proves that Case 1 does not occur.

Suppose we have Case 2: for a unit Z in $\mathbb{D}_U$, $AZ = \lambda Z$, $A\phi Z = \lambda \phi Z$. From (4.10) we obtain $(1 + \alpha\lambda)\lambda + \alpha\lambda^2 = 0$. From Theorem 1.2, this case is impossible.

Thus we have proven that non-Hopf real hypersurfaces do not satisfy our conditions.

Let us now suppose that M is Hopf, and write $A\xi = \alpha\xi$. Let $Y \in \mathbb{D}$ such that $AY = \lambda Y$. For any $Y \in \mathbb{D}$ from Condition (1.2), we obtain $\alpha\lambda A\phi Y + \alpha\lambda^2 \phi Y + \lambda \phi Y = 0$. From Theorem 2.1, calling μ the eigenvalue corresponding to ϕY, we get

$$\lambda(\alpha\mu + \alpha\lambda + 1) = 0. \tag{4.36}$$

Suppose $\alpha(\lambda + \mu) + 1 = 0$. From the value of μ appearing in Theorem 2.1 we get $2\alpha\lambda^2 + 2\lambda + \alpha = 0$, where α is locally constant. Thus, M has at most 3 distinct locally constant principal curvatures. When the discriminant of such an equation is $1 - 2\alpha^2 = 0$ we obtain only 2 distinct principal curvatures. Then from [2] , M is locally congruent to a geodesic hypersphere of radius r. Writing $\alpha = 2cot(2r)$, we obtain $cot(r) = \lambda = -\frac{1}{2\alpha}$. This yields $cot^2(r) = \frac{1}{2}$. Clearly these geodesic hyperspheres satisfy condition (1.1). Now take $X, Y \in \mathbb{D}$. Then $AX = cot(r)X$, $AY = cot(r)Y$. Moreover $A\xi = 2cot(2r)\xi$. Introducing X, Y in (4.1), we obtain $2cot(2r)cot(r)\nabla_X Y - 2cot(2r)cot(r)\nabla_Y X - 2cot(2r)A\nabla_X Y + 2cot(2r)A\nabla_Y X + (8cot^2(2r)cot(r) + cot(r))g(X, \phi Y)\xi$. If we take the scalar product with any $Z \in \mathbb{D}$ we obtain 0. Its scalar product with ξ gives $\{2cot(r)(cot(2r)+1)cot(r)\}g(X, \phi Y)$. As $2cot^2(r) = 1$, the above term equals zero. Thus, these geodesic hyperspheres also satisfy condition (1.2).

If there are 3 distinct principal curvatures, as they must be locally constant, from [6] , M is locally congruent to either a type A_2 or B real hypersurface. However, in our case, we see that the distributions corresponding to the principal curvatures different from α are not ϕ-invariant. This yields that M cannot be of type A_2. If M is of type B, we can write $\alpha = 2tan(2r)$, $\lambda_1 = -cot(r)$, $\lambda_2 = tan(r)$, but these values do not satisfy $2\alpha^2 + 2\lambda + \alpha = 0$.

Therefore, the last possibility is to have $\lambda = 0$. From Theorem 2.1, $A\phi Y = -\frac{2}{\alpha}\phi Y$. If we apply (4.1) to $X = Y$ and ϕY we have $-\alpha g(A\phi Y, \phi AY) + \alpha g(Y, \phi A^2\phi Y) + g(Y, \phi A\phi Y) = 0$. This yields $-\alpha\mu^2 - \mu = 0$. As $\mu \neq 0$, we get $\mu = -\frac{1}{\alpha}$ and arrive at a contradiction.

This finishes the proof of the Theorem. □

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