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On Kakutani–Krein and Maeda–Ogasawara spaces

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Abstract: Let E be an Archimedean Riesz space. It is shown that the Kakutani–Krein space of the center of the Dedekind completion of E and the Maeda–Ogasawara space of E are homeomorphic. By applying this, we can reprove a Banach Stone type theorem for $C^\infty(S)$ spaces, where S is a Stonean space.

Key words: Riesz space, universal completion, Kakutani–Krein space, Maeda–Ogasawara space

1. Introduction

For standard definitions and terminology of Riesz space theory, we refer to [7], [11], or [4]. The Riesz space of real valued continuous functions on a topological space is denoted by $C(X)$. A topological space X is called *extremely disconnected* if the closure of every open subset of X is also open. A compact extremely disconnected space is called *Stonean*.

Let E be a uniformly complete Riesz space with an order unit $e > 0$. The Kakutani–Krein representation theorem states [9] that there exists a unique (up to homeomorphism) compact Hausdorff space K such that E and $C(K)$ are Riesz isomorphic. We shall call K the *Kakutani–Krein space* of E.

Let S be an extremely disconnected space. A function f from S into $[-\infty, \infty]$ is called an *extended continuous function* if f is continuous and $f^{-1}(\mathbb{R})$ is dense in S, where $[-\infty, \infty]$ is equipped with the 2-point compactification of $\mathbb{R}$. The set of extended continuous functions is denoted by $C^\infty(S)$. If S is an extremely disconnected space and O is an open subset of S, then each extended continuous function f from O into $[-\infty, \infty]$ has a unique continuous extension $f : \overline{O} \to [-\infty, \infty]$. From this, it is easy to see that $C^\infty(S)$ is a Riesz space under point-wise order and the following algebraic operations:

$$f + g = \overline{(f+g)|_{f^{-1}(\mathbb{R})\cap g^{-1}(\mathbb{R})}} \quad \text{and} \quad \alpha f = \overline{(\alpha f)|_{f^{-1}(\mathbb{R})}}$$

for all f, $g \in C^\infty(S)$, $\alpha \in \mathbb{R}$. Note that $f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})$ is an open dense subset of X. The space $C^\infty(S)$ is laterally complete (that is, each nonempty disjoint subset of $C^\infty(S)$ has a supremum) and Dedekind complete. Namely, $C^\infty(S)$ is universally complete. Recall that a Riesz space E is called *universally complete* if it is Dedekind complete and the supremum of each nonempty disjoint subset of E exists. See [2] for details on the Riesz space $C^\infty(S)$.

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Let F and G be universally complete Riesz spaces and suppose that a Riesz space E is Riesz isomorphic to order dense Riesz subspaces of F and G. Then F and G are Riesz isomorphic spaces. For any Archimedean Riesz space E, there exists a unique (up to Riesz isomorphism) universally complete Riesz space E^u such that E is Riesz isomorphic to an order dense subspace of E. The space E^u is called the *universal completion* of E. For different construction of the universal completion, see [3], [5], and [12].

The Maeda–Ogasawara representation theorem ([8]; see also [10]) states that for any Archimedean Riesz space E there exists a unique (up to homeomorphism) Stonean space S such that $C^\infty(S)$ is Riesz isomorphic to the universal completion of E, and we shall call S the *Maeda–Ogasawara space*.

Let E be an Archimedean Riesz space. The center $Z(E)$ consists of all operators $T : E \to E$ such that

$$-\alpha I \leq T \leq \alpha I$$

for some $\alpha \geq 0$. It is well known that $Z(E)$ is a uniformly complete Riesz space with the order unit being the identity operator I on E, and so by the Kakutani–Krein representation theorem, $Z(E)$ and $C(K)$ are Riesz isomorphic spaces for a unique compact Hausdorff space K.

In this short paper we show that, for an Archimedean Riesz space E, the Kakutani–Krein space of $Z(E)$ and the Maeda–Ogasawara space are homeomorphic.

