Coloring hypercomplete and hyperpath graphs

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Abstract: Given a graph $G$ with an induced subgraph $H$ and a family $\mathcal{F}$ of graphs, we introduce a (hyper)graph $H_H(G; \mathcal{F}) = (V_H, E_H)$, the hyper-H (hyper)graph of $G$ with respect to $\mathcal{F}$, whose vertices are induced copies of $H$ in $G$, and $\{H_1, H_2, \ldots, H_r\} \in E_H$ if and only if the induced subgraph of $G$ by the set $\bigcup_{i=1}^r H_i$ is isomorphic to a graph $F$ in the family $\mathcal{F}$, and the integer $r$ is the least integer for $F$ with this property. When $H$ is a $k$-complete or a $k$-path of $G$, we abbreviate $H_K^k(G; \mathcal{F})$ and $H_P^k(G; \mathcal{F})$ to $H^k(G; \mathcal{F})$ and $H_P^k(G; \mathcal{F})$, respectively. Our motivation to introduce this new (hyper)graph operator on graphs comes from the fact that the graph $H^k(K_n; \{K_2k\})$ is isomorphic to the ordinary Kneser graph $K(n; k)$ whenever $2k \leq n$. As a generalization of the Lovász–Kneser theorem, we prove that $\chi(H^k(G; \{K_2k\})) = \chi(G) - 2k + 2$ for any graph $G$ with $\omega(G) = \chi(G)$ and any integer $k \leq \lfloor \omega(G)/2 \rfloor$. We determine the clique and fractional chromatic numbers of $H^k(G; \{K_2k\})$, and we consider the generalized Johnson graphs $\mathcal{H}_r(H; \{K_r+1\})$ and show that $\chi(\mathcal{H}_r(H; \{K_r+1\})) \leq \chi(H)$ for any graph $H$ and any integer $r < \omega(H)$. By way of application, we construct examples of graphs such that the gap between their chromatic and fractional chromatic numbers is arbitrarily large. We further analyze the chromatic number of hyperpath (hyper)graphs $H_P^k(G; P_m)$, and we provide upper bounds when $m = k + 1$ and $m = 2k$ in terms of the $k$-distance chromatic number of the source graph.

Key words: Kneser graphs and hypergraphs, chromatic and fractional chromatic numbers, hypercomplete and hyperpath (hyper)graphs

1. Introduction

Most surprised reactions (and equally appreciations) to Lovász’s proof of the Kneser conjecture are due to his method of proof that is now counted as the beginning of the discipline: topological combinatorics. Many other proofs with more topological flavors were made afterwards. Mainly, the methods of these proofs concentrate on the famous Borsuk–Ulam theorem or its combinatorial counterparts. However, we here are willing to provide more reasons to celebrate the proof of the Kneser conjecture after more than 30 years. We show that its presence may enable us to determine the chromatic numbers of many other graphs with no additional costs. Our departure begins by understanding the construction of Kneser graphs from complete graphs.

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To be more explicit, we recall that for given integers \( n, k \) with \( 2k - 1 \leq n \), the Kneser graph \( K(n; k) \) is the graph whose vertex set consists of all \( k \)-element subsets of \([n] := \{1, 2, \ldots, n\}\), where 2 vertices form an edge if and only if they are disjoint. The Kneser conjecture [8] now takes the following form in graph theory language that computes the chromatic number of the Kneser graph.

**Theorem 1.1 (Lovász–Kneser Theorem [11])** If \( k > 0 \) and \( n \geq 2k - 1 \), then \( \chi(K(n; k)) = n - 2k + 2 \).

Once we take \([n]\) as the set of vertices of the complete graph \( K_n \), it is easy to notice that a \( k \)-element subset of \([n]\) corresponds to an induced complete graph of order \( k \), and being disjoint for any 2 such sets means exactly that their set union induces a complete graph of order \( 2k \) in \( K_n \). Such an observation may naturally be generalized to an arbitrary graph by taking it as a source graph instead of \( K_n \). In other words, when we are given a graph \( G \), we may construct a new graph by taking its vertices to be all induced \( k \)-complete graphs in \( G \), where \( 2k \leq \omega(G) \), the clique number of \( G \), and declaring that any such 2 vertices form an edge if and only if the union of their vertices induces a \( 2k \)-complete graph in \( G \). Let us denote the resulting graph by \( \mathcal{H}_k(G; K_{2k}) \), which we call the \( k \)-hypercomplete graph of \( G \) with respect to the graph \( K_{2k} \) (see Definition 3.2 for a more explicit and general description). The next logical step is to vary the reference family within the above construction. In other words, we consider \( \mathcal{H}_k(G; \mathcal{F}) \), where \( \mathcal{F} \) is a family of graphs. Surprisingly, some of the constructions may end up being hypergraphs depending on the choice of \( \mathcal{F} \) as well as the parameter \( k \). As an example, we note that \( \mathcal{H}_k(K_n; K_{rk}) \) is just the Kneser \( r \)-hypergraph of the set system \(([n], \binom{[n]}{k})\) with respect to suitably chosen integers \( n, k, \) and \( r \) (see Section 2.2). The known computation of the chromatic number of \( \mathcal{H}_k(K_n; K_{rk}) \) can be easily generalized to other graphs, which proves the power of the hypercompletion operator in the case of hypergraphs.

**Theorem 4.2** Let \( G \) be a graph with \( \omega(G) = \chi(G) \). Then for any integers \( k \) and \( r \) satisfying \( 2k + 1 \leq n \) and \( 2 \leq r \leq \lceil \frac{\omega(G)}{k} \rceil \), we have

\[
\chi(\mathcal{H}_k(G; K_{rk})) = \left\lceil \frac{\chi(G) - r(k - 1)}{r - 1} \right\rceil.
\]

In the particular case where \( r = 2 \), we state the following result, which can be considered as the generalized Lovász–Kneser theorem:

**Corollary 4.3** If \( G \) is a graph with \( \chi(G) = \omega(G) \), then \( \chi(\mathcal{H}_k(G; K_{2k})) = \chi(G) - 2k + 2 \) for any \( k \leq \lceil \frac{\omega(G)}{2} \rceil \).

In fact, we show that \( \chi(\mathcal{H}_k(G; K_{2k})) \in [\omega(G) - 2k + 2, \chi(G) - 2k + 2] \) for any graph \( G \).

Even if every graph can be realized as an induced subgraph of some Kneser graph, we note that there exist graphs for which \( \mathcal{H}_k(G; K_{2k}) \) is not an induced subgraph of \( K(n; k) \) for any integer \( n \).

Almost all other computations regarding to Kneser (hyper)graphs can be carried out to hypercomplete (hyper)graphs (see Sections 4 and 4.1 for more details). For instance, the clique numbers or the fractional chromatic numbers can be derived similarly. Such results also enable us to provide examples of graphs such that there is a large gap between their chromatic and fractional chromatic numbers that may be of particular interest.

