On the centroid of prime semirings

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Abstract: We define and study the extended centroid of a prime semiring. We show that the extended centroid is a semifield and give some properties of the centroid of a right multiplicatively cancellable prime semiring.

Key words: Semiring, prime ideal, prime semiring, quotient semiring

1. Introduction

Semirings abound in the mathematical world around us. Indeed, the first mathematical structure we encounter—the set of natural numbers—is a semiring. Historically, semirings first appear implicitly in [3] and later in [8], [6], [10] and [7], in connection with the study of ideals of a ring. They also appear in [4] and [5] in connection with the axiomatization of the natural numbers and nonnegative rational numbers. Over the years, semirings have been studied by various researchers either in their own right, in an attempt to broaden techniques coming from semigroup theory or ring theory, or in connection with applications. In [9] Martindale first constructed for any prime ring $R$ a “ring of quotients” $Q$. After, Öztürk and Jun introduced the extended centroid of a prime Γ-ring and obtained some results in Γ-ring $M$ with derivation which was related to $Q$, and the quotient Γ-ring of $M$ [11, 12]. In this paper, we define and study the extended centroid of a prime semiring.

2. Preliminaries

A semiring is a nonempty set $R$ on which operations of addition and multiplication have been defined such that the following conditions are satisfied:

1. $(R, +)$ is a commutative monoid with identity element $0_R$;
2. $(R, \cdot)$ is a monoid with identity element $1_R$;
3. Multiplication distributes over addition from either side;
4. $0_{R}r = 0_{R} = r0_{R}$ for all $r \in R$;
5. $1_{R} \neq 0_{R}$.

An element $a$ of semiring $R$ is right multiplicatively cancellable if and only if $ba = ca$ only when $b = c$. Left multiplicatively cancellable elements are similarly defined. An element of $R$ is multiplicatively cancellable if and only if it is both left and right multiplicatively cancellable.

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An element of semiring \( R \) is a unit if and only if there exists element \( r_1 \) of \( R \) satisfying \( rr_1 = 1_R = r_1r \).

The set of all units of \( R \) is denoted by \( U(R) \). \( U(R) \) is submonoid of \( (R,) \) which is in fact a group. If \( U(R) = R \setminus \{0_R\} \) then \( R \) is division semiring. A commutative division semiring is semifield.

A nonzero element \( a \) of semiring \( R \) is a left zero divisor if and only if there exists nonzero element \( b \) of \( R \) satisfying \( ab = 0_R \). It is a right zero divisor if and only if there exists a nonzero element \( b \) of \( R \) satisfying \( ba = 0_R \). It is a zero divisor if and only if it is both a left and right zero divisor. A semiring \( R \) having no zero divisors is entire.

Let \( a \) be an element of semiring \( R \). An element \( b \) of \( R \) is additive inverse of \( a \) if and only if \( a + b = 0_R \).

A left ideal \( U \) of a semiring \( R \) is a nonempty subset of \( R \) satisfying the following conditions:

1. If \( a, b \in U \) then \( a + b \in U \);
2. If \( a \in U \) and \( r \in R \) then \( ra \in U \);
3. \( U \neq R \) (\( 1_R \notin U \)).

Note that ideals are proper, namely \( R \) is not an ideal of itself.

As in the case of rings, an ideal \( U \) of a semiring \( R \) is prime if and only if whenever \( HK \subseteq U \), for ideals \( H \) and \( K \) of \( R \), we must have either \( H \subseteq U \) or \( K \subseteq U \). \( R \) is a prime semiring if and only if ideal \( \{0_R\} \) of semiring \( R \) is prime.

**Proposition 1** [2, Proposition 7.4] The following conditions on ideal \( U \) of a semiring \( R \) are equivalent:

1. \( U \) is prime;
2. \( \{ar | r \in R\} \subseteq U \) if and only if \( a \in U \) or \( b \in U \);
3. If \( a \) and \( b \) are elements of \( R \) satisfying \((a)(b) \subseteq U \) then either \( a \in U \) or \( b \in U \).

