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# Continuous invariant averagings

**Djavvat KHADJIEV,*Abdullah ÇAVUŞ**
Department of Mathematics, Karadeniz Technical University, Trabzon, 61080, Turkey

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**Abstract:** Main results: For every equicontinuous almost periodic linear representation of a group in a complete locally convex space $L$ with the countability property, there exists the unique invariant averaging; it is continuous and is expressed by using the $L$-valued invariant mean of Bochner and von-Neumann. An analog of Wiener's approximation theorem for an equicontinuous almost periodic linear representation in a locally convex space with the countability property is proved.

**Key words:** Invariant averaging, invariant mean, almost periodic function

## 1. Introduction

The concept of the invariant averaging was introduced and investigated in the paper [14]. The following results were obtained in it: $(i)$. A group $G$ is amenable if and only if every almost periodic linear representation of $G$ in a quasi-complete locally convex space has an invariant averaging; $(ii)$. A locally compact group $G$ is compact if and only if every strongly continuous linear representation of $G$ in a quasi-complete locally convex space has an invariant averaging. The invariant averaging can be considered as an infinite-dimensional analog of Reynold's operator in the invariant theory [21, II.3.2]. Invariant averagings are closely connected with vector-valued invariant means, amenable groups, almost periodic functions, almost periodic linear representations of a group in locally convex spaces and uniformly equicontinuous actions of a group on compacts. The vector-valued invariant mean with values in a locally convex space was introduced and applied to investigation of vector-valued almost periodic functions by Bochner and von Neumann [1]. This investigation was continued in papers ([3]–[7], [10], [11], [17], [19], [20], [25]) and different applications were obtained. A survey on almost periodic functions and representations in locally convex spaces is given in [22].

The continuity property of an invariant averaging and uniqueness of an invariant averaging have important role for investigations of problems of harmonic analysis and dynamical systems. The present paper is devoted to a description of a class of almost periodic linear representations of a group with a continuous invariant averaging.

Our paper is organized as follows. In section 2, some definitions and results which will be used in the following sections are given. In particular, the theorem on equivalence of different conditions of precompactness for $L$-valued bounded functions is given (Theorem 2), where $L$ is a locally convex space. In section 3, the theorem on existence of invariant averaging for almost periodic representation of an amenable group in a locally convex space is obtained (Theorem 3). In section 4, the following theorem is obtained: for every

*Correspondence: haciyev@ktu.edu.tr
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equicontinuous almost periodic linear representation of a group $G$ in a complete locally convex space with the countability property, there exists the unique invariant averaging; it is continuous and is expressed by using the $L$-valued invariant mean of Bochner and von Neumann (Theorem 5). In section 5, an analog of Wiener's approximation theorem for an equicontinuous almost periodic linear representation in a locally convex space with the countability property is proved (Theorem 7).

## 2. Preliminaries

Let $L$ be a complex locally convex vector space and $H(L)$ be the group of all continuous linear operators $A : L \to L$ such that $A^{-1}$ exists and is continuous. Let $G$ be a group.

**Definition 1** *A homomorphism $\alpha : G \to H(L)$ is called a linear representation of a group $G$ in a locally convex space $L$.*

Let $\alpha$ be a linear representation of $G$ in $L$. For $x \in L$, let $\{\alpha(s)x, s \in G\} = \{y \in L : y = \alpha(t)x, t \in G\}$ be the $\alpha(G)$-orbit of $x$. Denote the convex hull of $\{\alpha(s)x, s \in G\}$ by $Co\{\alpha(s)x, s \in G\}$ and the closure of $Co\{\alpha(s)x, s \in G\}$ in $L$ by $\overline{Co\{\alpha(s)x, s \in G\}}$.

**Definition 2** [14, Definition 1.2] *A linear operator $M : L \to L$ is called an invariant averaging for $\alpha$ if:*

*(i)* $\alpha(t)M(x) = M(\alpha(t)x) = M(x)$ *for all* $x \in L$ *and all* $t \in G$*;*

*(ii)* $M(x) \in \overline{Co\{\alpha(s)x, s \in G\}}$ *for all* $x \in L$.

**Remark 1** *This definition differs from the definition of the "invariant average" in [3].*

Let $M$ be an invariant averaging for $\alpha$. Put $L^G = \{y \in L : \alpha(t)y = y, \forall t \in G\}$; that is $L^G$ is the set of all $\alpha(G)$-invariant points of $L$. Then $L^G$ is a closed linear subspace of $L$. By Definition 2, $M(x) \in L^G$ for all $x \in L$, $M(x) = x$ for all $x \in L^G$ and $M(M(x)) = M(x)$ for all $x \in L$. Hence $M$ is a projection operator onto $L^G$.

**Remark 2** *Linear operators which are similar to the invariant averaging (for example, the Reynolds operator), have an important role in the invariant theory [21, II.3.2]. Similar operators also appear and are useful in the harmonic analysis. For example, the linear operator $P_0$ in [15, Proposition 2] is an invariant averaging of a strongly continuous linear representation of the one-dimensional torus $T$ in a Banach space.*

Let $L$ be a complete complex locally convex space. Denote by $Z(L)$ a zero neighborhood basis of $L$ such that every $U \in Z(L)$ is an absolutely convex subset of $L$. Let $G$ be a group.