Shortly after Lovász’s proof of the Kneser conjecture, Schrijver found a vertex-critical induced subgraph of \( K(n; k) \), whose chromatic number is the same as that of \( K(n; k) \) [15]. We recall that a set \( S \subseteq [n] \) is called
stable if $i \in S$ implies $i+1 \notin S$, and if $n \in S$ implies $1 \notin S$. The stable Kneser graph (or the Schrijver graph) $SK(n; k)$ is defined to be the graph whose vertices are all stable $k$-subsets of $[n]$, 2 being adjacent if and only if they are disjoint. We generalize Schrijver’s construction to hypercomplete graphs by fixing a coloring of the source graph beforehand. Suppose that $\kappa$ is a proper $n$-coloring of $G$, where $\chi(G) = n$. When $2k \leq \omega(G)$, we call a $k$-complete $S$ of $G$ $\kappa$-stable if $\kappa(S) := \{\kappa(s) : s \in S\}$ is a stable set in $[n]$. We define the stable $k$-hypercomplete graph $SH_k^n(G; K_{2k})$ of $G$ with respect to $\kappa$ to be the graph with vertex set consisting of all $\kappa$-stable $k$-completes of $G$, where 2 such sets form an edge if and only if their set union induces a $2k$-complete in $G$. The following is not a surprise:

**Corollary 4.12** Let $\kappa$ be a proper $n$-coloring of a graph $G$ with $n = \chi(G) = \omega(G)$. Then, $\chi(SH_k^n(G; K_{2k})) = \chi(G) - 2k + 2$ for any $k \leq \lfloor \frac{n}{2} \rfloor$.

It should be noted that the subgraphs $SH_k^n(G; K_{2k})$ of $\mathcal{H}_k(G; K_{2k})$ need not be vertex-critical as opposed to the case of complete graphs. Furthermore, for any 2 distinct $n$-colorings $\kappa_1$ and $\kappa_2$ of $G$, the graphs $SH_k^{\kappa_1}(G; K_{2k})$ and $SH_k^{\kappa_2}(G; K_{2k})$ are not isomorphic in general.

Our final move is to study the chromatic number of hyperpath (hyper)graphs $\mathcal{H}(P_k(G; P_m))$. As opposed to hypercomplete graphs, there seems to be no general relation between $\chi(G)$ and $\chi(\mathcal{H}(P_k(G; P_m)))$. However, there is a simple upper bound involving the maximum degree of the source graph $\chi(\mathcal{H}(P_k(G; P_2k))) \leq 2\Delta(G) - 1$, which is indeed the best possible. On the other hand, by the use of the hypercompletion operator, we also provide upper bounds on $\chi(\mathcal{H}(P_k(G; P_m)))$ when $m = k + 1$ and $m = 2k$ in terms of the $k$-distance chromatic numbers of the source graph (see Section 5).

**Theorem 5.5** For any graph $G$ and any integer $k \geq 2$, we have $\chi(\mathcal{H}(P_k(G; P_{k+1}))) \leq \chi_k(G)$ and $\chi(\mathcal{H}(P_k(G; P_{2k}))) \leq \chi_{2k-1}(G) - 2k + 2$.

2. Preliminaries

We first recall some general notions and notations needed throughout the paper, and repeat some of the definitions mentioned in the introduction more formally.

2.1. Graphs

By a simple graph $G$, we will mean an undirected graph without loops or multiple edges. If $G$ is a graph, $V(G)$ and $E(G)$ (or simply $V$ and $E$) denote its vertex and edge sets. An edge between $u$ and $v$ is denoted by $e = uv$ or $e = \{u, v\}$ interchangeably. A graph $G = (V, E)$ is called the null-graph on $V$ whenever $E = \emptyset$. If $U \subset V$, the graph induced on $U$ is written as $G[U]$. We denote by $|G|$ and $||G||$ the order and size of $G$, while $d(v)$ denotes the degree of a given vertex $v \in V$. A graph $G = (V, E)$ is said to be $r$-regular if $d(v) = r$ for any $v \in V$.

The complement $\overline{G} := (V, \overline{E})$ of a graph $G = (V, E)$ is the graph on $V$ with $uv \in \overline{E}$ if and only if $uv \notin E$. Throughout the paper, $K_n$, $C_n$, and $P_n$ will denote the complete, cycle, and path graphs on $n$ vertices, respectively. The distance $d(x, y)$ between 2 vertices $x$ and $y$ is defined to be the length of a shortest path in $G$ between them.

A subset $S \subseteq V$ is called a complete of $G$ if $G[S]$ is isomorphic to a complete graph, and a $k$-complete of $G$ is a complete of cardinality $k$. The set of all $k$-completes of $G$ will be denoted by $C_k(G)$. In particular,
a complete that is maximal with respect to inclusion is called a clique of \( G \), and the largest \( k \) such that \( G \) has a \( k \)-clique is called the clique number of \( G \) and denoted by \( \omega(G) \). An independent set of \( G \) is a complete of \( \overline{G} \), and the independence number is defined by \( \alpha(G) := \omega(\overline{G}) \).

We recall that a mapping \( \kappa: V \to [m] \) is a (proper vertex) coloring of a graph \( G \) if \( \kappa(u) \neq \kappa(v) \) whenever \( uv \in E \), where \([m] := \{1, 2, \ldots, m\} \). The least integer \( m \) for which \( G \) admits a coloring with \( m \) colors is called the chromatic number of \( G \) and is denoted by \( \chi(G) \).

The line graph \( L(G) \) of a graph \( G \) is the graph on the edges, 2 of which form an edge if and only if they share a common vertex.

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be 2 nonempty graphs on (disjoint) sets of \( r \) and \( s \) vertices. The Cartesian product \( G_1 \square G_2 \) is the graph with vertex set \( V_1 \times V_2 \) and \((x_1, x_2)(y_1, y_2)\) is an edge if and only if either \( x_1y_1 \in E_1 \) and \( x_2 = y_2 \) or \( x_2y_2 \in E_2 \) and \( x_1 = y_1 \), whereas the tensor product (also known as the categorical or weak product) \( G_1 \times G_2 \) is the graph with vertex set \( V_1 \times V_2 \), where 2 vertices \((x_1, x_2)(y_1, y_2)\) are adjacent if and only if both \( x_1y_1 \in E_1 \) and \( x_2y_2 \in E_2 \). The strong product \( G_1 \boxtimes G_2 \) is the graph with vertex set \( V_1 \times V_2 \) and edge set \((E(G_1 \square G_2)) \cup (E(G_1 \times G_2)) \). Furthermore, their disjoint union \( G_1 \cup G_2 \) is the graph \( G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2) \), and their join \( G_1 \vee G_2 \) is the graph on \( n = r + s \) vertices obtained from \( G_1 \cup G_2 \) by inserting new edges from each vertex of \( G_1 \) to each vertex of \( G_2 \). We write \( nG \) to denote the disjoint union of \( n \) copies of \( G \). A (graph) homomorphism \( \phi: G_1 \to G_2 \) is a mapping from \( V_1 \) to \( V_2 \) such that \( \phi(x)\phi(y) \in E_2 \) whenever \( xy \in E_1 \).

For a given positive integer \( n \), we denote by \( \binom{[n]}{k} \) the set of \( k \)-subsets of \([n]\). The Kneser graph \( K(n; k) \) for \( 2k - 1 \leq n \) is defined to be the simple graph on \( \binom{[n]}{k} \), where 2 vertices form an edge if and only if they are disjoint. Note that \( K(n; 1) \) is the \( n \)-vertex complete graph, and \( K(2k - 1, k) \) is the null-graph.