If \( R \) and \( S \) are semirings then function \( \gamma : R \to S \) is a morphism of semirings if and only if

1. \( \gamma (0_R) = 0_S \),
2. \( \gamma (1_R) = 1_S \),
3. \( \gamma (r_1 + r_2) = \gamma (r_1) + \gamma (r_2) \) and \( \gamma (r_1r_2) = \gamma (r_1) \gamma (r_2) \) for all \( r_1, r_2 \in R \).

A morphism of semirings which is both injective and surjective is called isomorphism. If there exists isomorphism between semirings \( R \) and \( S \) we write \( R \cong S \). If \( \gamma : R \to S \) is a morphism of semirings then \( \text{Im} (\gamma) = \{\gamma(r) | r \in R\} \) is a subsemiring of \( S \).

Let \( R \) be a semiring. A left \( R \) semimodule is a commutative monoid \( (M, +) \) with additive identity \( 0_M \) for which we have function \( R \times M \to M \), denoted by \( (r, m) \mapsto rm \) and called scalar multiplication, which satisfy the following conditions for all elements \( r_1 \) and \( r_2 \) of \( R \) and all elements \( m_1 \) and \( m_2 \) of \( M \):

1. \( (r_1r_2)m = r_1(r_2m) \),
2. \( r(m_1 + m_2) = rm_1 + rm_2 \),
3. \( 1_Rm = m \),
4. \( r0_M = 0_M = 0_Rm \).

If \( R \) is a semiring and \( M \) and \( N \) are left \( R \) semimodules then a function \( \alpha \) from \( M \) to \( N \) is an \( R \) homomorphism if and only if the following conditions are satisfied:

1. \( (m_1 + m_2) \alpha = m_1 \alpha + m_2 \alpha \) for all \( m_1, m_2 \in M \);
2. \( (rm) \alpha = r(m \alpha) \) for all \( m \in M \) and \( r \in R \).
3. Extended centroid
Let $R$ be a prime semiring. Let us denote set of all nonzero ideals of $R$ and $R$ by $N = N(R)$. That is,
\[ N = N(R) = \{ U \mid \emptyset \neq U \text{ is ideal of } R \} \cup \{ R \} \]
$U$ and $R$ are regarded as right $R$ semimodules.

Define relation $\equiv_N$ on $M = \{ f : U \to R \mid U \subseteq N, f \text{ is right } R \text{ homomorphism} \}$ as follows: Let $f, g \in M$.