**Definition 3** *([23, Definition 11]) The function $f : G \to L$ is called bounded if, for every $U \in Z(L)$, there exists a real number $\lambda > 0$ such that $\lambda f(t) \in U$ for all $t \in G$.*

Denote by $B(G, L)$ the set of all bounded functions $f : G \to L$. $B(H, L)$ is a complex vector space. We consider the following topology in $B(G, L)$. For $U \in Z(L)$, put $\tau(U) = \{f \in B(G, L) : f(t) \in U, \forall t \in G\}$. Then $\{\tau(U), U \in Z(L)\}$ is a fundamental system of zero neighborhoods for some locally convex topology $\tau$ in

$B(G, L)$. Further, we consider $B(G, L)$ with the same topology $\tau$. In the case $L = C$, where $C$ is the field of complex numbers, the topology $\tau$ is the topology of the norm $\|x\| = \sup_{t \in G} |x(t)|$ in $B(G, C)$.

For brevity, we will say that a locally convex space $L$ has the countability property if there exists a sequence of zero neighborhoods with empty intersection. The following theorem is known.

**Theorem 1** *[23, Theorems 17, 18] If $L$ is a complete locally convex space with the countability property, so is $\{B(G, L), \tau\}$.*

For fixed $f \in B(G, L)$ and $a \in G$, let ${}_af$ be the $L$-valued function on $G$ such that ${}_af(x) = f(ax)$ for all $x \in G$. Let $f_a$ be the $L$-valued function on $G$ such that $f_a(x) = f(xa)$ for all $x \in G$. Then, for fixed $a, b \in G$, ${}_af_b$ is the function ${}_af_b(x) = f(axb)$ for all $x \in G$. Let $D_af$ be the function on $G \times G = G^2$ such that $(D_af)(x, y) = f(xay)$ for all $x, y \in G$.

For a proof of Theorem 4, we need the following theorem.

**Theorem 2** *Let $L$ be a complete locally convex space, $G$ is a group and let $f \in B(G, L)$. Then the following properties of $f$ are equivalent:*

$(i)$ *$\{f_a : a \in G\}$ is precompact in $B(G, L)$;*

$(ii)$ *$\{{}_af : a \in G\}$ is precompact in $B(G, L)$;*

$(iii)$ *$\{D_af : a \in G\}$ is precompact in $B(G^2, L)$.*

**Proof** This theorem is known in the case $L = C$, where $C$ is the field of complex numbers (see [9, Theorem 18. 1], [18, p.145]). For an arbitrary complete locally convex space $L$, a proof of this theorem is similar to the case $L = C$. □

## 3. Invariant averagings of almost periodic representations of an amenable group

Let $\alpha$ be a linear representation of a group $G$ in a locally convex space $L$.

**Definition 4** [18, p. 142] *An element $x \in L$ is called almost periodic if its orbit $\{\alpha(t)x, t \in G\}$ is precompact in $L$. A representation $\alpha$ will be called almost periodic if every element of $L$ is almost periodic.*

**Remark 3** *This is a variant of the definition of an almost periodic operator semigroup proposed by K. de Leeuw and I. Glicksberg [16].*

Let $C$ be the field of complex numbers and $B(G, C)$ is the set of all bounded complex functions on $G$. $B(G, C)$ is a Banach space with respect to the norm $\|x\| = \sup_{t \in G} |x(t)|$, where $x \in B(G, C)$. Let $B(G, C)'$ be the conjugate space of $B(G, C)$. For $\varphi \in B(G, C)'$, put $(Q_s)\varphi(x) = \varphi(x_s)$, where $s \in G, x_s(t) = x(s^{-1}t)$. Then $Q$ is a linear representation of $G$ in $B(G, C)'$. We consider $B(G, C)'$ with respect to the $w^*$-topology.

**Proposition 1** $(i)$ *$B(G, C)'$ is a quasi-complete locally convex space;*

$(ii)$ *The linear representation $Q$ is almost periodic.*

**Proof** $(i)$. According to Corollary 2 in [2, ch.III, 3.7], $B(G,C)'$ is a quasi-complete locally convex space with respect to the $w^*$-topology.

$(ii)$. We have $|(Q_s\varphi)x| = |\varphi(x_s)| \leq \|\varphi\|\|x_s\| = \|\varphi\|\|x\|$. Hence $\|Q_s\varphi\| \leq \|\varphi\|$ for all $s \in G$ and

$$\|\sum_{i=1}^{n} \lambda_i (Q_s\varphi)x\| \leq \sum_{i=1}^{n} \lambda_i \|(Q_s\varphi)x\| \leq \sum_{i=1}^{n} \lambda_i \|\varphi\|\|x\| = \|\varphi\|\|x\|$$

for all $\lambda_i \in R$ such that $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$. Hence $\overline{Co\{Q_s\varphi, s \in G\}}$ is bounded in $B(G,C)'$. This implies that it is compact with respect to the $w^*$-topology. Hence the linear representation $Q$ is almost periodic. □

**Theorem 3** *For a group $G$ the following conditions are equivalent:*

$(i)$ *$G$ is an amenable group;*

$(ii)$ *Every almost periodic representation of $G$ in a quasi-complete locally convex space has an invariant averaging;*

$(iii)$ *The linear representation $Q$ has an invariant averaging.*

**Proof** $(i) \leftrightarrow (ii)$ is given in [14, Theorem 2.1].

$(ii) \rightarrow (iii)$. By Proposition 1, $B(G,C)'$ is a quasi-complete locally convex space and the linear representation $Q$ is almost periodic. Hence $Q$ has an invariant averaging.