The fractional chromatic number \( \chi_f(G) \) of a graph \( G \) is defined to be the infimum of the fractions \( \frac{n}{k} \) such that there exists a graph homomorphism from \( G \) to \( K(n; k) \) (see \([2, 16]\)).

### 2.2. Hypergraphs

A hypergraph is a set system \( \mathcal{H} = (V, \mathcal{E}) \), where \( V \) is a set, the set of vertices of \( \mathcal{H} \), and \( \mathcal{E} \subseteq 2^V \). The hypergraphs that we consider may have loops; that is, \( \mathcal{E} \) may contain singletons as edges. However, we do not allow repeated edges in general. A hypergraph \( \mathcal{H} = (V, \mathcal{E}) \) is said to be simple if \( A \not\subseteq B \) and \( B \not\subseteq A \) for any 2 edges \( A, B \in \mathcal{E} \). We call a hypergraph \( r \)-uniform if all of its edges have the same cardinality \( r \).

When \( S \subseteq V \), the induced subhypergraph of \( \mathcal{H} \) by the set \( S \) is defined to be the hypergraph \( \mathcal{H}[S] := \{A \in \mathcal{E}: A \subseteq S\} \).

A mapping \( \kappa: V \to [m] \) is a (proper) coloring of a hypergraph \( \mathcal{H} = (V, \mathcal{E}) \) if none of the edges of \( \mathcal{H} \) are monochromatic under \( \kappa \). The least integer \( m \) for which \( \mathcal{H} \) admits a coloring with \( m \) colors is called the chromatic number of \( \mathcal{H} \) and denoted by \( \chi(\mathcal{H}) \). The \( r \)-colorability defect \( \text{cd}_r(\mathcal{H}) \) of a hypergraph \( \mathcal{H} \) is defined by

\[
\text{cd}_r(\mathcal{H}) := \min\{|S|: S \subseteq V \text{ and } \chi(\mathcal{H}[V\setminus S]) \leq r\}.
\]

Given a hypergraph \( \mathcal{H} = (V, \mathcal{E}) \), the Kneser \( r \)-hypergraph of \( \mathcal{H} \), denoted by \( \mathcal{K}_r(\mathcal{H}) \), has vertex set \( \mathcal{E} \), and an \( r \)-tuple \((A_1, \ldots, A_r)\) of edges of \( \mathcal{H} \) forms an edge iff \( A_i \cap A_j = \emptyset \) for all \( i \neq j \). We will simply write \( \mathcal{K}_r(n; k) \) when \( \mathcal{H} = ([n], \binom{[n]}{k}) \).
3. Hyper- $H$ (hyper)graphs

In this section, we introduce our main objects of study, hyper- $H$ (hyper)graphs; provide various examples; and construct some known (hyper)graphs as hyper- $H$ (hyper)graphs. We choose to state the definition as broadly as possible.

**Definition 3.1** Let $G$ be a graph and let $H$ be an induced subgraph of $G$ containing at least one edge. Then $G$ is said to be $H$-decomposable if there exists a family $\{H_1, \ldots, H_s\}$ consisting of induced subgraphs of $G$, each being isomorphic to $H$ such that $G[\bigcup_{i=1}^{s} H_i] \cong G$, and the integer $t$ is called the size of this decomposition. When $G$ is $H$-decomposable, the least integer $d$ for which $G$ admits an $H$-decomposition of size $d$ is called the $H$-width of $G$ and denoted by $w_H(G)$. In case $G$ is not $H$-decomposable, we set $w_H(G) := \infty$.

In particular, when $H$ is a $k$-complete for some $k \geq 2$, we call $w_{K_k}(G)$ the $k$-clique-width of $G$ and denote it by $cw_k(G)$. Note that $w_H(G) < \infty$ if and only if any vertex of $G$ is contained by at least one induced copy of $H$ in $G$. Similarly, when $H$ is a $k$-path for some $k \geq 2$, we call $w_{P_k}(G)$ the $k$-path-width of $G$ and denote it by $pw_k(G)$.

**Definition 3.2** Let $G$ be a graph and let $H$ be an induced subgraph of $G$ containing at least one edge. For a given family $\mathcal{F}$ of graphs, we define the hyper-$H$ hypergraph $\mathcal{H}_H(G; \mathcal{F}) = (\mathcal{V}_H(G), \mathcal{E}_H(G))$ of $G$ with respect to $\mathcal{F}$ by taking $\mathcal{V}_H(G)$ to be the set of induced copies of $H$ in $G$, and $\{A_1, \ldots, A_t\} \in \mathcal{E}_H(G)$ if and only if there exists $F \in \mathcal{F}$ satisfying $w_H(F) = t$ such that $G[\bigcup_{i=1}^{t} A_i] \cong F$.

When $\mathcal{H}_H(G; \mathcal{F})$ is just an ordinary graph, we call it as the hyper-$H$ graph of $G$ with respect to $\mathcal{F}$, which is only possible when $w_H(F) = 2$ for all $F \in \mathcal{F}$. We also abbreviate $\mathcal{H}_H(G; \{F\})$ to $\mathcal{H}_H(G; F)$.

We generally refer to $G$ as the source (or ground) graph of $\mathcal{H}_H(G; \mathcal{F})$, and to $\mathcal{F}$ as the reference family of our operator.

Throughout the paper we only deal with 2 particular cases: $H = K_k$ for some $2 \leq k \leq \omega(G)$ and $H = P_k$ for some $k \geq 2$, where we call the resulting hypergraphs the $k$-hypercomplete hypergraph and $k$-hyperpath hypergraph of $G$ and denote them by $\mathcal{H}_{K_k}(G; \mathcal{F})$ and $\mathcal{H}_{P_k}(G; \mathcal{F})$, respectively.

**Example 3.3** If we consider the graph $G$ illustrated in Figure 1, then $\mathcal{H}_2(G; K_{1,3})$ is a 3-uniform hypergraph on the set $\{1, 2, \ldots, 8\}$ with edges $\{1, 2, 4\}, \{2, 3, 5\}, \{4, 6, 7\},$ and $\{5, 7, 8\}$. We also note that $\mathcal{H}_{P_3}(G; K_{1,3}) \cong 4K_3$.

Figure 1. A graph $G$ with the corresponding edge labeling.
We remark that the hypergraph $H_k(G; \mathcal{F})$ does not have to be a simple hypergraph in general. For instance, $H_2(G; \{P_3, K_{1,3}\})$ contains both $\{1, 2\}$ and $\{1, 2, 4\}$ as edges for the graph $G$ of Figure 1. This may be overcome by assuming that if $F \in \mathcal{F}$ and $F'$ is an induced subgraph of $F$ with $cw_k(F') < \infty$, then $F' \notin \mathcal{F}$.

**Example 3.4** For a given graph $G$, its 2-hypercomplete graph with respect to the family $\{P_3, K_3\}$ is isomorphic to the line graph of $G$; that is, $H_2(G; \{P_3, K_3\}) \cong L(G)$. This easily follows from the fact that for any 2 intersecting edges in $G$, the set of vertices incident to these edges induces either a $P_3$ or a $K_3$.

In fact, our construction generalizes many of the known operators on graphs when $k = 2$. For instance, $H_2(G; P_3)$ is the Gallai's graph of $G$, $H_2(G; P_4)$ is the wing-graph of $G$, and $H_2(G; \{P_4, C_4\})$ is the Chvatal's graph of $G$ (see [3] for details).