\[ f \equiv_N g \iff \text{there exists } K \in N \text{ and } K \subseteq U \cap V \text{ such that } f = g \text{ on } K \]
where $U \in N$ and $V \in N$ are domains of $f$ and $g$ respectively. Hence $\equiv_N$ is reflexive. Let $f, g \in M$ where $U \in N$ and $V \in N$ are domains of $f$ and $g$ respectively. Suppose $f \equiv_N g$. Then there exists $K \in N$ and $K \subseteq U \cap V$ such that $f = g$ on $K$. Hence $\equiv_N$ is symmetric. Let $f, g, h \in M$ where $U \subseteq N$, $V \subseteq N$ and $H \subseteq N$ are domains of $f$, $g$ and $h$ respectively. Suppose $f \equiv_N g$ and $g \equiv_N h$. Then there exist $K_1, K_2 \in N$ and $K_1 \subseteq U \cap V$ and $K_2 \subseteq V \cap H$ such that $f = g$ on $K_1$ and $g = h$ on $K_2$. Thus, $\{ 0_R \} \neq K = K_1 \cap K_2 \subseteq (U \cap V) \cap (V \cap H) \subseteq U \cap H$ and $f = h$ on $K$. This implies then $f \equiv_N h$. Hence $\equiv_N$ is transitive. Consequently, $\equiv_N$ is an equivalence relation on $M$. This gives a chance for us to get partition of $M$. We then denote the equivalence class by $\hat{f} = [U, f]$, where $\hat{f} := \{ g : V \to R \mid f \equiv_N g \}$ and denote by $Q_r$ set of all equivalence classes. That is,
\[ Q_r = \{ \hat{f} \mid f : U \to R \text{ right } R \text{ homomorphism and } U \in N \} \]
Now we define an addition "$+$" on $Q_r$ as
\[ \hat{f} + \hat{g} = \hat{f + g} \]
for all $\hat{f}, \hat{g} \in Q_r$. Let $\hat{f}, \hat{g} \in Q_r$, where $U \subseteq N$ and $V \subseteq N$ are domains of $f$ and $g$, respectively. Therefore $f + g : U \cap V \to R$ is a right $R$ homomorphism. Assume that $f_1 \equiv_N f_2$ and $g_1 \equiv_N g_2$, where $U_1, U_2, V_1$ and $V_2$ are domains of $f_1, f_2, g_1$ and $g_2$, respectively. Then $\exists K_1 \subseteq U_1 \cap U_2$, such that $f_1 = f_2$ on $K_1$ and $\exists K_2 \subseteq V_1 \cap V_2$ such that $g_1 = g_2$ on $K_2$. Taking $K = K_1 \cap K_2$. Then $K \neq \{ 0_R \}$ and
\[ K = K_1 \cap K_2 \subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2) \]
\[ = (U_1 \cap V_1) \cap (U_2 \cap V_2) \]
For any $x \in K$, we have $(f_1 + g_1)(x) = f_1(x) + g_1(x) = f_2(x) + g_2(x) = (f_2 + g_2)(x)$, and so $f_1 + g_1 = f_2 + g_2$ on $K$. Therefore $f_1 + g_1 \equiv_N f_2 + g_2$ where $f_1 + g_1 : U_1 \cap V_1 \to R$ and $f_2 + g_2 : U_2 \cap V_2 \to R$ are right $R$ homomorphisms. That is, addition $"+"$ is well-defined. Now we prove that $Q_r$ is a commutative monoid.
Let $\hat{f}, \hat{g}, \hat{h} \in Q_r$, where $U \subseteq N, V \subseteq N$ and $H \subseteq N$ are domains of $f$, $g$ and $h$ respectively. Since $U \cap (V \cap H) = (U \cap V) \cap H$, we get for all $x \in U \cap (V \cap H)$,
\[ [f + (g + h)](x) = f(x) + (g + h)(x) = f(x) + [g(x) + h(x)] = [f(x) + g(x)] + h(x) = (f + g)(x) + h(x) = [(f + g) + h](x) \]
Hence \( f + (g + h) = (f + g) + h \) on \( U \cap (V \cap H) \). That is, \( \widehat{f + (g + h)} = (\widehat{f + g}) + \widehat{h} \).

Taking \( \widehat{\theta} \in Q_r \), where \( \theta : R \to R, x \mapsto 0_R \) for all \( x \in R \). Let \( \widehat{f} \in Q_r \), where \( U \in N \) is domain of \( f \). Since \( U \subseteq U \cap R \), we get for all \( x \in U \),

\[
(f + \theta) (x) = f (x) + \theta (x) = f (x) + 0_R = f (x)
\]

and

\[
(\theta + f) (x) = \theta (x) + f (x) = 0_R + f (x) = f (x).
\]

Thus, \( \widehat{f + \theta} = \widehat{\theta} + \widehat{f} = \widehat{f} \). Hence \( \widehat{\theta} \) is the additive identity in \( Q_r \).

Finally, for any elements \( \widehat{f}, \widehat{g} \in Q_r \) where \( U \in N \) and \( V \in N \) are domains of \( f \) and \( g \) respectively, we have for all \( x \in U \cap V = V \cap U \),

\[
(f + g) (x) = f (x) + g (x) = g (x) + f (x) = (g + f) (x).
\]

That is, \( \widehat{f + g} = \widehat{g} + \widehat{f} \). Therefore \((Q_r, +)\) commutative monoid.