$(iii) \rightarrow (i)$. Assume that $Q$ has an invariant averaging $M$. Then $\overline{Co\{Q_s\mu, s \in G\}}$ has a $G$-invariant element $M\mu$ for all $\mu \in B(G,C)'$. Let $\mu \in B(G,C)'$ be a mean on $B(G,C)$ that is a linear functional on $B(G,C)$ such that $\mu(f) \geq 0$ for all $f \geq 0$ and $\mu(1) = 1$. Then there exists a net $\{\mu_\nu\}$ in $\overline{Co\{Q_s\mu, s \in G\}}$ such that $\mu_\nu(f) \rightarrow (M\mu)(f)$ for all $f \in B(G,C)$, where $\mu_\nu(f) = \sum_{j=1}^{m_\nu} a_{\nu j}\mu(f((s_{\nu j}^{-1}t)))$, $\sum_{j=1}^{m_\nu} a_{\nu j} = 1$, $s_{\nu j} \in G$ and $a_{\nu j} \geq 0$ for all $\nu, j$. Since $\mu_\nu(f) \geq 0$ for all $f \geq 0$ and $\mu_\nu(1) = 1$, we obtain that $(M\mu)(f) \geq 0$ for all $f \geq 0$ and $(M\mu)(1) = 1$ that is $M\mu$ is an left invariant mean on $G$. Hence $G$ is amenable. □

Assume that the linear representation $\alpha$ in $L$ has an invariant averaging. Then according to Definition 2, $\overline{Co\{\alpha(s)x, s \in G\}}$ contains an $\alpha(G)$-invariant point for all $x \in L$. It is very important (in particular, in ergodic theory) to know when $\overline{Co\{\alpha(s)x, s \in G\}}$ has a unique $\alpha(G)$-invariant point.

**Proposition 2** *Let $\alpha$ be a linear representation of a group $G$ in a locally convex space $L$. Assume that $\alpha$ has a continuous invariant averaging $M$. Then*

$(i)$ *$\overline{Co\{\alpha(s)x, s \in G\}}$ contains the unique $\alpha(G)$-invariant point for every $x \in L$;*

$(ii)$ *every invariant averaging of the linear representation $\alpha$ is equal to $M$.*

**Proof** $(i)$. This assertion is proved in [14, Proposition 1.2].

$(ii)$. Let $M_1$ be an arbitrary invariant averaging of $\alpha$. Then, by Definition 2, $M_1(x) \in \overline{Co\{\alpha(s)x, s \in G\}}$ and $M_1(x) \in L^G$. According to assertion $(i)$, the set $\overline{Co\{\alpha(s)x, s \in G\}}$ has the unique $\alpha(G)$-invariant point. Since $M(x)$ and $M_1(x)$ are $\alpha(G)$-invariant elements of $\overline{Co\{\alpha(s)x, s \in G\}}$, we obtain that $M(x) = M_1(x)$ for all $x \in L$. Hence $M = M_1$. □

**4. Invariant averaging of an equicontinuous almost periodic linear representation**

Let $\alpha$ be a linear representation of a group $G$ in a locally convex space $L$.

**Definition 5** [13, p. 232] *A linear representation $\alpha$ of $G$ in a locally convex space $L$ will be called equicontinuous if, for every zero neighborhood $U$ of $L$, there exists a zero neighborhood $V$ such that $\alpha(t)V \subseteq U$ for all $t \in G$.*

*Example.* We define linear representations $\alpha_l$ and $\alpha_r$ of $G$ in $B(G,L)$ as follows: for $s \in G$ and $f \in B(G,L)$, let $(\alpha_l(s)f)(x) = f(s^{-1}x)$ and $(\alpha_r(s)f)(x) = f(xs)$. Then $\alpha_l$ and $\alpha_r$ are equicontinuous linear representations of $G$ in $B(G,L)$.

**Definition 6** *A linear representation $\alpha$ of $G$ in a quasi-complete locally convex space $L$ will be called equicontinuous almost periodic if*

*(i) the set $\{\alpha(t)x, t \in G\}$ is conditionally compact for any $x \in L$;*

*(ii) the family $\{\alpha(t), t \in G\}$ is equicontinuous in $L$.*

**Remark 4** *For a Banach space $L$, every almost periodic linear representation of a group $G$ in $L$ is equicontinuous (see [18, p.142]).*

Let $\alpha$ be a linear representation of $G$ in a locally convex space $L$. Denote by $A_p(L)$ the set of all almost periodic elements of $L$. It is obvious that $L^G \subseteq A_p(L)$.