**Proposition 3.5** If $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$, then $H_k(K_n; K_{2k}) \cong K(n; k)$.

**Proof** Once we assume that $V(K_n) = [n]$, the claim follows, since 2 $k$-completes $A, B$ of $K_n$ are disjoint if and only if $K_n[A \cup B] \cong K_{2k}$. □

**Remark 3.6** It is well known that any graph can be obtained as an induced subgraph of some Kneser graph [5]; however, for an arbitrary graph $G$, it is not true that the graph $H_k(G; K_{2k})$ is an induced subgraph of $K(n; k)$ for some $n \geq 2k$. For instance, if we consider the graph $G$ illustrated in Figure 3.6, there exists no integer $n$ such that $H_2(G; K_4)$ is an induced subgraph of $K(n; 2)$. This is clearly due to the fact that being disjoint is not enough to form an edge of the hypercomplete graph for any 2 completes.

We may readily extend the construction of Kneser graphs as hypercomplete graphs to a general graph family obtained from the complete graphs by the same manner.
Suppose that $n, r, s \in \mathbb{N}$ are given with $1 \leq s < r \leq \lfloor \frac{n}{2} \rfloor$. We then define $IK(n, r; s)$ to be the graph whose vertices are the $r$-element subsets of $[n]$, 2 of which form an edge if and only if their intersection is a set with $s$-element. We omit the proof of the following obvious fact.

**Proposition 3.7** Let $n, r, s$ be integers with $1 \leq s < r \leq \lfloor \frac{n}{2} \rfloor$. Then $H_r(K_n; K_{2r-s}) = IK(n, r; s)$.

We particularly note that the graph $H_k(K_n; K_{k+1})$ is the well-known Johnson graph $J(n; k)$, whose vertices are $k$-element subsets of $[n]$, and 2 such sets form an edge if and only if their intersection has exactly $k-1$ elements (see the discussion after Theorem 4.13).

It may be expected that our construction will always produce simple graphs if we choose the reference family to be the one consisting of a single complete graph. However, this is not the case. As an example, we consider $H_k(K_n; K_{rk})$ for integers satisfying $2k+1 < n$ and $2 < r \leq \lfloor \frac{n}{k} \rfloor$. On the other hand, it should be noted that the hypergraph $H_k(G; K_{rk})$ is not in general isomorphic to the Kneser $r$-hypergraph of $(V, C_k(G))$ for an arbitrary graph $G = (V, E)$ and integers with $2k+1 < \omega(G)$ and $2 < r \leq \lfloor \frac{\omega(G)}{k} \rfloor$. Moreover, $H_k(G; K_{rk})$ does not have to be an induced subhypergraph of $K_r(n; k)$ for some $n$.

One other family of graphs that we will construct as hypercomplete graphs are stable Kneser graphs, also known as Schrijver graphs [15].

In order to construct $SK(n; k)$ as a hypercomplete graph, we first point out that any stable subset $S$ of $[n]$ corresponds to an independent set in the cycle $C_n$. When $n \geq 4$, we call $SK_n := \overline{C_n}$ the stable complete graph of order $n$.

**Proposition 3.8** For all integers $n, k$ with $1 < k \leq \lfloor \frac{n}{2} \rfloor$, we have

$$
SK(n; k) \cong \begin{cases} 
H_k(SK_2; SK_{2k}), & \text{if } n = 2k, \\
H_k(SK; \{P_{r_1} \cup \ldots \cup P_{r_s} : r_1 + \ldots + r_s = 2k, r_i \geq 1\}), & \text{otherwise}.
\end{cases}
$$

**Proof** It is enough to show that if $A$ and $B$ are two stable $k$-subset of $[n] = V(C_n)$, then $A \cap B = \emptyset$ if and only if $C_n[A \cup B] \cong P_{r_1} \cup \ldots \cup P_{r_s}$ for some partition of $2k$, which trivially holds.

4. Generalized Lovász–Kneser theorem

The generalization of the Lovász–Kneser theorem to Kneser hypergraphs was the main subject of many recent papers [1, 9, 10, 13, 17]. There are 2 main approaches that differ on whether to allow edges consisting
of multisubsets of the ground set together with intersection multiplicities. We here only consider Kneser hypergraphs of hypergraphs whose edges consist of ordinary subsets of the vertex set without multiplicities.

In this restricted context, there is a well-known lower bound due to Kőnig [9, 12].

**Theorem 4.1** ([9]) For any (finite) hypergraph \( H = (V, E) \) and any \( r \geq 2 \), we have \( \chi(K_r(H)) \geq \frac{1}{r-1} cd_r(H) \).

We should also note that the upper bound in the specific case of \( \chi(H_k(K_n; K_{rk})) \) was first constructed by Erdős [4] (see also Ziegler [17]), which is just a modification of Kneser’s upper bound for \( K(n; k) \), and the lower bound is the simple extension of Dol’nikov’s 2-colorability defect [12].

**Theorem 4.2** Let \( G \) be a graph with \( \omega(G) = \chi(G) = n \). Then for any integers \( k \) and \( r \) satisfying \( 2k + 1 \leq n \) and \( 2 \leq r \leq \left\lfloor \frac{n}{k} \right\rfloor \), we have

\[
\chi(H_k(G; K_{rk})) = \left\lfloor \frac{n - r(k - 1)}{r - 1} \right\rfloor.
\]

**Proof** The lower bound follows from Theorem 4.1, since \( \omega(G) = n \), implying that \( H_k(K_n; K_{rk}) \) is an induced subhypergraph of \( H_k(G; K_{rk}) \). For the upper bound, we simply follow Erdős. We let \( \lambda : V(G) \to [n] \) be a proper coloring of \( G \) and define \( \Lambda : V(H_k(G; K_{rk})) \to [m] \) by

\[
\Lambda(A) := \min\{ \frac{1}{r - 1} \min\{ \lambda(A) \}, m \},
\]

where \( \lambda(A) := \{ \lambda(a) : a \in A \} \) and \( m := \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil \). Suppose that \( \{A_1, \ldots, A_r\} \) is a monochromatic edge of \( H_k(G; K_{rk}) \) under \( \Lambda \). If \( \Lambda(A_j) = i < m \) for all \( 1 \leq j \leq r \), then there must exist a vertex \( a_j \in A_j \) with \( \lambda(a_j) = i \) for each \( j \in [r] \). However, this is impossible since \( G[\bigcup_{j=1}^r A_j] \cong K_{rk} \). On the other hand, if \( \Lambda(A_j) = m \) for each \( j \in [r] \), it follows that \( \bigcup_{j=1}^r \Lambda(A_j) \subseteq \{ (r - 1)m, (r - 1)m + 1, \ldots, n \} \). This is also impossible, since the set on the left-hand side contains exactly \( rk \) elements by \( G[\bigcup_{j=1}^r A_j] \cong K_{rk} \), while on the right-hand side there are at most \( rk - r + 1 \) elements.

**Corollary 4.3** If \( G \) is a graph with \( \chi(G) = \omega(G) \), then \( \chi(H_k(G; K_{2k})) = \chi(G) - 2k + 2 \) for any \( k \leq \left\lfloor \frac{\omega(G)}{2} \right\rfloor \).