Now we define a multiplication \( .\cdot \) on \( Q_r \) as

\[
\widehat{fg} = \widehat{fg},
\]

for all \( \widehat{f}, \widehat{g} \in Q_r \). Let \( \widehat{f}, \widehat{g} \in Q_r \) where \( U \in N \) and \( V \in N \) are domains of \( f \) and \( g \), respectively. Therefore \( fg : VU \to R \) is a right \( R \) homomorphism. Assume that \( f_1 \equiv_N f_2 \) and \( g_1 \equiv_N g_2 \) where \( U_1, U_2, V_1 \) and \( V_2 \) are domains of \( f_1, f_2, g_1 \) and \( g_2 \) respectively. Then \( \exists K_1 \subseteq U_1 \cap U_2 \) such that \( f_1 = f_2 \) on \( K_1 \) and \( \exists K_2 \subseteq V_1 \cap V_2 \) such that \( g_1 = g_2 \) on \( K_2 \). Also \( V_1 U_1 \cap V_2 U_2 \subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2) = (U_1 \cap V_1) \cap (U_2 \cap V_2) \) and there exists \( \{0_R\} \neq K \in N \) such that \( K \subseteq V_1 U_1 \cap V_2 U_2 \). For any \( x \in K \), \( x \in V_1 U_1 \cap V_2 U_2 \). So that \( x \in V_1 U_1 \) and \( x \in V_2 U_2 \). Then, \( x = \sum_{\text{finite}} a_i b_i, a_i \in V_1 \cap V_2 \) and \( b_i \in U_1 \cap U_2 \). Therefore

\[
(f_1 g_1) (x) = f_1 (g_1 (x)) = f_1 \left( g_1 \left( \sum_{\text{finite}} a_i b_i \right) \right) = f_1 \left( \sum_{\text{finite}} g_1 (a_i) b_i \right) = \sum_{\text{finite}} f_1 (g_1 (a_i) b_i) = f_2 \left( \sum_{\text{finite}} g_2 (a_i) b_i \right) = f_2 (g_2 (x)) = (f_2 g_2) (x),
\]

and so \( f_1 g_1 = f_2 g_2 \) on \( K \). Hence, \( f_1 g_1 \equiv_N f_2 g_2 \). That is, \( .\cdot \) is well-defined. Now we will prove that \((Q_r, .)\) is a monoid. Let \( \widehat{f}, \widehat{g}, \widehat{h} \in Q_r \) where \( U \in N \), \( V \in N \) and \( H \in N \) are domains of \( f \), \( g \) and \( h \), respectively. Since \( H (VU) = (HV) U \), we get for all \( x \in H (VU) \),

\[
[(fg) h] (x) = (fg) (h (x)) = f (g (h (x))) = f ((gh) (x)) = (f (gh)) (x),
\]

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Hence \((fg)h = f(gh)\) on \(H(VU)\). That is, \(\left(\hat{f}\hat{g}\right)\hat{h} = \hat{f}\left(\hat{g}\hat{h}\right)\).

Taking \(\hat{1} \in Q_r\) where \(1 : R \rightarrow R, x \mapsto x\) for all \(x \in R\). Let \(\hat{f} \in Q_r\), where \(U \in N\) is domain of \(f\).

Since \(RU \subseteq U\), we get for all \(x \in RU\), \((f1)(x) = f(1(x)) = f(x)\) and \((1f)(x) = 1(f(x)) = f(x)\). Thus, \(\hat{f}\hat{1} = \hat{1}\hat{f} = \hat{f}\). Hence \(\hat{1}\) is the multiplicative identity in \(Q_r\). Therefore \((Q_r, \cdot)\) is a monoid.

Let \(\hat{f}, \hat{g}, \hat{h} \in Q_r\) where \(U \in N\), \(V \in N\) and \(H \in N\) are domains of \(f\), \(g\) and \(h\) respectively. Since \((V \cap H)U \subseteq VU \cap HU\), we get for all \(x \in (V \cap H)U\),

\[
[f(g + h)](x) = f((g + h)(x)) = f(g(x) + h(x)) = f(g(x)) + f(h(x)) = (fg + fh)(x).
\]

Hence \(f(g + h) = fg + fh\) on \((V \cap H)U\). That is, \(\hat{f}\left(\hat{g} + \hat{h}\right) = \hat{f}\hat{g} + \hat{f}\hat{h}\). Similarly, \(\left(\hat{f} + \hat{g}\right)\hat{h} = \hat{f}\hat{h} + \hat{g}\hat{h}\).

