Quasinormability and diametral dimension

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Abstract: Two versions of diametral dimension are shown to coincide for quasinormable Fréchet spaces. The diametral dimension is determined by a single bounded subset in certain cases.

Key words and phrases: Diametral dimension, Fréchet spaces, Köthe spaces

1. Introduction

The set \( \Delta(E) \) of sequences \( (\xi_n) \) such that for each neighborhood \( U \) of zero of a locally convex space \( E \) there is another such neighborhood with \( \lim \xi_n d_n(V, U) = 0 \), where \( d_n(V, U) \) is the \( n \)-th diameter of \( V \) with respect to \( U \), is called the diametral dimension of \( E \). ([3], [6], [7], [8]). Another version is the set \( \Delta_b(E) \) of all sequences \( (\xi_n) \) such that for each neighborhood \( U \) and each bounded subset \( B \) we have \( \lim \xi_n d_n(B, U) = 0 \). \( \Delta_b(E) \) is less frequently used than \( \Delta(E) \). We always have \( c_0 \subset \Delta(E) \subset \Delta_b(E) \). In [6] Mitiagin claimed that \( \Delta(E) = \Delta_b(E) \) holds for every Fréchet space (\( F \)-space) \( E \), referring for the proof to a forthcoming joint paper. However, there is an example of a Köthe space \( \lambda(A) \), which is a Montel space but fails to be a Schwartz space. In this case we have

\[ \Delta(\lambda(A)) = c_0 \subset \ell_\infty \subset \Delta_b(\lambda(A)). \]

On the other hand, if \( E \) is a locally convex space with a bounded subset that is not precompact, we have \( \Delta(E) = \Delta_b(E) = c_0 \).

We recall that a Fréchet-Montel space (\( FM \)-space) is a Fréchet-Schwartz space (\( FS \)-space) if and only if \( E \) is quasinormable [3]. There is an extensive literature concerning quasinormability (cf. [1]). We want to single out a remarkable result of Meise and Vogt [5], which states that an \( F \)-space is quasinormable if and only if it is isomorphic to a quotient space of the complete tensor product \( \ell^1(I) \tilde{\otimes}_\pi \lambda(A) \), where \( I \) is a suitable index set and \( \lambda(A) \) a suitable Köthe-Schwartz space.

We recall the definition of the \( n \)-th diameter

\[ d_n(A, B) = \inf \inf \{ \alpha > 0 : A \subset \alpha B + L \} \]

where \( A \) and \( B \) are subsets of a locally convex space \( E \) with \( A \subset \rho B \) for some \( \rho > 0 \). The second infimum is taken over all subspaces \( L \) of \( E \) with dimension not exceeding \( n \in N \).

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**Proposition 1** If \( E \) is a quasinormable metrisable space, then \( \Delta(E) = \Delta_b(E) \).

**Proof** For \( (\xi_n) \in \Delta_b(E) \) we find \( \delta_n \geq \delta_{n+1} > 0 \) with \( \lim \xi_n \delta_n = 0 \). By quasinormability we can choose a base of absolutely convex, closed neighborhoods \( U_1 \supset U_2 \supset \cdots \) such that for each \( k \) and \( \delta_n > 0 \) there is a bounded subset \( B_{k,n} \) with

\[
U_{k+1} \subset B_{k,n} + \delta_n U_k.
\]

In particular,

\[
U_{n+1} \subset B_{k,n} + \delta_n U_n
\]

holds for each \( n \geq k \). To see how much of \( B_{k,n} \) we need in the above inclusion we observe that for each \( x \in U_{n+1} \) there is a \( b \in B_{k,n} \) with \( \|x - b\| \leq \delta_n \). This means if we replace \( B_{k,n} \) with

\[
B_{k,n} \cap (1 + \delta_0)U_n
\]

the above inclusion still holds. Therefore, we will assume without loss of generality that

\[
B_{k,n} \subset (1 + \delta_0)U_n
\]

in the above inclusion.