In general, the condition \( \chi(G) = \omega(G) \) in the statement of Theorem 4.2 cannot be dropped. For this, we consider the graph \( G \) depicted in Figure 4. Note that \( \chi(G) = 5 \) and \( \omega(G) = 4 \), while \( \chi(H_2(G; K_4)) = 2 \).

We may, however, obtain a lower and an upper bound in the general case in the view of Theorem 4.2.

**Corollary 4.4** For any graph \( G \) and for any integers \( k \) and \( r \) satisfying \( 2k + 1 \leq n \) and \( 2 \leq r \leq \left\lfloor \frac{n}{k} \right\rfloor \), we have

\[
\left\lfloor \frac{\omega(G) - r(k - 1)}{r - 1} \right\rfloor \leq \chi(H_k(G; K_{rk})) \leq \left\lfloor \frac{\chi(G) - r(k - 1)}{r - 1} \right\rfloor.
\]

**Proof** Assume that \( G \) is a graph with \( \chi(G) = m > n = \omega(G) \). We then have \( \chi(H_k(G; K_{rk})) \geq \chi(H_k(K_n; K_{rk})) \), since \( \omega(G) = n \). On the other hand, let \( H := G \cup K_m \); that is, \( H \) is the disjoint union of \( G \) and \( K_m \). We note that \( \chi(H) = \omega(H) = m \); hence, \( \chi(H_k(H; K_{rk})) \geq \chi(H_k(G; K_{rk})) \), since \( G \) is an induced subgraph of \( H \).
When \( r = 2 \), the Corollary 4.3 simply translates that \( \chi(H_k(G; K_{2k})) \in [w(G) - 2k + 2, \chi(G) - 2k + 2] \) for any \( k \leq \left\lfloor \frac{\chi(G)}{2} \right\rfloor \).

**Example 4.5** We define \( H_m \) to be the join of \( s \)-copies of the 2m-cycle. If we assume that \( m, s \geq 3 \), then we have \( \chi(H_2(H_m^s; K_{2r})) = \left\lceil \frac{2r - s}{r - 1} \right\rceil \) for any \( s \geq r > 2 \) by Theorem 4.2.

**Example 4.6** Assume that \( G = C_5 \vee C_5 \), the join of the 5-cycle with itself, so \( \omega(G) = 4 \) and \( \chi(G) = 6 \); hence, \( 2 \leq \chi(H_2(G; K_4)) \leq 4 \) by Corollary 4.3. However, the chromatic number of \( H_2(G; K_4) \) is indeed equal to 3. To ensure that, we let \( \{a, b, c, d, e\} \) and \( \{A, B, C, D, E\} \) denote the set of vertices of two 5-cycles respectively with edges in cyclic fashion. We then note that the graph \( H_2(G; K_4) \) is disconnected with 2 connected components; one is isomorphic to the complete bipartite graph \( K_{5,5} \), induced by the set of vertices \( U := \{ab, bc, cd, de, ae, AB, BC, CD, DE, AE\} \), and the other component is isomorphic to \( C_5 \times C_5 \), the tensor product of \( C_5 \) with itself, induced by the rest of the vertices of \( H_2(G; K_4) \).

The above analysis can naturally be generalized to higher order odd cycles. In other words, if we set \( G_r := C_{2r+1} \vee C_{2r+1} \) for \( r \geq 2 \), then the vertices of \( H_2(G_r; K_4) \) corresponding to the initial edges of the copies of \( C_{2r+1} \) in \( G_r \) form a complete bipartite graph \( K_{2r+1,2r+1} \), and the rest of the vertices induce the component being isomorphic to the tensor product \( C_{2r+1} \times C_{2r+1} \). Therefore, \( \chi(H_2(G_r; K_4)) = 3 \).

**Proposition 4.7** If \( G \) is a graph and \( 2k \leq \omega(G) \), then \( \omega(H_k(G; K_{2k})) = \left\lceil \frac{\omega(G)}{k} \right\rceil \).

**Proof** Let \( W = \{A_1, \ldots, A_s\} \) be a complete of \( H_k(G; K_{2k}) \). Then \( G[A_i \cup A_j] \cong K_{2k} \) for all \( i, j \in [s] \) with \( i \neq j \). Therefore, we must have \( G[\cup_{i=1}^s A_i] \cong K_{sk} \), which in turn implies that \( \omega(H_k(G; K_{2k})) = \left\lceil \frac{\omega(G)}{k} \right\rceil \) as claimed.

Our next computation concerns the independence numbers of hypercomplete graphs \( H_k(G; K_{2k}) \). We first point out that the classical result of Erdös–Ko–Rado [7] takes the following simple form in the language of hypercomplete graphs.

**Theorem 4.8** (Erdös–Ko–Rado Theorem) If \( n \geq 2k \), then

\[
\omega(H_k(K_n; \{K_{k+1}, K_{k+2}, \ldots, K_{2k-1}\})) = \binom{n-1}{k-1}.
\]
Therefore, we have \( \alpha(\mathcal{H}_k(K_n; K_{2k})) = \binom{n-1}{k-1} \) whenever \( 2k \leq n \). It seems that we are not that lucky for the general case. We are only able to propose the computation with respect to particularly chosen graphs. We remark that what really differs from the specific case of complete graphs is that the consideration of the maximal intersecting families of \( k \)-completes is not enough to deduce the result for an arbitrary graph. Furthermore, the maximal independent sets do not have to be uniquely determined, as opposed to the cases of many generalizations of the Erdős–Ko–Rado theorem [6].

Let \( G = (V, E) \) be a graph. We denote the strong product graph \( G \boxtimes K_m \) by \( G^{[m]} \) for any \( m \geq 2 \). As an example, the graph \( P_3^{[2]} \) is depicted in Figure 5.

![Figure 5](image_url)

**Figure 5.** The graph \( P_3^{[2]} \) and its 2-hypercompletion.

The reason why we consider the strong product of a graph with complete graphs is that \( \omega(G^{[m]}) = m \cdot \omega(G) \), while \( \alpha(G^{[m]}) = \alpha(G) \) for any graph \( G \) and \( m \geq 2 \).

**Proposition 4.9** For any \( 2 \leq k \leq m \) and \( n \geq 2 \), we have

\[
\alpha(\mathcal{H}_k(P_n^{[m]}; K_{2k})) = \begin{cases} 
(n-1) \cdot \binom{2m-1}{k-1} - \left\lfloor \frac{n-1}{2} \right\rfloor \cdot \binom{m-1}{k-1}, & \text{if } n \text{ is even}, \\
(n-1) \cdot \binom{2m-1}{k-1} - \left\lfloor \frac{n-2}{2} \right\rfloor \cdot \binom{m-1}{k-1}, & \text{if } n \text{ is odd}.
\end{cases} \quad (4.10)
\]