Suppose that \(\hat{\theta} = \hat{1}\). Then, \(\theta \equiv_N 1\). That is, there exist \(K \in N\) and \(K \subseteq R \cap R\) such that \(\theta = 1\) on \(K\). Therefore, \(\theta(x) = 1(x)\) for all \(x \in K\). This is a contradiction with \(\{0_R\} \neq K\). Thus, \(\hat{\theta} \neq \hat{1}\).

Now we prove that \(\hat{f} \hat{\theta} = \hat{\theta} \hat{f} = \hat{\theta f}\) for all \(\hat{f} \in Q_r\). Since \(RU \subseteq RU \cap R\) and \(f\theta = \theta\) on \(RU\) where \(U \in N\) is domain of \(f\), we get \(\hat{f} \hat{\theta} = \hat{\theta} \hat{f} = \hat{\theta f}\). Similarly \(\hat{f}\hat{f} = \hat{\theta}\). Thus, \((Q_r, +, \cdot)\) is a semiring.

**Theorem 1** Let \(R\) be prime semiring and the relation \(\equiv_N\) on \(M\) defined by \((*)\). If \(R\) is a right multiplicatively cancellable semiring, then \(R\) may be embedded in \(Q_r\) as a subsemiring.

**Proof** Let \(a \in R\). Define \(\lambda : R \rightarrow R\) by \(\lambda (a) = ar\) for all \(r \in R\). It is clear that \(\lambda\) is a right \(R\) homomorphism. Since \(R \in N\), then \(\lambda \in M\). So \(\lambda\) defines element \(\hat{\lambda}\) of \(Q_r\). Hence we may define \(\lambda : R \rightarrow Q_r\) by \(\lambda (a) = \hat{\lambda}_a\) for \(a \in R\). Clearly \(\lambda\) is well-defined. Let \(\lambda (a) = \lambda (b)\) for any \(a, b \in R\). Then, \(\hat{\lambda}_a = \hat{\lambda}_b\), i.e., \(\lambda_a \equiv_N \lambda_b\). Hence there exists \(K \subseteq N\) and \(K \subseteq R \cap R\) such that \(\lambda_a = \lambda_b\) on \(K\). Therefore, \(\lambda_a(x) = \lambda_b(x)\) for all \(x \in K\). So, \(ax = bx\) for all \(x \in K\). Since \(R\) is right multiplicatively cancellable semiring, we get \(a = b\). That is, \(\lambda\) is injective. In order to prove \(\lambda\) is a homomorphism, let \(a, b \in R\). Then

\[
\lambda_{a+b}(x) = (a+b)x = ax + bx = \lambda_a(x) + \lambda_b(x)
\]

and

\[
\lambda_{ab}(x) = (ab)x = a(bx) = \lambda_a(bx) = \lambda_a(\lambda_b(x)) = (\lambda_a \lambda_b)(x),
\]

for all \(x \in R\). It follows that \(\lambda_{a+b} = \lambda_a + \lambda_b\) and \(\lambda_{ab} = \lambda_a \lambda_b\). Hence

\[
\lambda(a + b) = \hat{\lambda}_{a+b} = \hat{\lambda}_a + \hat{\lambda}_b = \hat{\lambda}_{a+b} = \lambda(a) + \lambda(b)
\]

and

\[
\lambda(ab) = \hat{\lambda}_{ab} = \hat{\lambda}_a \hat{\lambda}_b = \hat{\lambda}_{a+b} = \lambda(a) \lambda(b).
\]

Also, \(\lambda(0_R) = \hat{\theta}\) and \(\lambda(1_R) = \hat{1}\). Therefore \(R\) is a subsemiring of \(Q_r\). \(\square\)

Therefore \(R\) is a subsemiring of \(Q_r\), we call \(Q_r\) the right quotient semiring of \(R\). For purposes of convenience, we use \(q\) instead of \(\hat{q}\) in \(Q_r\).
Lemma 1 Let $R$ be a right multiplicatively cancellable prime semiring. For each nonzero $q \in Q_r$, there is nonzero ideal $U$ of $R$ such that $q(U) \subseteq R$.