**Proposition 3** *Let $\alpha$ be a linear representation of $G$ in a locally convex space $L$. Then $A_p(L)$ is a linear subspace of $L$.*

**Proof** Let $U \in Z(L)$ be an arbitrary zero neighborhood. For $U$, there exists $V \in Z(L)$ such that $V + V \subset U$. Let $x, y \in L$ be almost periodic points. Then, for $V$, there exist finite $V$-meshes $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_m\}$ of $\alpha(G)$-orbits $\{\alpha(t)x, t \in G\}$ and $\{\alpha(t)y, t \in G\}$, respectively. Hence, for every $t \in G$, there exist $a_k$ and $b_l$ such that $\alpha(t)x - a_k \in V$ and $\alpha(t)y - b_l \in V$. We have $\alpha(t)(x+y) - (a_k + b_l) = (\alpha(t)x - a_k) + (\alpha(t)y - b_l) \in V + V \subset U$. Hence the set $\{a_k + b_l : 1 \le k \le n, 1 \le l \le m\}$ is an $U$-mesh of the set $\alpha(t)(x+y)$. This means that $x + y$ is an almost periodic point. Similarly, if $x$ is almost periodic and $\lambda \in C$, then $\lambda x$ is almost periodic. □

**Proposition 4** *Let $\alpha$ be an equicontinuous linear representation of $G$ in a complete locally convex space $L$. Then $A_p(L)$ is a closed linear subspace of $L$.*

**Proof** Let $\{x_\nu\}$ be a convergent net in $L$ such that $x_\nu \in A_p(L)$ and $\lim_\nu x_\nu = x \in L$. Prove that $x \in A_p(L)$. Let $U$ be a zero neighborhood of $L$. Then there exist zero neighborhoods $V$ and $V'$ of $L$ such that $V + V \subseteq U$ and $\alpha(t)V' \subseteq V$ for all $t \in G$. Since $x_\nu \to x$, there exists an element $x_{\nu_0}$ of the net $\{x_\nu\}$ such that $x - x_{\nu_0} \in V'$. We have $\alpha(t)(x - x_{\nu_0}) \in \alpha(t)V' \subseteq V$ for all $t \in G$. Since $x_{\nu_0}$ is an almost periodic element of $L$, there exists a finite subset $\{y_1, y_2, \cdots, y_m\}$ of $L$ with the following property: for any $t \in G$, there exists $k \in \{1, 2, \ldots, m\}$ such that $\alpha(t)x_{\nu_0} - y_k \in V$. This implies $\alpha(t)x - y_k = (\alpha(t)x - \alpha(t)x_{\nu_0}) + (\alpha(t)x_{\nu_0} - y_k) \in V + V \subseteq U$. This means that the set $\{y_1, y_2, \cdots, y_m\}$ is an $U$-mesh of $\{\alpha(t)x, t \in G\}$. Hence $\{\alpha(t)x, t \in G\}$ is precompact and $x \in A_p(L)$. □

Let $x \in L$ be fixed. Put $F_x(t) = \alpha(t)x$. Then $F_x : G \to L$ is an $L$-valued function on $G$.

**Definition 7** *([1, Definition 1]). A function* $f \in B(G,L)$ *is called almost periodic if the sets* $\{f_a : a \in G\}$ *and* $\{{}_af : a \in G\}$ *are precompact in* $B(G,L)$.

**Theorem 4** *Let* $\alpha$ *be an equicontinuous almost periodic linear representation of* $G$ *in a complete locally convex space* $L$ *and* $x \in L$. *Then* $F_x(t) = \alpha(t)x \in B(G,L)$ *and it is an almost periodic function on* $G$ *for every* $x \in L$.

**Proof** Prove that $F_x \in B(G,L)$ for every $x \in L$. Since every element of $L$ is almost periodic, according to Definition 3, the set $\{\alpha(t)x, t \in G\}$ is precompact in $L$. Hence it is bounded in $L$ and $F_x \in B(G,L)$.

Let $x \in L$ be fixed. Prove that $\{F_x(ta) : a \in G\}$ is precompact in $B(G,L)$. Since the family $\{\alpha(t), t \in G\}$ is equicontinuous in $L$, for every zero neighborhood $U \in Z(L)$, there exists $V \in Z(L)$ such that $\alpha(t)V \subset U$ for all $t \in G$. Since the set $\{\alpha(a)x, a \in G\}$ is precompact in $L$, there exists a finite subset $\{a_1, \ldots, a_n\}$ of $G$ such that, for every $a \in G$, there exists $k \in \{1, 2, \ldots, n\}$ satisfying the condition $\alpha(a)x - \alpha(a_k)x \in V$. Then $\alpha(t)(\alpha(a)x - \alpha(a_k)x) = (\alpha(ta)x - \alpha(ta_k)x) \in \alpha(t)V \subset U$ for all $t \in G$. This means that the set $\{\alpha(ta_k)x : k = 1, 2, \ldots, n\}$ is an $U$-mesh of $\{F_x(ta) = \alpha(ta)x : a \in G\}$. Hence $\{F_x(ta) = \alpha(ta)x : a \in G\}$ is precompact in $B(G,L)$ and the property $(i)$ of Theorem 3 holds for the function $F_x(t) = \alpha(t)x$. Hence, by the equivalence $(i) \leftrightarrow (ii)$ in Theorem 3, $F_x(t)$ is almost periodic in the sense of the Definition 7. □

The $L$-valued invariant mean in $B(G,L)$ was introduced by S. Bochner and J. von Neumann in the paper [1]. Denote it by $M_{NB}$.

**Theorem 5** *Let* $L$ *be a complete locally convex space with the countability property and* $\alpha$ *is an equicontinuous almost periodic linear representation of a group* $G$ *in* $L$. *Then there exists the unique invariant averaging* $M$ *of* $\alpha$, *it is continuous and* $M(x) = M_{NB}(\alpha(t)x)$ *for all* $x \in L$.