**Proof** We verify the claim by the help of Theorem 4.8 together with an analysis of \( \mathcal{H}_k(P_n^{[m]}; K_{2k}) \). We write \( V(P_n^{[m]}) = \{(i, j) : i \in [n] \text{ and } j \in [m]\} \), and we define \( V_i := \{(i, r), (i + 1, r) : r \in [m]\} \) and \( R_i := P_n^{[m]}[V_i] \) for \( i \in [n-1] \). Similarly, we let \( U_j := \{(j, r) : r \in [m]\} \) and \( T_j := P_n^{[m]}[U_j] \) for \( 2 \leq j \leq n-1 \). We note that \( R_i \cong K_{2m} \) and \( T_j \cong K_m \). We then consider the families of \( k \)-completes of \( P_n^{[m]} \) defined by

\[ A_{2r-1} := \{A \in C_k(R_{2r-1}) : (2r - 1, 1) \in A\} \quad \text{and} \quad A_{2r} := \{A \in C_k(R_{2r}) \setminus C_k(T_{2r}) : (2r, 1) \in A\} \]

whenever \( 2r - 1, 2r \in [n-1] \). When \( n \) is even, it is straightforward to show that \( \cup_{i=1}^{n-1} A_i \) is an independent set in \( \mathcal{H}_k(P_n^{[m]}; K_{2k}) \) with the requested cardinality. When \( n \) is odd, we consider the independent set \( \cup_{i=1}^{n-2} A_i \cup A_n \), where \( W_n := \{(n-1, r), (n, r) : r \in [m]\} \), \( S_n := P_n^{[m]}[W_n] \) and \( A_n := \{A \in C_k(S_n) : (n, 1) \in A\} \). This proves the lower bound.

For the upper bound, we simply note that the contribution of \( k \)-completes contained by any 2 consecutive \( 2m \)-cliques \( R_i \) and \( R_{i+1} \) to an independent set of \( \mathcal{H}_k(P_n^{[m]}; K_{2k}) \) can not be greater than \( 2\binom{2m-1}{k-1} - \binom{m-1}{k-1} \). \( \square \)

Our next aim is to prove an analogue of Schrijver’s result [15] for hypercomplete graphs.

**Definition 4.11** Suppose that \( \kappa \) is a proper \( n \)-coloring of \( G \), where \( \chi(G) = n \). When \( 2k \leq \omega(G) \), we call a \( k \)-complete \( S \) of \( G \) \( \kappa \)-stable if \( \kappa(S) := \{\kappa(s) : s \in S\} \) is a stable set in \([n]\). We define the stable \( k \)-hypercomplete
graph $SH^\kappa_k(G;K_{2k})$ of $G$ with respect to $\kappa$ to be the graph with vertex set consisting of all $\kappa$-stable $k$-completes of $G$, where 2 such sets form an edge if and only if their set union induces a $2k$-complete in $G$.

Corollary 4.12 Let $\kappa$ be a proper $n$-coloring of a graph $G$ with $n = \chi(G) = \omega(G)$. Then $\chi(SH^\kappa_k(G;K_{2k})) = \chi(G) - 2k + 2$ for any $k \leq \lfloor \frac{n}{2} \rfloor$.

Proof The upper bound is provided by the fact that $SH^\kappa_k(G;K_{2k})$ is an induced subgraph of $H_k(G;K_{2k})$, and the lower bound is a consequence of Schrijver’s theorem, since $\omega(G) = n$.

As noted earlier, the graphs $SH^\kappa_k(G;K_{2k})$ and $SH^\kappa_2(G;K_{2k})$ are not in general isomorphic for any distinct 2 $n$-colorings $\kappa_1$ and $\kappa_2$ of $G$. Furthermore, these subgraphs need not be always vertex-critical. For instance, we consider two 4-colorings $\kappa_1: V(P_3^{[2]} \rightarrow [4]$ of $P_3^{[2]}$ depicted in Figure 6. Note that neither $SH^\kappa_4(P_3^{[2]};K_4)$ nor $SH^\kappa_4(P_3^{[2]};K_4)$ is vertex-critical.

![Diagram](image)

**Figure 6.** Two 4-colorings of $P_3^{[2]}$, and the corresponding stable 2-hypercompletions $SH^\kappa_4(P_3^{[2]};K_4)$ and $SH^\kappa_4(P_3^{[2]};K_4)$ of $P_3^{[2]}$, in which the isolated vertices $bd$ and $dc$ are deleted.

As we have seen so far, our computations are only valid with respect to an even order complete graph taken as the reference family. There is one more exceptional case that we examine next.

Theorem 4.13 $\omega(G) - k + 1 \leq \chi(H_k(G;K_{k+1})) \leq \chi(G)$ for any graph $G$ and any integer $k < \omega(G)$.

Proof Suppose that $G$ is a graph with $\omega(G) = n \leq m = \chi(G)$. Since $G$ contains an induced $K_n$, the graph $H_k(G;K_{k+1})$ must contain an induced $K_{n-k+1}$; thus, $\chi(H_k(G;K_{k+1})) \geq n - k + 1$. Let $\lambda: V(G) \rightarrow \{0,1,\ldots,m-1\}$ be a proper $m$-coloring of $G$. We define a coloring $\Lambda: V(H_k(G;K_{k+1})) \rightarrow \{0,1,\ldots,m-1\}$ of $H_k(G;K_{k+1})$ by $\Lambda(\{x_1,\ldots,x_k\}) := \sum_{i=1}^{k} \lambda(x_i) \pmod{m}$ for any $k$-complete $\{x_1,\ldots,x_k\}$ of $G$. Suppose that $\Lambda(A) = \Lambda(B)$ for 2 $k$-completes of $G$. If $G[A \cup B] \cong K_{k+1}$, then we must have $|A \cap B| = k - 1$. We may therefore let $A = \{x_1,\ldots,x_{k-1},y\}$ and $B = \{x_1,\ldots,x_{k-1},z\}$. However, the equality $\Lambda(A) = \Lambda(B)$ implies that $\lambda(y) = \lambda(z)$; that is, $yz \notin E(G)$. Therefore, $G[A \cup B] \not\cong K_{k+1}$, a contradiction. So, $\lambda$ is a proper coloring of $H_k(G;K_{k+1})$ as claimed.

In the particular case where $k = 2$, Theorem 4.13 forces $n - 1 \leq \chi(H_2(G;K_3)) \leq n$ whenever $\chi(G) = \omega(G) = n \geq 2$ for a given graph $G$.

We also remark that both bounds are attainable in this special case. For example, if $G = K_{2r}$, then $\chi(H_2(K_{2r};K_3)) = 2r - 1$, and similarly, $\chi(H_2(K_{2r-1};K_3)) = 2r - 1$ for $r \geq 2$, as the resulting hypercomplete graphs simply correspond to the line graphs of complete graphs. Furthermore, the graphs $H_k(G;K_{k+1})$ can be considered as the generalized Johnson graphs, since $H_k(G;K_{k+1}) \cong J(n;k)$, when $G = K_n$ as noted earlier.
Example 4.14 We let $H_r := C_{2r} \lor C_{2r}$ for $r \geq 2$, and we consider the graph $\mathcal{H}_2(H_r; K_3)$. By Theorem 4.13, we must have $3 \leq \chi(\mathcal{H}_2(H_r; K_3)) \leq 4$. In fact, we claim that $\chi(\mathcal{H}_2(H_r; K_3)) = 3$. To verify that, we first note that the graph $\mathcal{H}_2(H_r; K_3)$ is a connected graph with $4r(r+1)$ vertices. If $U$ is the set of vertices of $\mathcal{H}_2(H_r; K_3)$ corresponding to the initial edges of copies of $C_{2r}$, then $U$ is an independent set. On the other hand, the subgraph of $\mathcal{H}_2(H_r; K_3)$ induced by the set $V(\mathcal{H}_2(H_r; K_3)) \setminus U$ is isomorphic to the Cartesian product $C_{2r} \square C_{2r}$, from which the claim follows.