**Proof** Clear. 

Lemma 2 Let $R$ be a right multiplicatively cancellable prime semiring. Then the quotient semiring $Q_r$ of $R$ is a prime semiring.

**Proof** Let $p$, $q \in Q_r$ be such that $pQ_r q = \theta$. If $p \neq \theta \neq q$, then there exist nonzero ideals $U$ and $V$ of $R$ such that $p(U) \subseteq R$ and $q(V) \subseteq R$. Since $p \neq \theta \neq q$, there exist nonzero elements $x \in U$ and $y \in V$ such that $p(x) \neq 0_R \neq q(y)$. Noticing that $R$ is a subsemiring of $Q_r$, we have

$$p(x) R q(y) \subseteq p(x) Q_r q(y) = \{0_R\}$$

and so $p(x) R q(y) = \{0_R\}$. This is a contradiction. Hence $p = \theta$ or $q = \theta$, ending the proof. 

Definition 1 The set

$$C := \{g \in Q_r \mid gf = fg \text{ for all } f \in Q_r\}$$

is called the extended centroid of a semiring $R$.

It is clear that $C$ is a subsemiring of $Q_r$. Let $\theta \neq c \in C$. Assume that $cf = \theta$ for any $f \in C$. Then, $gcf = \theta$ for any $g \in Q_r$. Hence, $cgf = \theta$, i.e., $cQ_r f = \theta$. Since $\theta 

Theorem 2 The center $C$ of $Q_r$ is a semifield.

**Proof** We prove that $C$ is a semifield such that every $\theta \neq c \in C$ is inverse. Let $\theta \neq c \in C$. Since $c \in Q_r$, there exists $\{0_R\} \neq U \in N$ such that $cU \subseteq R$. Since $R$ is a prime semiring, $cU \neq \{0_R\}$. Taking $\{0_R\} \neq V = cU$ ideal of $R$. Moreover, $d : V \to R$, $ca \mapsto a$ is a $R$ homomorphism. Then, $d = [d, V] \in Q_r$. Also, $dc = 1$ since $dc(a) = a = 1(a)$ for all $a \in U$. Thus, $\theta \neq c \in C$ is inverse. That is, $C$ is a semifield.

We now let $S = RC$, a subsemiring of $Q_r$ containing $R$. We shall call $S$ the central closure of $R$. The same proof used in showing that $Q_r$ was prime may be employed to show that $S$ is prime.

Proposition 2 Let $R$ be right multiplicatively cancellable prime semiring and $S$ be the central closure of $R$. Then $S$ is a right multiplicatively cancellable prime semiring.

**Proof** Since $S = \left\{ \sum_{finite} r_i c_i \mid \text{ for all } r_i \in R, c_i \in C \right\}$, the proof of proposition is sufficient for the finite sums with $i = 1$ to 2. Now, let $ad = bd, a = r_1 c_1 + r_2 c_2, b = r'_1 c'_1 + r'_2 c'_2, 0 \neq d = r''_1 c''_1 + r''_2 c''_2 \in S$, for all $r_1, r_2, r'_1, r'_2, r''_1, r''_2 \in R, c_1, c_2, c'_1, c'_2, c''_1, c''_2 \in C$ and $c_1 = [f_1, U_1], c_2 = [f_2, U_2], c'_1 = [g_1, V_1], c'_2 = [g_2, V_2], c''_1 = [h_1, W_1], c''_2 = [h_2, W_2]$. We have that

$$([\lambda_{r_1}, R][f_1, U_1] + [\lambda_{r_2}, R][f_2, U_2])([\lambda_{c''_1}, R][h_1, W_1] + [\lambda_{c''_2}, R][h_2, W_2])$$