**Proof** Let $x \in L$. We consider the function $F_x(t) = \alpha(t)x$. By Theorem 4, $F_x(t)$ is an almost periodic function on $G$. Then, according to [1], there exists the $L$-valued invariant mean $M_{NB}(F_x(t))$ of $F_x(t) = \alpha(t)x$. Put $M(x) = M_{NB}(\alpha(t)x)$. Then $M : L \to L$ is a mapping. Prove that $M$ is a continuous invariant averaging of $\alpha$.

By properties 2) and 3) of Theorem 16 in [1], $M(x+y) = M_{NB}(\alpha(t)(x+y)) = M_{NB}(\alpha(t)x) + M_{NB}(\alpha(t)y) = M(x) + M(y)$ and $M(\lambda x) = M_{NB}(\alpha(t)(\lambda x)) = M_{NB}(\lambda\alpha(t)x) = \lambda M_{NB}(\alpha(t)x) = \lambda M(x)$, where $\lambda \in C$. Hence $M$ is a linear operator on $L$.

By Theorem 13 in [1], for $F_x(t) = \alpha(t)x$ and any zero neighborhood $V \in Z(L)$ there exist real numbers $r_1, r_2, \ldots, r_m$ and a finite subset $\{a_1, a_2, \ldots, a_m\}$ of $G$ such that $r_1 + r_2 + \cdots + r_m = 1$, where $r_i \geq 0$ for all $i \in \{1, 2, \ldots, m\}$, and

$$r_1F_x(pa_1t) + r_2F_x(pa_2t) + \cdots + r_mF_x(pa_mt) - M(x) \in V \tag{1}$$

for all $p, t \in G$. For the identity element $p = t = e$ of $G$, we have $r_1\alpha(a_1)x + r_2\alpha(a_2)x + \cdots + r_m\alpha(a_m)x - M(x) \in V$. This implies that $M(x) \in Co^-\{\alpha(s)x, s \in G\}$. Prove that $M(\alpha(s)x) = \alpha(s)M(x) = M(x)$ for all $s \in G$. By property 5) of Theorem 16 in [1], we have $M(\alpha(s)x) = M_{NB}(\alpha(t)\alpha(s)x) = M_{NB}(\alpha(ts)x) = M_{NB}(\alpha(t)x) = M(x)$ for all $s \in G$.

Now we prove that $\alpha(s)M(x) = M(x)$ for all $s \in G$. Let $U \in Z(L)$ be any zero neighborhood. Then there exists a zero neighborhood $W \in Z(L)$ such that $W + W \subseteq U$. Since the linear representation $\alpha$ is equicontinuous, for $W \in Z(L)$, there exists a zero neighborhood $V \in Z(L)$ such that $\alpha(s)V \subseteq W$ for all $s \in G$.

By Equation (12), we obtain $\alpha(s)(r_1F_x(pa_1t)+r_2F_x(pa_2t)+\cdots+r_mF_x(pa_mt))-\alpha(s)M(x)\in\alpha(s)V\subseteq W$ for all $s,p,t\in G$. Using $\alpha(s)F_x(t)=\alpha(s)\alpha(t)x=\alpha(st)x=F_x(st)$, we have

$$(r_1F_x(spa_1t)+r_2F_x(spa_2t)+\cdots+r_mF_x(spa_mt)-\alpha(s)M(x)\in\alpha(s)V\subseteq W \tag{2}$$

for all $s,p,t\in G$. Put $p=e$ in Equation (2), where $e$ is the identity element of $G$. Then we have

$$(r_1F_x(sa_1t)+r_2F_x(sa_2t)+\cdots+r_mF_x(sa_mt)-\alpha(s)M(x)\in\alpha(s)V\subseteq W \tag{3}$$

for all $s,t\in G$. Using Equation (1) and Equation (3), we obtain

$$M(x)-\alpha(s)M(x)=(M(x)-(r_1F_x(sa_1t)+r_2F_x(sa_2t)+\cdots+r_mF_x(sa_mt)))+$$
$$((r_1F_x(sa_1t)+r_2F_x(sa_2t)+\cdots+r_mF_x(sa_mt))-\alpha(s)M(x))\in V+W\subseteq W+W\subseteq U$$

for all $s\in G$. Since $U\in Z(L)$ be an arbitrary zero neighborhood, we obtain that $M(x)=\alpha(s)M(x)$ for all $s\in G$.

Prove that $M$ is a continuous linear operator. Let $U\in Z(L)$ be an arbitrary zero neighborhood. Then there exists $V\in Z(L)$ such that $V+V\subseteq U$. By the property 8) of Theorem 16 in [1], for all $f(t),h(t)\in B(G,L)$ such that $f(t)-h(t)\in V,\forall t\in G$, we have

$$M_{NB}(f(t))-M_{NB}(h(t))\in U. \tag{4}$$

Since $\alpha$ is equicontinuous, for the zero neighborhood $V$, there exists a zero neighborhood $W$ such that $\alpha(t)W\subseteq V$. Let $x,y\in L$ such that $x-y\in W$. Then $F_x(t)-F_y(t)=\alpha(t)(x-y)\in\alpha(t)W\subseteq V$ for all $t\in G$. By Equation (4), we obtain that $M(x)-M(y)=M_{NB}(F_x(t))-M_{NB}(F_y(t))\in U$. Thus, for an arbitrary zero neighborhood $U\in Z(L)$, there exists a zero neighborhood $W\in Z(L)$ such that $x-y\in W$ implies $M(x)-M(y)\in U$. This means that the linear operator $M$ is continuous.

