Remarks on the paper ”On some new inequalities for convex functions” by M. Tunç

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Received: 03.07.2012 • Accepted: 03.10.2012 • Published Online: 26.08.2013 • Printed: 23.09.2013

Abstract: In this note, we slightly generalize Theorem 2 in the paper by M. Tunç and point out that the assumption of Theorem 3 is not sufficient.

A misuse of the term 'mean' is also discussed.

Key words: Convex function, mean

In the paper [3] the author proves the following theorem:

**Theorem 1 (Theorem 2, [3])** If \( f, g : [a, b] \rightarrow \mathbb{R} \) are convex, then

\[
\frac{1}{(b-a)^2} \int_a^b (b-x)[f(a)g(x) + f(x)g(a)]dx + \frac{1}{(b-a)^2} \int_a^b (x-a)[f(b)g(x) + f(x)g(b)]dx \leq \frac{M(a,b)}{3} + \frac{N(a,b)}{6} + \frac{1}{b-a} \int_a^b f(x)g(x)dx,
\]

where \( M(a,b) = f(a)g(a) + f(b)g(b), N(a,b) = f(a)g(b) + f(b)g(a) \).

In fact, this theorem can be restated as follows:

**Theorem 2** If \( f, g : [a, b] \rightarrow \mathbb{R} \) are of the same convexity (i.e. both convex or both concave), then (1) holds. If \( f \) and \( g \) are of opposite convexity, then (1) is reversed.

**Proof** Since for \( a < x < b \) we have

\[
x = \frac{b-x}{b-a} a + \frac{x-a}{b-a} b,
\]

the inequality

\[
\left( f(x) - \frac{b-x}{b-a} f(a) - \frac{x-a}{b-a} f(b) \right) \left( g(x) - \frac{b-x}{b-a} g(a) - \frac{x-a}{b-a} g(b) \right) \geq 0
\]

(2)
holds if \( f \) and \( g \) are of the same convexity, else (2) is reversed.

Integrating the above inequality over the interval \([a, b]\), we obtain the desired result.

Theorem 3 in [3] requires correction.

**Theorem 3**

Let \( f, g : [a, b] \to \mathbb{M} \) be convex, nonnegative functions. Then

\[
\frac{1}{b-a} \int_a^b \left( f\left(\frac{a+b}{2}\right) + g\left(\frac{a+b}{2}\right) \right) dx \leq \frac{1}{2(b-a)} \int_a^b f(x)g(x)dx + \frac{M}{12} + \frac{N}{6} + f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right),
\]

(3)

\( M \) and \( N \) being as in Theorem 1).

The original version does not contain the nonnegativity assumption, but then it is easy to produce a counterexample: let \( g \) be convex and \( f(x) \equiv -1 \). Then the inequality (3) becomes

\[
\frac{1}{b-a} \int_a^b g(x)dx \geq \frac{g(a) + g(b)}{2}
\]

- obviously opposite to the right-hand side of the Hermite-Hadamard inequality.

As Theorem 3 is not valid in the general case, we cannot trust Proposition 5 in [3] in the case \( a, b < 0 \) (especially because \( x^n \) is not convex in the interval \((-\infty, 0)\) for odd \( n \)).

We feel obliged to comment on the use of the term 'mean' in section 3: in the mathematical literature the word 'mean' denotes a function taking values between the extremities of its argument(s). The attempt to extend the definition of the geometric, arithmetic, logarithmic, and generalised logarithmic means is only partially successful.

\[
A(a, b) = \frac{a+b}{2}, \quad G(a, b) = \sqrt{ab}, \quad L(a, b) = \frac{b-a}{\ln|b| - \ln|a|},
\]

\[
L_n(a, b) = \left( \frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right)^{\frac{1}{n}}, \quad K(a, b) = \sqrt{\frac{a^2 + b^2}{2}}
\]

(4)

define means for positive \( a, b \). Clearly, the expressions above make sense for some other arguments, but usually their values do not lie between the arguments: \( G(-1, -4) = 2, \quad L(e^2, -e) = e^2 + e, \quad \lim_{a \to -\infty} L_2(a, 1) = \infty \) etc., thus calling them 'means' should be regarded as a mistake.

Tracing back the cited literature for the source of this misuse, we see that the process started in the paper by Dragomir and Agarwal ([1]), the mistake was reproduced by Kirmaci ([2]) and, consequently, by Tunç.

It is worth noting that the extended logarithmic means \( L_n \) and power means \( M_n(a, b) = \left(a^n, b^n\right)^{1/n} \) can be extended to the real line in the case of positive real exponents. To this end, let \( f_n(x) = \text{sgn}(x)|x|^n \). Then \( f_n \) is a strictly increasing, odd function and we can define

\[
L_n(a, b) = f_n^{-1}\left( \frac{1}{b-a} \int_a^b f_n(t)dt \right) \quad \text{and} \quad M_n(a, b) = f_n^{-1}\left( \frac{f_n(a) + f_n(b)}{2} \right).
\]

Both definitions match the original ones for positive arguments and define means (in the case of odd natural \( n \), \( L_n \) matches the original definition). Unfortunately the power functions (with the exception of \( n = 1 \)) cannot be extended to a bijection preserving the convexity.
Clearly this method cannot be applied to negative exponents.

References

