$L^p$ solutions of infinite time interval BSDEs and the corresponding $g$-expectations and $g$-martingales

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Abstract: In this paper we study the existence and uniqueness theorem for $L^p$ ($1 < p < 2$) solutions for a class of infinite time interval backward stochastic differential equations (BSDEs). Furthermore, we introduce generalized $g$-expectations and generalized $g$-martingales via the $L^p$ solutions and prove the stability theorem of generalized $g$-expectations.

Key words: Backward stochastic differential equation (BSDE), comparison theorem, generalized $g$-expectation, generalized $g$-martingale

1. Introduction
The theory of backward stochastic differential equations (BSDEs) was developed by Pardoux and Peng [24], from which we know that there exists a unique adapted and square integrable solution to a BSDE of the type

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s dW_s, \quad t \in [0, T],$$

(1)

provided the function $g$ (also called the generator) is Lipschitz in both variables $y$ and $z$, and $\xi$ and $(g(t, 0, 0))_{0 \leq t \leq T}$ are square integrable. Later, many researchers developed the theory of BSDEs and their applications in a series of papers (for example, see Briand et al. [3], Hu and Peng [16], Lepeltier and San Martin [19], Pardoux [22, 23], El Karoui et al. [13] and the references therein) under some other assumptions on the coefficients but for a fixed terminal time $T > 0$. Let us mention the contribution of Lepeltier and San Martin [19], which dealt with the quadratic of growth generator $g$ in $z$ and got the existence and uniqueness result in $L^2$. Let us mention also that when the generator $g$ is Lipschitz continuous, a result of El Karoui et al. [13] provides for a solution when the data $\xi$ and $\{(g(t, 0, 0))_{t \in [0, T]}\}$ are in $L^p$ even for $p \in (1, 2)$. In 2003, Briand et al. [3] was devoted to the generalization of this result to the case of a monotone generator for BSDEs on a fixed time interval.

In 1997, Peng [27] introduced the notions of $g$-expectation and $g$-martingale via the $L^2$ solution of BSDE (1). Peng’s $g$-expectation is a kind of nonlinear expectation, which can be considered as a nonlinear extension of the well-known Girsanov transformations. The original motivation for studying Peng’s $g$-expectation comes from the theory of expected utility. Since the notion of Peng’s $g$-expectation was introduced, many properties of Peng’s $g$-expectation have been studied by Briand et al. [2], Chen [4], Chen and Wang [5], Chen and Epstein
In this paper, we investigate generalized $g$-expectations and generalized $g$-martingales via $L^p$ ($1 < p < 2$) solutions of infinite time interval BSDEs. One difficulty of this problem is how to study the existence and uniqueness of BSDE (1) when $T \equiv \infty$ in $L^p$. In fact, such a problem in $L^p$ ($1 < p \leq 2$) has been investigated by Briand et al. [3], Peng [26], Pardoux [22], Darling and Pardoux [11], Pardoux and Zhang [25] and other researchers under the assumption that terminal value $\xi = 0$ or $E[e^{\rho T}|\xi|^p] < \infty$ for some constant $\rho$ and random terminal time $T$ (i.e. $T$ is a stopping time).

Let us mention the contribution of Briand et al. [3] which dealt with a monotone generator $g$ in $y$ and got the existence and uniqueness result in $L^p$ ($1 < p < 2$) on a random time interval. Furthermore, Briand et al. [3] strongly pointed out that their existence and uniqueness result covered the case of $T \equiv \infty$ (see the first paragraph of Section 5 and Remark 5.3 in [3]).

Let us mention also the contribution of Hu and Tessitore [17]. In 2007, Hu and Tessitore [17] studied the existence and uniqueness of mild solutions to a possibly degenerate elliptic partial differential equation

$$Lu(x) + \psi(x, u(x), \nabla u(x)G(x)) - \lambda u(x) = 0$$

in Hilbert spaces. The main tool was existence, uniqueness and regular dependence on parameters of a bounded solution to a suitable BSDE with a random terminal time $T$.