Hence the linear representation $\alpha$ has a continuous invariant averaging $M$. By Proposition 2, the linear representation $\alpha$ has the unique invariant averaging and it is equal to $M$. The theorem is completed. □

**Corollary 1** *Let $\alpha$ be an almost periodic linear representation of a group $G$ in a Banach space $L$. Then there exists the unique invariant averaging $M$ of $\alpha$, it is continuous and $M(x)=M_{NB}(\alpha(t)x)$ for all $x\in L$.*

**Proof** Every almost periodic linear representation of a group $G$ in a Banach space $L$ is equicontinuous (see [18, p.142]). Hence, this corollary follows from Theorem 5. □

## 5. The spectrum of an element in an equicontinuous almost periodic representation

Let $C$ be the field of complex numbers and $T$ is its multiplicative subgroup $\{\lambda\in C:\mid\lambda\mid=1\}$. In this section, $G$ is an abelian group with the discrete topology and $\hat{G}$ is the group of all characters of $G$ (that is the set of all homomorphisms of $G$ to $T$).

**Proposition 5** *Let $\alpha$ be an equicontinuous almost periodic representation of $G$ in a quasi-complete locally convex space $L$ and $\chi\in\hat{G}$. Then $\beta=\chi^{-1}\alpha:G\to G(L)$, where $\beta(t)x=\chi^{-1}(t)\alpha(t)x,(x\in L)$, is an equicontinuous almost periodic representation.*

**Proof** Since $\alpha$ is almost periodic, the set $A = \overline{\{\alpha(t)x, t \in G\}}$ is compact for every $x \in L$. Using continuity of the mapping $F : C \times L \to L$, where $F(\lambda, a) = \lambda a, \lambda \in C, a \in L$, we find that the set $B = \{\lambda a, \lambda \in C, |\lambda| = 1, a \in A\}$ is compact. On account of $\{\chi^{-1}(t)\alpha(t)x, t \in T\} \subset B$ the set $\overline{\{\chi^{-1}(t)\alpha(t)x\}}$ is compact. Hence $\chi^{-1}\alpha$ is an almost periodic representation.

Prove that the linear representation $\beta$ is equicontinuous. Since $\alpha$ is an equicontinuous linear representation of $G$ in a quasi-complete locally convex space $L$, for every balanced zero neighborhood $U$ of $L$, there exists a zero neighborhood $V$ such that $\alpha(t)V \subseteq U$ for all $t \in G$. By $\left|\chi^{-1}(t)\right| = 1$ for all $t \in G$, we obtain $\beta(t)V = \chi^{-1}(t)\alpha(t)V \subseteq \chi^{-1}(t)U \subseteq U$ for all $t \in G$. □

Let $L$ be a complete locally convex space with the countability property and $\alpha$ is an equicontinuous almost periodic linear representation of a group $G$ in $L$ and $\chi \in \hat{G}$. By Proposition 5 and Theorem 5, the unique invariant averaging of the linear representation $\chi^{-1}\alpha$ exists and it is continuous. Denote it by $M_\chi$.

**Proposition 6** *The linear operator $M_\chi$ has the following properties:*

$$\alpha(s)M_\chi(x) = M_\chi(\alpha(s)x) = \chi(s)M_\chi(x)$$

*for all $x \in L$ and $s \in H$.*

**Proof** Using the definition of an invariant averaging, we find $\chi^{-1}(t)\alpha(t)M_\chi(x) = M_\chi(\chi^{-1}(t)\alpha(t)x) = M_\chi(x)$. Using linearity of $M_\chi$, we obtain $\alpha(t)M_\chi(x) = M_\chi(\alpha(t)x) = \chi(t)M_\chi(x)$. □

**Proposition 7** *Let $L$ be a complete locally convex space with the countability property and $\alpha$ is an equicontinuous almost periodic linear representation of a group $G$ in $L$. Assume that an element $y \in L, y \neq 0$, satisfies the condition $\alpha(t)y = \gamma(t)y$ for some $\gamma \in \hat{G}$. Then $M_\gamma(y) = y$ and $M_\chi(y) = 0$ for all $\chi \in \hat{G}$ such that $\chi \neq \gamma$.*

**Proof** For the element $y$ we have $\gamma^{-1}(t)\alpha(t)y = y$. Hence $y$ is an $G$-invariant point for the representation $\gamma^{-1}\alpha$. Then $M_\gamma(y) = y$. Let $\chi \in \hat{G}$ and $\chi \neq \gamma$. Then $\chi^{-1}(t)\alpha(t)y = \chi^{-1}(t)\gamma(t)y$. Consider the linear representation $\beta = \chi^{-1}\alpha$ in $L$. We have $\beta(t)y = \chi^{-1}(t)\alpha(t)y = \chi^{-1}\gamma(t)y$. The closed convex hull of $Gy$ is

$$V(y) = \overline{\left\{(\lambda_1\chi^{-1}(t_1)\gamma(t_1) + \cdots + \lambda_n\chi^{-1}(t_n)\gamma(t_n))y, \lambda_k \geq 0, \sum_{k=1}^{n} \lambda_k = 1, \lambda_i \in R, t_i \in G\right\}}.$$