This implies that

\[
B_k = \bigcup_{n=k}^{\infty} B_{k,n}
\]

is a bounded set and therefore for all \( n \geq k \) we obtain

\[
U_{k+1} \subset B_k + \delta_n U_k.
\]

Using the definition of the \( n \)-th diameter, from the above inclusion for \( n \geq k \) we get

\[
d_n(U_{k+1}, U_k) \leq d_n(B_k, U_k) + \delta_n.
\]

Hence

\[
\lim \xi_n d_n(U_{k+1}, U_k) = 0.
\]

Our result implies that for an \( F \)-space we have \( \Delta(E) \neq \Delta_b(E) \) if and only if \( E \) is a Montel but not a Schwartz space.

In certain cases it is sufficient to consider a single bounded subset of \( E \) to determine \( \Delta(E) \). We will call an absolutely convex bounded subset \( B \) of an \( F \)-space \( E \) a prominent set if \( \lim \xi_n d_n(B, U_k) = 0 \) for every \( k \) implies \( (\xi_n) \in \Delta(E) \). If \( E \) has a prominent set \( B \) then since

\[
\Delta(E) = \{ (\xi_n) : \lim \xi_n d_n(B, U_k) = 0, \quad k = 1, 2, \ldots \}
\]

the diametral dimension as a space of sequences is an \( F \)-space itself. For an exponent sequence \( 0 < \alpha_n \leq \alpha_{n+1} \leq \cdots \) with \( \lim \alpha_n = \infty \) the unit ball \( B_1 \) of \( \ell_1 \) is a prominent set of the finite-type power series space \( A_1(\alpha) \) (cf. for example [6], [8]). We will generalize this result in what follows.

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Following [10], we call a Banach space \((\ell, \|\|\|)\) of scalar sequences an admissible space if \(\|e_n\| = 1\) and for \(a \in \ell_\infty, \ x \in \ell\) we have \(ax = (a_n x_n) \in \ell\) and \(\|ax\| \leq \|a\|_\infty \|x\|\). As usual \(e_n\) is that sequence with 1 as the \(n\)-th term and zero elsewhere. The classical sequence spaces \(\ell_p, 1 \leq p \leq \infty\) and \(c_0\) are the most well-known examples of admissible spaces.

Let \(A\) be a Köthe set and \(\lambda^t(A)\) be the space of all sequences \(x = (x_n)\) such that \((x_n a_n^k) \in \ell\) for each \(k\). Equipped with the seminorms, \(\|x\|_k = \|a^k x\|\), \(\lambda^t(A)\) is an \(F\)-space. \(\lambda^1(A)\) is of course the usual Köthe space \(\lambda(A)\). In fact the spaces \(\lambda^{r_1}(A), \lambda^{\infty}(A), 1 \leq p \leq \infty\), are also quite well known.

A Köthe space \(\lambda(A)\) is called a smooth sequence space of finite type [8] (or a \(G_1\)-space) if \(0 < a_{n+1}^k \leq a_n^k\) and for each \(k\) there is a \(j\) with \((a_n^k/(a_n^j)^2) \in \ell_\infty\).

**Proposition 2** Let \(\lambda(A)\) be a \(G_1\)-space and \(\ell\) an admissible space with closed unit ball \(B_\ell\). Then \(B_\ell\) is a prominent subset of \(\lambda^t(A)\).

**Proof** Let

\[
U_k = \{(x_n) \in \lambda^t(A) : \|(x_n a_n^k)\| \leq 1\}.
\]

By [10], Prop. 1, we have the basic inequality

\[
\inf \left\{ \frac{a_n^i}{a_n^j} : i \leq n \right\} \leq d_n(U_j, U_k) \leq \sup \left\{ \frac{a_n^i}{a_n^j} : i \geq n \right\}
\]

We note that both sides of this inequality are independent of \(\ell\). With the same argument we can easily show

\[
d_n(B_\ell, U_k) = a_n^k.
\]

Now for \(k\) given we choose \(j\) such that for some \(\rho > 0\) we have \(a_n^k \leq \rho (a_n^j)^2\) for all \(n \in N\). From the above inequality we obtain

\[
d_n(U_j, U_k) \leq \rho \sup \{a_n^j : i \geq n\} = \rho a_n^j.
\]

Therefore,

\[
d_n(U_j, U_k) \leq \rho d_n(B_\ell, U_j).
\]

This shows

\[
\Delta(\lambda^t(A)) = \Delta_\rho(\lambda^t(A)) = \{(\xi_n) : \lim \xi_n d_n(B_\ell, U_j) = 0 \text{ for all } j \in N\}.
\]

Of course, a finite type power series space \(\Lambda_1(\alpha)\) is a \(G_1\)-space. In this special case the closed unit ball \(B_\rho\) of \(\ell_\rho\) or \(B_0\) of \(c_0\) is a prominent subset of \(\Lambda_1^{r_1}(\alpha)\), or of \(\Lambda_1^{\infty}(\alpha)\).