In 2000, Chen and Wang [5] obtained the existence and uniqueness theorem for $L^2$ solutions of infinite time interval BSDEs when $T \equiv \infty$, by the martingale representation theorem and fixed point theorem. But in $L^p$ ($1 < p < 2$), there is no martingale representation theorem. In order to get rid of this difficulty, we give a new a priori estimate (Lemma 3.1). The main idea of this a priori estimate comes from Proposition 3.2 in Briand et al. [3]. Using this a priori estimate, we study the existence and uniqueness of $L^p$ solutions to infinite time interval BSDEs. In fact, the difference between [3] and this paper is not the time horizon over which the problem is formulated but the assumptions on the function that appear in BSDE (1) (this paper’s $g$ and [3]’s $f$), in which $\lambda$ and $\mu$ appearing in (H2) of [3] are constant, while our $\alpha$ and $\beta$ are integrable Lipschitz functions on time $t$. These integrability conditions are introduced in [5]. In this paper, we also introduce generalized $g$-expectations and generalized $g$-martingales via $L^p$ solutions of infinite time interval BSDEs. Furthermore, we give the stability theorem of generalized $g$-expectations.

This paper is organized as follows. In Section 2, we introduce some notations, assumptions and lemmas. In Section 3, we prove the existence and uniqueness theorem for $L^p$ solutions of infinite time interval BSDEs. In Section 4, we introduce generalized $g$-expectations and generalized $g$-martingales via $L^p$ solutions of infinite time interval BSDEs and prove the stability theorem of generalized $g$-expectations.

2. Preliminaries

In this section, we shall present some notations, assumptions and lemmas that are used in this paper.

Let $(\Omega, \mathcal{F}, P)$ be a completed probability space, $(W_t)_{t \geq 0}$ be a $d$-dimensional standard Brownian motion defined on this space and $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration generated by Brownian motion $(W_t)_{t \geq 0}$, that is,

$$\mathcal{F}_t := \sigma\{W_s; s \leq t\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the set of all $P$-null subsets. Furthermore, we assume $\mathcal{F} := \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t\right)$. 

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For simplicity, we just consider the case that \( d = 1 \), but our method can be easily extended to the other cases.

We consider the following spaces:

\[
L^p(\Omega, \mathcal{F}, P) := \{ \xi : \xi \text{ is } \mathcal{F}\text{-measurable random variable such that } E[|\xi|^p] < \infty, p \geq 1 \}; \\
\mathcal{L}(\Omega, \mathcal{F}, P) := \bigcup_{p > 1} L^p(\Omega, \mathcal{F}, P); \\
S^p(\mathbb{R}) := \{ V : V_t \text{ is } \mathcal{F}_t\text{-adapted process such that } E[\sup_{t \geq 0} |V_t|^p] < \infty, p \geq 1 \}; \\
S(\mathbb{R}) := \bigcup_{p > 1} S^p(\mathbb{R}); \\
L^p(\mathbb{R}) := \{ V : V_t \text{ is } \mathcal{F}_t\text{-adapted process such that } E\left[\left(\int_0^\infty |V_s|^2 ds\right)^{\frac{p}{2}}\right] < \infty, p \geq 1 \}; \\
L(\mathbb{R}) := \bigcup_{p > 1} L^p(\mathbb{R}).
\]

In the sequel, we assume that \( 1 < p < 2 \).

Consider the following infinite time interval BSDE:

\[
Y_t = \xi + \int_t^\infty g(s, Y_s, Z_s)ds + V_\infty - V_t - \int_t^\infty Z_s dW_s. \tag{2}
\]

Let

\[
g : \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}
\]

such that for any \((y, z) \in \mathbb{R} \times \mathbb{R}\), \(g(\cdot, y, z)\) is \(\mathcal{F}_t\)-progressively measurable. We make the following assumptions:

(A.1) \( E\left[\left(\int_0^\infty |g(t, 0, 0)|dt\right)^2\right] < \infty \);

(A.2) There exists two positive non-random functions \(\alpha(t)\) and \(\beta(t)\), such that for all \(y_1, y_2 \in \mathbb{R}\), \(z_1, z_2 \in \mathbb{R}\),

\[
|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq \alpha(t)|y_1 - y_2| + \beta(t)|z_1 - z_2|,
\]

where \(\alpha(t)\) and \(\beta(t)\) satisfy that \(\int_0^\infty \alpha(t)dt < \infty\), \(\int_0^\infty \beta(t)dt < \infty\), \(\int_0^\infty \beta^2(t)dt < \infty\);