4.1. Fractional chromatic number and examples

In this subsection, we prove the fractional analogue of the generalized Lovász–Kneser theorem that turns out to have no difference from the specific case of complete graphs.

Theorem 4.15 If $G$ is a graph with $\chi(G) = \omega(G) = n$, then $\chi_f(\mathcal{H}_k(G; K_{2k})) = \frac{n}{k}$ for any $k \leq \lfloor \frac{\omega(G)}{2} \rfloor$.

Proof For the upper bound, we construct an explicit graph homomorphism from $\mathcal{H}_k(G; K_{2k})$ to $\mathcal{H}_k(K_n; K_{2k})$. Since $\omega(G) = n$, a copy of $K_n$ is contained in $G$ as an induced subgraph. We let $\kappa : V(G) \to [n]$ be a proper $n$-coloring of $G$, and we define $\psi : V(G) \to V(K_n) \subseteq V(G)$ by $\psi(v) = x$ if and only if $\kappa(v) = \kappa(x)$.

If we define $\Psi : V(\mathcal{H}_k(G; K_{2k})) \to V(\mathcal{H}_k(K_n; K_{2k}))$ by $\Psi(A) := \{\psi(a) : a \in A\}$, we claim that $\Psi$ is a graph homomorphism. We first note that if $a, b \in A$, then we necessarily have $\psi(a) \neq \psi(b)$ so that $\Psi(A)$ is a $k$-complete in $K_n$. On the other hand, if $G[A \lor B] \cong K_{2k}$ for some $A, B \in V(\mathcal{H}_k(G; K_{2k}))$, then $\Psi(A) \cap \Psi(B) = \emptyset$. We therefore conclude that $\chi_f(\mathcal{H}_k(G; K_{2k})) \leq \frac{n}{k}$.

For the other direction, we simply note that $\chi_f(\mathcal{H}_k(G; K_{2k})) \geq \chi_f(\mathcal{H}_k(K_n; K_{2k})) = \frac{n}{k}$, since $G$ contains an induced $K_n$.

Following the same steps in the proof of Corollary 4.3, we can easily obtain the following by Theorem 4.15.

Corollary 4.16 $\chi_f(\mathcal{H}_k(G; K_{2k})) \in \left[\frac{\omega(G)}{k}, \frac{\chi(G)}{k}\right]$ for any graph $G$ and any integer $k \leq \lfloor \frac{\omega(G)}{2} \rfloor$.

Before providing an example, we recall that $\chi_f(G \lor H) = \max\{\chi_f(G), \chi_f(H)\}$ for any graphs $G$ and $H$ [14], and $\chi_f(G \times C_s) = \min\{\chi_f(G), \chi_f(C_s)\}$ for any graph $G$ and any integer $s \geq 3$ by a recent result of Alon and Lubetzky [2].

Example 4.17 Consider the graph $G_r := C_{2r+1} \lor C_{2r+1}$ for $r \geq 2$. We know from Example 4.6 that $\mathcal{H}_2(G_r; K_4) \cong K_{2r+1, 2r+1} \lor (C_{2r+1} \times C_{2r+1})$. By Corollary 4.16, we have $\chi_f(\mathcal{H}_2(G_r; K_4)) \in [2, 3]$. On the other hand, we note that $\chi_f(K_{2r+1, 2r+1}) = 2$ and $\chi_f(C_{2r+1}) = 2 + \frac{1}{r}$, since both $K_{2r+1, 2r+1}$ and $C_{2r+1}$ are vertex-transitive ([14]). By the same reason together with Alon and Lubetzky’s result, we conclude that $\chi_f(\mathcal{H}_2(G_r; K_4)) = 2 + \frac{1}{r}$ for any $r \geq 2$.

We next provide examples of graphs such that there is a large gap between their chromatic and fractional chromatic numbers, as we promised. Let us first mention an already existing example. The Kneser graph $\mathcal{H}_k(K_{2k-1}; K_{2k})$ contains no triangle while having chromatic number $k + 1$ for $k \geq 2$. By Corollary 4.3 and Theorem 4.15, any graph (other than complete graphs) whose clique and chromatic numbers are equal to $3k - 1$ will produce a new example of a graph with the desired property.
Example 4.18 We may consider the complete $t$-partite graphs $K_{n_1, \ldots, n_t}$ with $n_i \geq 2$. By choosing $t = 3k - 1$, we conclude that $\chi(H_k(K_{n_1, \ldots, n_{3k-1}}; K_{2k})) = k+1$, while it contains no triangle and $\chi_f(H_k(K_{n_1, \ldots, n_{3k-1}}; K_{2k})) = 3 - \frac{1}{k}$.

Example 4.19 Let $m, s \geq 2$ be given. Recall that $H^s_m$ is the join of $s$-copies of the $2m$-cycle. If we choose $k > 2$ and $s = 2k + 1$, then $\chi(H_k(H^2_{2k+1}; K_{2k})) = 2k + 4$ by Corollary 4.3, $\omega(H_k(H^2_{2k+1}; K_{2k})) = 4$ by Proposition 4.7, and $\chi_f(H_k(H^2_{2k+1}; K_{2k})) = 4 + \frac{2}{k}$ by Theorem 4.15.

5. Hyperpath hypergraphs

In this section, we provide a study of the chromatic number of hyperpath (hyper)graphs $\mathcal{HP}_k(G; P_m)$, where we implicitly assume that the source graph $G$ is always $P_m$-dense in the sense that any vertex of $G$ is contained in an induced $m$-path of $G$, since otherwise we have $\mathcal{HP}_k(G; P_m) \cong \mathcal{HP}_k(G - x; P_m)$, if $x \in V(G)$ is not contained by any $m$-path of $G$. We note that $\mathcal{HP}_k(G; P_m)$ is a simple graph whenever $k < m \leq 2k$. For notational purposes, if $A = \{x_1, \ldots, x_k\}$ is a vertex of $\mathcal{HP}_k(G; P_m)$, then we assume that $x_ix_{i+1} \in E(G)$ for any $1 \leq i < k$, and we call $x_1$ and $x_k$ the ends of $A$.

We first show that every graph can be obtained as an induced subgraph of a hyperpath graph, for which we first need a definition.

Definition 5.1 Let $G = (V, E)$ be a graph and let $S = \{x_1, x_2, \ldots, x_n\} \subseteq V$ be given. For any given $k \geq 2$, we define the whisker $W^S_k(G)$ of $G$ of length $k$ with respect to $S$ to be the graph whose vertex set is $V^S_k(G) := V \cup \{x_i^j : 1 \leq j \leq k$ and $i \in [n]\}$ and whose edge set is $E^S_k(G) := E \cup \{x_i^jx_i^{j+1} : 1 \leq j < k$ and $i \in [n]\}$. In particular, we write $W_k(G)$ when $S = V$.

Lemma 5.2 For any given graph $G$, the hyperpath graph $\mathcal{HP}_k(W_k(G); P_{2k})$ contains $G$ as an induced subgraph for any $k \geq 2$.