$$= ([\lambda_{c''_1}, R][g_1, V_1] + [\lambda_{c''_2}, R][g_2, V_2])([\lambda_{c''_1}, R][h_1, W_1] + [\lambda_{c''_2}, R][h_2, W_2]).$$
Then,
\[
([\lambda_{i_1}, f_1, U_1] + [\lambda_{i_2}, f_2, U_2])([\lambda_{i_1}^{-1} h_1, W_1] + [\lambda_{i_2}^{-1} h_2, W_2]) = ([\lambda_{i_1}^{-1} g_1, V_1] + [\lambda_{i_2}^{-1} g_2, V_2])([\lambda_{i_1}^{-1} h_1, W_1] + [\lambda_{i_2}^{-1} h_2, W_2])
\]
\[
([\lambda_{i_1}, f_1 + \lambda_{i_2}, f_2, U_1 \cap U_2])([\lambda_{i_1}^{-1} h_1 + \lambda_{i_2}^{-1} h_2, W_1 \cap W_2]) = ([\lambda_{i_1}^{-1} g_1 + \lambda_{i_2}^{-1} g_2, V_1 \cap V_2])([\lambda_{i_1}^{-1} h_1 + \lambda_{i_2}^{-1} h_2, W_1 \cap W_2])
\]
\[
([\lambda_{i_1}, f_1 + \lambda_{i_2}, f_2])([\lambda_{i_1}^{-1} h_1 + \lambda_{i_2}^{-1} h_2], (W_1 \cap W_2)(U_1 \cap U_2)) = ([\lambda_{i_1}^{-1} g_1 + \lambda_{i_2}^{-1} g_2])([\lambda_{i_1}^{-1} h_1 + \lambda_{i_2}^{-1} h_2], (W_1 \cap W_2)(V_1 \cap V_2)).
\]

Thus, there exist \( K \in N \) and \( K \subseteq (W_1 \cap W_2)(U_1 \cap U_2) \cap (W_1 \cap W_2)(V_1 \cap V_2) \) such that \((\lambda_{i_1}, f_1 + \lambda_{i_2}, f_2)(\lambda_{i_1}^{-1} h_1 + \lambda_{i_2}^{-1} h_2) = (\lambda_{i_1}^{-1} g_1 + \lambda_{i_2}^{-1} g_2)(\lambda_{i_1}^{-1} h_1 + \lambda_{i_2}^{-1} h_2)\) on \( K \). \( \forall x \in K \),
\[
((\lambda_{i_1}, f_1 + \lambda_{i_2}, f_2)(\lambda_{i_1}^{-1} h_1 + \lambda_{i_2}^{-1} h_2))(x) = ((\lambda_{i_1}^{-1} g_1 + \lambda_{i_2}^{-1} g_2)(\lambda_{i_1}^{-1} h_1 + \lambda_{i_2}^{-1} h_2))(x)
\]
\[
((\lambda_{i_1}, f_1 + \lambda_{i_2}, f_2)(\lambda_{i_1}^{-1} h_1 + \lambda_{i_2}^{-1} h_2))(x) = ((\lambda_{i_1}^{-1} g_1 + \lambda_{i_2}^{-1} g_2)(\lambda_{i_1}^{-1} h_1 + \lambda_{i_2}^{-1} h_2))(x)
\]