(A.3) There exists some constant \(T \in [0, \infty)\) such that

\[
E\left[\left(\int_0^T |g(t, 0, 0)|dt\right)^p\right] < \infty,
\]

\[
E\left[\int_T^\infty |g(t, 0, 0)|dt\right] < \infty.
\]

(A.4) \( (V_t)_{t \geq 0} \) is an RCLL process (i.e. \( (V_t)_{t \geq 0} \) has sample paths which are right continuous with left limits) with \( (V_t)_{t \geq 0} \in S^2(\mathbb{R}) \).

The following lemmas are very useful in this paper.

**Lemma 2.1** Let \( \{K_t\}_{t \geq 0} \) and \( \{H_t\}_{t \geq 0} \) be two progressively measurable processes with values in \( \mathbb{R} \) such that \( P\text{-a.s.}, \)

\[
\int_0^\infty (|K_t| + |H_t|^2)dt < +\infty.
\]
We consider the $\mathbb{R}$-valued semi-martingale $\{X_t\}_{t \geq 0}$ defined by

$$X_t = X_0 + \int_0^t K_s ds + \int_0^t H_s dW_s, \quad 0 \leq t \leq \infty.$$ 

Then, for any $p \geq 1$, we have

$$|X_\infty|^p \geq |X_0|^p + p \int_0^\infty |X_s|^{p-1} \frac{\mathbf{1}_{|X_s| \neq 0}}{|X_s|} K_s ds + p \int_0^\infty |X_s|^{p-1} \frac{\mathbf{1}_{(X_s,\neq 0)}}{|X_s|} H_s dW_s,$$

where $c(p) = \frac{p(1-p)}{2}$, $1 \wedge (p-1) := \min\{1, p-1\}$.

The proof of Lemma 2.1 is very similar to that of Lemma 2.2 in [3]. It is almost verbatim adapted from [3]. Now we briefly give the idea of the proof of Lemma 2.1. Since the function $x \mapsto |x|^p$ is not smooth enough (for $p \in [1, 2]$) to apply Itô’s formula, we use an approximation. Let $e > 0$ and let us consider the function $u_{\varepsilon}(x) := (|x|^2 + \varepsilon^2)^{\frac{1}{2}}$. Obviously, it is a smooth function. Itô’s formula leads to the following equality:

$$u_\varepsilon^p(X_\infty) = u_\varepsilon^p(X_0) + \int_0^\infty u_\varepsilon^{p-2}(X_s) X_s K_s ds + \int_0^\infty u_\varepsilon^{p-2}(X_s) X_s H_s dW_s + \frac{1}{2p} \int_0^\infty \int_0^\infty u_\varepsilon^{p-4}(X_s) X_s^2 s dW_s ds.$$

Letting $\varepsilon \to 0$ in (4) and applying convergence, we can obtain (3).

**Lemma 2.2** If $(Y, Z)$ is a solution of the following BSDE:

$$Y_t = \xi + \int_t^\infty g(s, Y_s, Z_s) ds - \int_t^\infty Z_s dW_s, \quad 0 \leq t \leq \infty,$$

then we have

$$|Y_t|^p \geq |\xi|^p + p \int_t^\infty |Y_s|^{p-1} \frac{\mathbf{1}_{(Y_s, \neq 0)}}{|Y_s|} Z_s^2 ds,$$

and

$$|Y_t|^p \leq |\xi|^p + p \int_t^\infty |Y_s|^{p-1} \frac{\mathbf{1}_{(Y_s, \neq 0)}}{|Y_s|} g(s, Y_s, Z_s) ds.$$

**Proof** Noting that

$$Y_t = Y_0 - \int_0^t g(s, Y_s, Z_s) ds + \int_0^t Z_s dW_s, \quad 0 \leq t \leq \infty,$$

then, together with (3), we obtain (6). \hfill \Box

### 3. Existence and uniqueness

In this section, we prove the existence and uniqueness theorem for $L^p$ solutions of infinite time interval BSDEs which generalizes the result of [5] and give the corresponding comparison theorem.