Hence $V(y)$ is a closed subset of $\{\lambda y, \lambda \in C\}$, containing $y$. Since $\beta$ is an almost periodic representation, $V(y)$ contains an $G$-invariant point for $\beta$. Let $z = \lambda_0 y$ be an $G$-invariant point for some $\lambda_0 \in C$. If $\lambda_0 \neq 0$ then $\chi^{-1}(t)\alpha(t)z = \lambda_0\chi^{-1}\gamma(t)y = \lambda_0 y$ for all $t \in G$. Hence $\chi^{-1}(t)\gamma(t)y = y$. Then $\chi^{-1}(t)\gamma(t) = 1$ for all $t \in G$. But this is a contradiction. Therefore $\lambda_0 = 0$. Then $z = 0$ and $M_\chi(y) = 0$. □

**Corollary 2** *The linear operator $M_\chi$ has the following properties:*

(i) *$M_\chi^2 = M_\chi$ for all $\chi \in \hat{G}$;*

(ii) *$M_\chi M_\gamma = 0$ for all $\chi, \gamma \in \hat{G}$ such that $\chi \neq \gamma$.*

**Proof** A proof follows from Proposition 7. □

Let $M_{NB}$ is the invariant mean defined in the paper [1]. It is defined an element $M_{NB}(\chi^{-1}(t)f(t))$ of $L$ for every $\chi \in \hat{G}$ and every $f \in B(G, L)$.

**Corollary 3** *Let $L$ be a complete locally convex space with the countability property and $\alpha$ is an equicontinuous almost periodic linear representation of a group $G$ in $L$. Then:*

(i) $M_\chi(x) = M_{NB}(\chi^{-1}(t)\alpha(t)x)$ *for every* $x \in L$ *and every* $\chi \in \hat{G}$*;*

(ii) *the linear operator* $M_\chi$ *is continuous on* $L$ *for every* $\chi \in \hat{G}$*.*

**Proof** A proof follows from Theorem 5. □

Let $L$ be a complete locally convex space with the countability property and $\alpha$ is an equicontinuous almost periodic linear representation of a group $G$ in $L$. Then for every $x \in L$ and every $\chi \in \hat{G}$ is defined the element $M_\chi(x) \in L$.

**Definition 8** *The set* $\left\{\chi \in \hat{G} : M_\chi(x) \neq 0\right\}$ *will be called the spectrum of the element* $x \in L$ *and denoted by* $Spec(x)$.

**Remark 5** *By theorem 25 in [1], the spectrum* $\left\{\chi \in \hat{H} : M_{NB}(\chi^{-1}(t)F(t)) \neq 0\right\}$ *is countable for every* $F \in L_b^G$.

**Remark 6** *Countability of spectrums of almost periodic functions and weak almost periodic functions with values in a Banach space is well known ([12], [17]). But there exists a weak almost periodic function of Besicovitch with uncountable spectrum [12]. The remarkable criterion for the countability of the spectrum of a scalar almost periodic function with values in a Banach space was obtained in the paper [8].*

Let $L$ be a complete locally convex space with the countability property and $\alpha$ is an equicontinuous almost periodic linear representation of a group $H$ in $L$. We put $Q_\chi = \{y \in Q : \alpha(t)y = \chi(t)y, \forall t \in H\}$ for a closed $\alpha(H)$-invariant subspace $Q$ of $L$ and $\chi \in \hat{H}$. $Q_\chi$ is a closed $\alpha(H)$-invariant subspace of $Q$. For $x \in L$ denote by $L(x)$ the smallest closed $\alpha(H)$-invariant subspace of $L$, containing $x$. For $L(x)$ and $\chi \in \hat{H}$ we consider the subspace $L_\chi(x)$. Denote by $\sum_{\chi \in \hat{H}} L_\chi(x)$ the algebraic sum of all subspaces $L_\chi(x), \chi \in \hat{H}$.

**Theorem 6** *Let $L$ be a complete locally convex space with the countability property and $\alpha$ is an equicontinuous almost periodic linear representation of a group $H$ in $L$. Then* $\overline{\sum_{\chi \in \hat{H}} L_\chi(x)} = L(x)$ *for every* $x \in L$*.*

**Proof** It is obviously that $L_\chi(x) \subset L(x)$ for all $\chi \in \hat{H}$. Therefore $\overline{\sum_{\chi \in \hat{H}} L_\chi(x)} \subset L(x)$ for every $x \in L$. Prove the inverse inclusion. Using theorems 21, 23 of the paper [1] and corollary 3, we obtain that the function $F_x(t) = \alpha(t)x$ is a limit of functions $F_\nu(t)$ in the topology of $L_b^H$, where $F_\nu(t)$ has a form $F_\nu(t) = \sum_{k=1}^{m} r_k \chi_k(t) M_{\chi_k}(x), r_k \in R, 0 \leq r_k \leq 1$. Putting $t = 0$, we obtain that the element $x$ is a limit of elements $\sum_{k=1}^{m} r_k M_{\chi_k}(x)$. Hence $x \in \overline{\sum_{\chi \in \hat{H}} L_\chi(x)}$ and $L(x) = \overline{\sum_{\chi \in \hat{H}} L_\chi(x)}$. □

Let $E$ be a closed $\alpha(H)$-invariant subspace of $L$. Put $SpecE = \cup_{x \in E} \mathrm{Spec}(x)$.