2. The result

The Dedekind completion of an Archimedean Riesz space E is denoted by E^δ, and the universal completion of it is denoted by E^u. We note that for a Stonean space, the universal completion of $C(S)^u$ is $C^\infty(S)$.

Lemma 2.1 *Let S be a Stonean space. Then the Kakutani-Krein space of $Z(C(S)^u)$ is S. That is, $Z(C(S)^u)$ and $C(S)$ are Riesz isomorphic spaces.*

Proof The proof goes along similar lines as the proof of Theorem 2.63 of [1]. □

We are now in a position to state and prove our main result.

Theorem 2.2 *Let E be an Archimedean Riesz space. Then the Kakutani–Krein space of $Z(E^\delta)$ and the Maeda–Ogasawara space of E are homeomorphic spaces, where E^δ denotes the Dedekind completion of E.*

Proof Let E^u be the universal completion of E. We note that E^u is also the universal completion of E^δ. Let $T \in Z(E^u)$, so $|T| \leq \lambda I$ for some $\lambda \geq 0$. Let $x \in E^\delta$ be given. Then $|T(x)| \leq \lambda x$ in E^u. Since E^δ is an ideal in E^u (see [3]), we have $T(x) \in E^\delta$, so $T(E^\delta) \subset E^\delta$. This implies, following Theorem 2.63 of [1], that $Z(E^u)$ and $Z(E^\delta)$ are Riesz isomorphic spaces. By the Maeda–Ogasawara representation theorem, we have that, if S is the Maeda–Ogasawara space, then E^u and $C^\infty(S)$ are Riesz isomorphic, where $Z(E^u)$ is Riesz isomorphic to $C(S)$. Let K be the Kakutani–Krein space of $Z(E^\delta)$, so that $Z(E^\delta)$ and $C(K)$ are Riesz isomorphic spaces. Hence, $C(S)$ and $C(K)$ are Riesz isomorphic spaces. By the Banach–Stone theorem, S and K are homeomorphic. This completes the proof. □

A proof of the following theorem can be found in ([2], p. 309). We can give a shorter and different proof of this fact as follows.

Theorem 2.3 *Let S and K be extremely disconnected spaces. Then the following are equivalent.*

i.) S and K are homeomorphic.

ii.) $C^\infty(S)$ and $C^\infty(K)$ are Riesz isomorphic spaces.

Proof Suppose that (ii) holds, i.e. $C^{\infty}(S)$ and $C^{\infty}(K)$ are Riesz isomorphic spaces. Then $Z(C^{\infty}(S))$ and $Z(C^{\infty}(K))$ are Riesz isomorphic. It is obvious that $C(S)$ is Riesz isomorphic to $Z(C^{\infty}(S))$ and $C(K)$ is Riesz isomorphic to $Z(C^{\infty}(K))$, so $C(K)$ is Riesz isomorphic to $C(S)$. Now, by the Banach–Stone theorem, S and K are homeomorphic. The converse implication is straightforward. □

Let E be a uniformly complete Riesz space. In [6] it was proven that $Z(E)$ is Riesz and algebraic isomorphic to $C_b(prime(E))$, where $prime(E)$ is the topological space on E with the hull-kernel topology, such that

$$prime(E) = \{P : P \quad \text{is proper prime ideal of} \quad E\}$$

equipped with the topology having a basis

$$\{\{P \in prime(E) : x \notin P\} : x \in E\}.$$

Since $C_b(prime(E))$ is Riesz and algebraic isomorphic to $C(S_E)$ for a unique compact Hausdorff space S_E (up to homeomorphism), it follows that E is Dedekind complete if and only if S_E is Stonean, and thus we have the following.

Theorem 2.4 *Let E be an Archimedean Riesz space. Then the Maeda–Ogasawara space of E defined above is homeomorphic to S_{E^u}.*

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