**Theorem 3.1** Under assumptions (A.2)--(A.4), if $\xi \in L^p(\Omega, \mathcal{F}, P)$, then BSDE (2) has a unique solution $(Y, Z) \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{L}^p(\mathbb{R})$.

In order to prove Theorem 3.1, we give an a priori estimate.
Lemma 3.1  Suppose that (A.2) holds for \( g \). Furthermore, each \( \phi_i \) satisfies that

\[
E \left[ \left( \int_0^\infty |\phi_i(s)| ds \right)^p \right] < \infty.
\]

Let \( \xi_i \in L^p(\Omega, \mathcal{F}, P), \ (Y^1, Z^i) \in \mathcal{S}^p(\mathbb{R}) \times \mathcal{L}^p(\mathbb{R}) \) satisfy the following BSDEs:

\[
Y^1_t = \xi_t + \int_t^\infty [g(s, Y^1_s, Z^i_s) + \phi_i(s)] ds - \int_t^\infty Z^i_s dW_s, \quad i = 1, 2.
\]

Then

\[
E \left[ \sup_{s \geq 0} |Y^1_s - Y^2_s|^p + \left( \int_0^\infty |Z^1_s - Z^2_s|^2 ds \right)^{\frac{p}{2}} \right] \leq C_p E \left[ |\xi_1 - \xi_2|^p + \left( \int_0^\infty |\phi_1(s) - \phi_2(s)| ds \right)^p \right],
\]

where \( C_p \) is a positive constant depending only on \( p \).

**Proof**  It is easy to check that

\[
\int_0^\infty \left( g(s, Y^1_s, Z^i_s) - g(s, Y^2_s, Z^i_s) + \phi_i(s) \right)^2 ds < \infty,
\]

so applying Itô’s formula to \( (Y^1_s - Y^2_s)^2 \), we have

\[
\begin{align*}
|Y^1_s - Y^2_s|^2 &= |\xi_1 - \xi_2|^2 + 2 \int_0^s \langle Y^1_t - Y^2_t \rangle \left( g(s, Y^1_s, Z^i_s) - g(s, Y^2_s, Z^i_s) + \phi_i(s) \right) ds \\
&\quad - 2 \int_0^s \langle Y^1_t - Y^2_t \rangle \left( Z^1_t - Z^2_t \right) dW_t.
\end{align*}
\]

From the Lipschitz assumption (A.2) on \( g \), we have

\[
\begin{align*}
2 \langle Y^1_s - Y^2_s \rangle \left( g(s, Y^1_s, Z^i_s) - g(s, Y^2_s, Z^i_s) \right) &\leq 2\alpha(s) |Y^1_s - Y^2_s|^2 + 2\beta(s) |Z^1_s - Z^2_s| |Y^1_s - Y^2_s| \\
&\quad + \frac{1}{2} |Z^1_s - Z^2_s|^2, \quad s \geq 0.
\end{align*}
\]

It follows that

\[
\begin{align*}
\frac{1}{2} \int_0^\infty |Z^1_s - Z^2_s|^2 ds &\leq \left[ 1 + 2 \left( \int_0^\infty \alpha(s) ds + \int_0^\infty \beta^2(s) ds \right) \right] \sup_{s \geq 0} |Y^1_s - Y^2_s|^2 \\
&\quad + 2 \int_0^\infty |Y^1_s - Y^2_s| |\phi_1(s) - \phi_2(s)| ds + 2 \int_0^\infty \langle Y^1_s - Y^2_s \rangle \left( Z^1_s - Z^2_s \right) dW_t.
\end{align*}
\]