**Proposition 8** *Let $L$ be a complete locally convex space with the countability property and $\alpha$ is an equicontinuous almost periodic linear representation of a group $H$ in $L$ and $x, y \in L$. Assume that $M_\chi(x) = M_\chi(y)$ for all $\chi \in \hat{H}$. Then $x = y$.*

**Proof** According to Theorem 6 the element $x - y$ is a limit of elements of a form $\sum_{k=1}^{m} r_k M_{\chi_k}(x - y) = 0$. Hence $x = y$. □

**Proposition 9** *Let $L$ be a complete locally convex space with the countability property and $\alpha$ is an equicontinuous almost periodic linear representation of a group $H$ in $L$, $y \in L$ and $\lim_{\nu \in \Omega} y_\nu = y$ for some direction $y_\nu, \nu \in \Omega$, in $L$. Then $Spec(y) \subset \cup_{\nu \in \Omega} Spec(y_\nu)$.*

**Proof** Put $P = \cup_{\nu \in \Omega} \mathrm{Spec}(y_\nu)$. Let $B$ be a fundamental system of neighborhoods of the zero in $L$ and $W \in B$. Then there exists $\nu_0 \in \Omega$ such that $y - y_\nu \in W$ for all $\nu \geq \nu_0$. Let $\chi \notin P$. Then $M_\chi(y_\nu) = 0$ for all $\nu \in \Omega$. Since $M_\chi$ is continuous, for every $U \in B$ there exists $W \in B$ such that $M_\chi(W) \subset U$. For $\chi \notin P$ and for all $\nu \geq \nu_0$ we have $M_\chi(y) = M_\chi(y - y_\nu) + M_\chi(y_\nu) = M_\chi(y - y_\nu) \in U$. Hence $M_\chi(y) = 0$. This means that $\chi \notin \mathrm{Spec}(y)$. Thus $\mathrm{Spec}(y) \subset \cup_\Omega \mathrm{Spec}(y_\nu)$. □

**Proposition 10** *Let $L$ be a complete locally convex space with the countability property and $\alpha$ is an equicontinuous almost periodic linear representation of a group $H$ in $L$. Then $Spec(x) = SpecL(x)$.*

**Proof** The inclusion $\mathrm{Spec}(x) \subset SpecL(x)$ is obvious. Prove the converse inclusion. According to Proposition 5, we have $M_\chi(\alpha(s)x) = \chi(s) M_\chi(x)$. Using this equality, we obtain that $M_\chi(\alpha(s)x) = 0$ if and only if $M_\chi(x) = 0$. Hence $\mathrm{Spec}(x) = \mathrm{Spec}(\alpha(s)x)$ for all $s \in H$. Let $y = \sum_{k=1}^{n} \lambda_k \alpha(s_k)x$ and $\chi \notin \mathrm{Spec}(x)$. Then $M_\chi(\alpha(s_k)x) = 0$ for all $k = 1, \ldots, n$. Hence $M_\chi(y) = \sum_{k=1}^{n} \lambda_k M_\chi(\alpha(s_k)x) = 0$. Thus $\mathrm{Spec}(y) \subset \mathrm{Spec}(x)$. Let $z \in L(x)$. Then according to Theorem 6 there exists a direction $\{y_\nu, \nu \in \Omega\}$, where $y_\nu \in \sum_{\chi \in \hat{H}} L_\chi(x)$, such that $\lim_\Omega y_\nu = z$. Using Proposition 9, we find $\mathrm{Spec}(z) \subset \cup_\Omega \mathrm{Spec}(y_\nu) \subset \mathrm{Spec}(x)$. Thus $SpecL(x) = \mathrm{Spec}(x)$. □

**Theorem 7** *Let $L$ be a complete locally convex space with the countability property and $\alpha$ is an equicontinuous almost periodic linear representation of a group $H$ in $L$ and $x \in L$. Assume that $dim L_\chi \leq 1$ for all $\chi \in \hat{H}$. Then $L(x) = L$ if and only if $Spec(x) = SpecL$.*

**Proof** Let $L(x) = L$. According to Proposition 10, we have $\mathrm{Spec}(x) = SpecL(x) = SpecL$. Prove the converse statement. Let $\mathrm{Spec}(x) = SpecL$. Assume that $y \in L$. According to Theorem 6 there exists a direction $\{y_{\nu, \nu \in \Omega}\}$ such that $\lim_\Omega y_\nu = y$, where $y_\nu$ has a form $y_\nu = \sum_{k=1}^{n} r_k^{(\nu)} M_{\chi_k^{(\nu)}}(y)$ for some $\chi_k^{(\nu)} \in \mathrm{Spec}(y), r_k^{(\nu)} \in R$. Since $dim L_\chi \leq 1$ and $\mathrm{Spec}(x) = SpecL$, we obtain that $M_\chi(y) \in L(x)$ for all $\chi \in SpecL$. Then $y_\nu \in L(x)$ and $y \in L(x)$. Thus $L(x) = L$. □

**Remark 7** *Theorem 7 is an analog of the tauberian theorem of Wiener ([24, ch.II]).*

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