and so \(\lambda_{i_1}, f_1 + \lambda_{i_2}, f_2 = \lambda_{i_1}^{-1} g_1 + \lambda_{i_2}^{-1} g_2\) on \((\lambda_{i_1}^{-1} h_1 + \lambda_{i_2}^{-1} h_2)(K)\). Since \(h_1\) and \(h_2\) are right \(R\) homomorphisms, and \(K \subseteq (W_1 \cap W_2)(U_1 \cap U_2) \cap (W_1 \cap W_2)(V_1 \cap V_2)\), we get \(h_1(K) \subseteq h_1(W_1 \cap W_2)(U_1 \cap U_2) \cap h_1(W_1 \cap W_2)(V_1 \cap V_2) \subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2)\) and \(h_2(K) \subseteq h_2(W_1 \cap W_2)(U_1 \cap U_2) \cap h_2(W_1 \cap W_2)(V_1 \cap V_2) \subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2)\). So that \(Rh_1(K) + Rh_2(K) \subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2)\). If we use that \(h_1(K)\) and \(h_2(K)\) are right ideals, then \(Rh_1(K) + Rh_2(K)\) is an ideal of \(R\). We assume that \(Rh_1(K) + Rh_2(K) = \{0_R\}\). Therefore we get \(Rh_1(K) \cap Rh_2(K) \subseteq Rh_1(K) + Rh_2(K) = \{0_R\}\). Since \(R\) is a prime semiring, \(Rh_1(K) = \{0_R\}\) or \(Rh_2(K) = \{0_R\}\). Hence we obtain that \((\lambda_{i_1}^{-1} h_1 + \lambda_{i_2}^{-1} h_2)(K) = \{0_R\}\) for all \(r_1^o, r_2^o \in R\) from \((\lambda_{i_1}^{-1} h_1)(K) = \{0_R\}\) and \((\lambda_{i_2}^{-1} h_2)(K) = \{0_R\}\), this contradicts with \(d \neq 0\). So \(Rh_1(K) + Rh_2(K)\) must be nonzero. Therefore \(\lambda_{i_1}, f_1 + \lambda_{i_2}, f_2 = \lambda_{i_1}^{-1} g_1 + \lambda_{i_2}^{-1} g_2\) on \(Rh_1(K) + Rh_2(K)\). Hence \([\lambda_{i_1}, R][f_1, U_1] + [\lambda_{i_2}, R][f_2, U_2] = [\lambda_{i_1}, R][g_1, V_1] + [\lambda_{i_2}, R][g_2, V_2]\). That is, \(r_1c_1 + r_2c_2 = r_1'c_1' + r_2'c_2'\) and so \(a = b\), ending the proof. \(\square\)

**Theorem 3** Let \(R\) be a right multiplicatively cancellable prime semiring. If \(a\) and \(b\) are nonzero elements in \(S\) such that \(axb = bx\) for all \(x \in R\) and \(S\) is left cancellable then there exists \(q \in C\) such that \(qa = b\).

**Proof** We may assume that \(a \neq \theta\) and \(b \neq \theta\). Let \(U\) be a nonzero ideal of \(R\) such that \(aU \subseteq R\) and \(bU \subseteq R\), and \(V = UAU\). We define a mapping \(f : V \rightarrow R\), \(f \left( \sum_i x_i a y_i \right) = \sum_i x_i b y_i\), \(x_i, y_i \in U\). Let \(\sum_i x_i a y_i = \sum_i x_i' a y_i'\), \(x_i, y_i, x_i', y_i' \in U\). Then,
\[
\left( \sum_i x_i a y_i \right) b = \left( \sum_i x_i' a y_i' \right) b
\]
\[
\sum_i x_i (ayi) = \sum_i x_i' (ayi')
\]

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\[ \sum_i x_i (by_i a) = \sum_i x'_i (by'_i a) \]
\[ (\sum_i x_i by_i) a = \left( \sum_i x'_i by'_i \right) a. \]

Since \( S \) is a right multiplicatively cancellable semiring by Proposition 2, we get \( \sum_i x_i by_i = \sum_i x'_i by'_i \). That is, \( f \) is well-defined. \( f \) is a right \( R \) homomorphism. Because \( f((xay)r) = f((xa)(yr)) = xa(yr) = (xby)r = f(xay)r \) for all \( x, y \in U \) and \( r \in R \). Let \( q \) denote the element of \( Q \), determined by \( f \) and let \( p \) be any element of \( Q_r \), with \( pK \subseteq R \) for some nonzero ideal \( K \) of \( R \). For \( x, y \in U \) and \( z \in K \) we have \( (qp)(zxay) = q((pz)xay) = (pz)xby = p(zxby) = pq(zxay) \). Since \( KU \subseteq K \cap U \) and \( pq = qp \) on \( KU \), we get \( qp = pq \) for all \( p \in Q_r \), and so \( q \in C \). Since \( S \) is multiplicatively cancellable, \( xqay = xby \), for all \( x, y \in U \) implies \( qa = b \).

References