Since \( 2 \int_0^\infty |Y^1_s - Y^2_s| |\phi_1(s) - \phi_2(s)| ds \leq \sup_{s \geq 0} |Y^1_s - Y^2_s|^2 + \left( \int_0^\infty |\phi_1(s) - \phi_2(s)| ds \right)^2 \), we have

\[
\begin{align*}
\int_0^\infty |Z^1_s - Z^2_s|^2 ds &\leq 4 \left[ 1 + \left( \int_0^\infty \alpha(s) ds + \int_0^\infty \beta^2(s) ds \right) \sup_{s \geq 0} |Y^1_s - Y^2_s|^2 \right] \\
&\quad + 4 \left( \int_0^\infty |\phi_1(s) - \phi_2(s)| ds \right)^2 + \int_0^\infty \langle Y^1_s - Y^2_s \rangle \left( Z^1_s - Z^2_s \right) dW_t.
\end{align*}
\]

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Using the fact that if \( b, a_i \geq 0 \) and \( b \leq \sum_{i=1}^{n} a_i \), then \( b^p \leq \sum_{i=1}^{n} a_i^p \) for any \( p \in (0, 1) \) (see, e.g., Kuang [18, page 132]), we have

\[
\left( \int_0^\infty |Z_1^s - Z_2^s|^2 \, ds \right)^{\frac{p}{2}} \leq c_p \left( \sup_{s \geq 0} |Y_1^s - Y_2^s|^p + \left( \int_0^\infty |\phi_1(s) - \phi_2(s)| \, ds \right)^p \right)
\]

where \( c_p \) is a positive constant depending only on \( p \). By the Burkholder-Davis-Gundy inequality (see, e.g., Barlow et al. [1, Table 4.1 page 162]), we get

\[
c_p E \left[ \int_0^\infty (Y_1^s - Y_2^s) (Z_1^s - Z_2^s) \, dW_s \right] \leq d_p E \left[ \left( \int_0^\infty |Y_1^s - Y_2^s|^2 |Z_1^s - Z_2^s|^2 \, ds \right)^{\frac{p}{2}} \right]
\]

and thus

\[
c_p E \left[ \int_0^\infty (Y_1^s - Y_2^s) (Z_1^s - Z_2^s) \, dW_s \right] \leq \frac{1}{2} E \left[ \left( \int_0^\infty |Z_1^s - Z_2^s|^2 \, ds \right)^{\frac{p}{2}} \right]
\]

where \( d_p \) is a positive constant depending only on \( p \). From (7) and (8), we have

\[
E \left[ \left( \int_0^\infty |Z_1^s - Z_2^s|^2 \, ds \right)^{\frac{p}{2}} \right] \leq C E \left[ \sup_{s \geq 0} |Y_1^s - Y_2^s|^p + \left( \int_0^\infty |\phi_1(s) - \phi_2(s)| \, ds \right)^p \right],
\]

where \( C \) is a positive constant depending only on \( p \).

Now, we prove that

\[
E \left[ \sup_{s \geq 0} |Y_1^s - Y_2^s|^p \right] \leq C' E \left[ |\xi_1 - \xi_2|^p + \left( \int_0^\infty |\phi_1(s) - \phi_2(s)| \, ds \right)^p \right],
\]

where \( C' \) is a positive constant depending only on \( p \). The proof of (10) is similar to that of Proposition 3.2 of Briand et al. [3]. Let us fix \( \theta(t) := \alpha(t) + \frac{\beta(t)}{p} \) and define \( \xi := e^{\int_0^t \theta(s) \, ds} \xi_t, Y_i := e^{\int_0^t \theta(s) \, ds} Y_i, Z_i := e^{\int_0^t \theta(s) \, ds} Z_i, i = 1, 2 \), which solve the following BSDEs, respectively:

\[
\overline{Y}_t = \xi_t + \int_t^\infty \left[ \overline{\sigma} \left( s, \overline{Y}_s, \overline{Z}_s \right) + e^{\int_t^s \theta(r) \, dr} \phi_1(s) \right] \, ds - \int_t^\infty \overline{Z}_s \, dW_s, \quad i = 1, 2,
\]

where \( \overline{\sigma}(t, y, z) := -e^{\int_0^t \theta(s) \, ds} g \left( t, e^{-\int_0^t \theta(s) \, ds} y, e^{-\int_0^t \theta(s) \, ds} z \right) - \theta(t) y \).