We will now give a necessary and sufficient condition for a bounded subset to be prominent.

**Proposition 3** Let \(B\) be a bounded subset of an \(F\)-space \(E\). \(B\) is a prominent set if and only if for each \(k\) there is a \(p\) and \(\rho > 0\) such that

\[
d_n(U_p, U_k) \leq \rho d_n(B, U_p)
\]

for all \(n \in N\).
Proof Sufficiency follows immediately from definitions of $\Delta(E)$ and $\Delta_b(E)$. Let us now assume $B$ is a prominent subset of $E$. In this case, the diametral dimension is

$$\Delta(E) = \lambda^w(B_E)$$

where

$$B_E = \{(d_n(B, U_k)) : k = 1, 2, \ldots\}$$

and so $\Delta(E)$ is itself an $F$-space. On the other hand, from the definition of $\Delta(E)$, for a given $k$ we have

$$\Delta(E) \subset \cup_{p \geq k} \{(\xi_n) : \lim_{n} d_n(U_p, U_k) = 0\}$$

The right-hand side of the above inclusion is an $LB$-space and the canonical inclusion map is sequentially closed. Therefore, by the Grothendieck factorization theorem we can find $m > k$ such that

$$\Delta(E) \subset \{(\xi_n) : \lim_{n} d_n(U_m, U_k) = 0\}.$$ 

This implies the existence of $j$ and $\rho > 0$ with

$$d_n(U_m, U_k) \leq \rho d_n(B, U_j), \quad n \in N.$$ 

Finally we choose $p = \max\{m, j\}$. 

Let $\lambda(A)$ now be a smooth sequence space of infinite type [8]. This means $0 < a_n^k \leq a_{n+1}^k \leq \ldots$ and for each $k$ there is $a_j$ with $((a_j^k)^2/a_i^k) \in \ell_\infty$. To avoid the trivial case $\lambda(A) = \ell_1$ we will also assume $\lim_{n \to \infty} a_n^k = \infty$ for every $k$. This implies that $\lambda(A)$ is a Schwartz space, and so

$$\Delta(\lambda(A)) = \Delta_b(\lambda(A))$$

by Proposition 1. Let $B$ be a prominent subset of $\lambda(A)$. Since a bounded set, which contains a prominent subset, is itself prominent, we can assume without loss of generality [2] that

$$B = \{(\xi_n) : \Sigma|\xi_n|a_n \leq 1\}$$

where $(a_n)$ is some sequence such that for each $k$ there is a $\rho_k > 0$ with $a_n^k \leq \rho_k a_n$ for all $n \in N$. By Prop. 3. for each $k$ there is a $\rho > 0$ and $p \geq k$ with

$$d_n(U_p, U_k) \leq \rho d_n(B, U_p).$$

We choose $m$ so that $((a_m^p)^2/a_i^m) \in \ell_\infty$. By the basic inequality

$$d_n(B, U_p) \leq \inf\{a_i^p/a_i : i \geq n\}$$

but

$$\frac{a_i^p}{a_i} \leq \frac{a_i^m}{(a_i^p)^2} \leq \frac{c \rho m}{(a_i^p)^2}$$

for some constant $c > 0$.

Applying the left-hand side of the basic inequality we have

$$\frac{a_i^k}{a_n^i} \leq d_n(U_p, U_k).$$

Hence $(a_n^k) \in \ell_\infty$, which is a contradiction. So in contrast to Prop. 2 we have the following result.
Proposition 4 A smooth sequence space of infinite type that is also a Schwartz space has no prominent subset.

In particular, an infinite type power series space $\Lambda_\infty(\alpha)$ has no prominent subset although $\Delta(\Lambda_\infty(\alpha)) = \Delta_b(\Lambda_\infty(\alpha))$.

References