Proof Denote by $A_i$ the vertex of $\mathcal{HP}_k(W_k(G); P_{2k})$ induced by the set of vertices $\{x_1, \ldots, x_k\}$; then $A_iA_j \in E(\mathcal{HP}_k(W_k(G); P_{2k}))$ if and only if $x_i^jx_i^{j+1} \in E(G)$, from which the claim follows.

As opposed to the case of hypercompletion $\mathcal{H}_k(G; K_{2k})$, the chromatic number of $\mathcal{HP}_k(G; P_{2k})$ may well exceed that of $G$. This is easily verified when $k$ is even, since, for example, the graph $\mathcal{HP}_2(G_0; P_4)$ has chromatic number 3. When $k$ is odd, it is rather illusive to construct such examples. However, we provide one such example as follows.

Example 5.3 Let $G$ be the graph depicted in Figure 7a. It is straightforward to check that $\chi(G) = 3$, while the graph $\mathcal{HP}_3(G; P_5)$ contains a 4-complete illustrated in Figure 7b; hence, it has a chromatic number of at least 4.

There is still a chance that $\chi(G)$ and $\chi(\mathcal{HP}_k(G; P_{2k}))$ may well be equal.

Proposition 5.4 If $G$ is bipartite and $k \geq 3$ is an odd integer, then $\chi(\mathcal{HP}_k(G; P_m)) = 2$ for any $m \geq 2k$.

Proof Let $V(G) = U_1 \cup U_2$ be the bipartition. Since $k$ is odd, any $k$-path in $G$ has its ends in either $U_1$ or $U_2$. Therefore, if we color any such $k$-path by $i$ if its ends are contained by $U_i$ for $i = 1, 2$, this is definitely a proper coloring of $\mathcal{HP}_k(G; P_m)$, \hfill $\Box$. 

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Even if there seems to be no general relation between \( \chi(G) \) and \( \chi(H(P_k; P_m)) \), we can bound the latter by the \( k \)-distance chromatic number of the source graph for a particularly chosen \( m \). We recall that a \( k \)-distance coloring of a graph \( G \) is a vertex coloring of it, in which 2 vertices lying at a distance of less than or equal to \( k \) must be assigned different colors. The minimum number of colors needed in such a coloring is called the \( k \)-distance chromatic number of \( G \) and is denoted by \( \chi_k(G) \). In terms of graph operations, \( \chi_k(G) \) is equal to the chromatic number of \( G^k \), the \( k \)-th power of \( G \), which is the graph on the same set of vertices such that any 2 vertices whose distance is at most \( k \) are adjacent.

**Theorem 5.5** For any graph \( G \) and any integer \( k \geq 2 \), we have \( \chi(H(P_k; P_{k+1})) \leq \chi_k(G) \) and \( \chi(H(P_k; P_{2k})) \leq \chi_{2k-1}(G) - 2k + 2 \).

**Proof** If \( A \) is a \( k \)-path in \( G \), then it is a \( k \)-complete of \( G^r \) when \( k - 1 \leq r \leq 2k \). Therefore, the inclusion maps \( H(P_k; P_{k+1}) \hookrightarrow H(G^k; K_{k+1}) \) and \( H(P_k; P_{2k}) \hookrightarrow H(G^{2k-1}; K_{2k}) \) are graph homomorphisms; hence, claims follow from Theorem 4.13 and Corollary 4.3.

**Theorem 5.6** For any graph \( G \) and any integer \( k \geq 2 \), we have \( \chi(H(P_k; P_{2k})) \leq 2\Delta(G) - 1 \), and this bound is tight.

**Proof** Suppose \( \Delta = \Delta(G) \) and let \( A = \{x_1, \ldots, x_k\} \) be a vertex of \( H(P_k; P_{2k}) \). If \( B \) is any other vertex of \( H(P_k; P_{2k}) \) satisfying \( AB \in E(H(P_k; P_{2k})) \), then we must have either \( N_G(x_1) \cap B \neq \emptyset \) or \( N_G(x_k) \cap B \neq \emptyset \). When \( N_G(x_1) \cap B \neq \emptyset \), we say that \( B \) is a neighbor of \( A \) at \( x_1 \), and similarly \( B \) is a neighbor of \( A \) at \( x_k \) if \( N_G(x_k) \cap B \neq \emptyset \). Now, if \( B_1, \ldots, B_r \) are neighbors of \( A \) at \( x_1 \), we must have \( B_i \cap B_j = \emptyset \) for any distinct \( i, j \in [r] \) in order to maximize the degree of \( A \) in \( H(P_k; P_{2k}) \); thus, \( r \leq \Delta - 1 \), which in turn implies the claim by Brooks’ theorem.

For tightness, let \( D(r, k) \) be the graph defined for \( r > 1 \) on the set \( V(D(r, k)) = V_1 \cup V_2 \cup V_3 \), where \( V_1 = \{x_1, \ldots, x_k\} \), \( V_2 = \{y^1_1, y^1_2, \ldots, y^{r-1}_1\} \) and \( V_3 = \{z^1_1, \ldots, z^1_k, \ldots, z^{r-1}_1, \ldots, z^{r-1}_k\} \) with the set of edges:

\[
E(D(r, k)) = \{x_ix_{i+1} : 1 \leq i < k\} \cup \{y^i_qy^i_{i+1} : 1 \leq i < k \text{ and } 1 \leq s \leq r - 1\} \\
\cup \{z^i_sz^i_{s+1} : 1 \leq i < k \text{ and } 1 \leq s \leq r - 1\} \cup \{x_1y^1_1 : 1 \leq s \leq r - 1\} \\
\cup \{x_kz^1_1 : 1 \leq s \leq r - 1\} \cup \{y^i_qz^i_s : 1 \leq i \leq r - 1\} \\
\cup \{y^p_{i+1}y^q_i : 1 \leq p < q \leq r - 1\} \cup \{z^p_{i+1}z^q_i : 1 \leq p < q \leq r - 1\}.
\]
Note that $\Delta(D(r,k)) = r$, and it is easy to check that the hyperpath graph $\mathcal{HP}_k(D(r,k);P_{2k})$ contains a $(2r-1)$-complete.

There are other ways to bound $\chi(\mathcal{HP}_k(G;P_{2k}))$ that we state next, while noting that the computation of $\chi(\mathcal{HP}_2(G^{k-1};P_4))$ or $\chi(\mathcal{H}_2(G;K_4-e))$ seems to be as difficult as that of $\chi(\mathcal{HP}_k(G;P_{2k}))$, where $K_4-e$ denotes the complete graph on 4 vertices with an edge removed (see Figure 3).

**Proposition 5.7** Let $G$ be a graph. Then $\chi(\mathcal{HP}_k(G;P_{2k})) \leq \chi(\mathcal{HP}_2(G^{k-1};P_4))$ for any $k \geq 2$ and $\chi(\mathcal{HP}_k(G;P_{2k})) \leq \chi(\mathcal{H}_2(G;K_4-e))$ for any $k > 2$.

**Proof** If $A = \{x_1, \ldots, x_k\}$ is a vertex of $\mathcal{HP}_k(G;P_{2k})$, then the mapping $\Psi(A) := \{x_1, x_k\}$ defines a graph homomorphism in either case.

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**References**