By Lemma 2.2, we can get the inequality

\[
\begin{align*}
&\left( \overline{Y}_t^2 - \overline{Y}_s^2 \right)^p + 2(p-1) \int_t^s \left( \overline{Y}_r^2 - \overline{Z}_r^2 \right)^{p-2} \left( \overline{Y}_r - \overline{Z}_r \right) \, dr \sup_{r \neq s} |\overline{Z}_r - \overline{Z}_s| \, ds \\
\leq &\left( \xi_1 - \xi_2 \right)^p + p \int_t^s \left( \overline{Y}_r^2 - \overline{Y}_s^2 \right)^{p-1} \frac{\overline{Y}_r - \overline{Z}_r}{|\overline{Y}_r - \overline{Z}_r|} \, dr \sup_{r \neq s} |\overline{Z}_r - \overline{Z}_s| \left( \overline{\sigma} \left( s, \overline{Y}_s, \overline{Z}_s \right) - \overline{\sigma} \left( s, \overline{Y}_s, \overline{Z}_s \right) \right) \, ds \\
+ & p \int_t^s \left( \overline{Y}_r^2 - \overline{Y}_s^2 \right)^{p-1} e^{\int_t^s \theta(r) \, dr} \left( \phi_1(s) - \phi_2(s) \right) \, ds \\
- & p \int_t^s \left( \overline{Y}_r^2 - \overline{Y}_s^2 \right)^{p-1} \frac{\overline{Y}_r - \overline{Z}_r}{|\overline{Y}_r - \overline{Z}_r|} \, dr \left( \overline{Z}_r^2 - \overline{Z}_s^2 \right) \, dW_s.
\end{align*}
\]
From the Lipschitz assumption (A.2) on \( g \) and with the help of

\[
\theta(t) := a(t) + \frac{\beta^2(t)}{p-1},
\]

\[
\mathbf{Y}_t^i := e^{\int_0^t \theta(s) \, ds} \mathbf{Y}_t^i, \quad \mathbf{Z}_t^i := e^{\int_0^t \theta(s) \, ds} \mathbf{Z}_t^i, \quad i = 1, 2
\]

and

\[
\mathbf{Y}(t, y, z) := e^{\int_0^t \theta(s) \, ds} \left( t, e^{-\int_0^t \theta(s) \, ds} y, e^{-\int_0^t \theta(s) \, ds} z \right) - \theta(t) y,
\]

we have

\[
\begin{align*}
    & p \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right|^{p-1} e^{\int_0^t \theta(r) \, dr} \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| \\
    & = p \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right|^{p-1} e^{\int_0^t \theta(r) \, dr} \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| \\
    & = \rho \theta(x) \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right|^{p-1} e^{\int_0^t \theta(r) \, dr} \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| \\
    & = \rho \theta(x) \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right|^{p-1} \left| \mathbf{Z}_s^1 - \mathbf{Z}_s^2 \right| - \frac{\rho^2 \theta(x)}{p+1} \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right|^{p}. \\
\end{align*}
\]

Noting that

\[
\begin{align*}
    & \rho \beta(x) \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right|^{p-1} \left| \mathbf{Z}_s^1 - \mathbf{Z}_s^2 \right| \\
    & = \rho \beta(x) \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right|^{p-1} \left| \mathbf{Z}_s^1 - \mathbf{Z}_s^2 \right| \\
    & \leq \rho \beta(x) \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right|^{p} + \rho \beta(x) \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right|^{p-2} \left| \mathbf{Z}_s^1 - \mathbf{Z}_s^2 \right|^2,
\end{align*}
\]

(where the inequality comes from the fact that if \( a, b \geq 0 \), then \( ab \leq a^2 + b^2 \)), we have

\[
\begin{align*}
    & p \int_t^\infty \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right|^{p-1} e^{\int_0^t \theta(r) \, dr} \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| ds \\
    & \leq \frac{\rho \beta(x)}{p+1} \int_t^\infty \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right|^{p-2} \left| \mathbf{Z}_s^1 - \mathbf{Z}_s^2 \right|^2 ds. \\
\end{align*}
\]

Thus from (11) and (13), we obtain the following inequality:

\[
\begin{align*}
    & \xi_1 - \xi_2 + p \int_t^\infty \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right|^{p-1} e^{\int_0^t \theta(r) \, dr} \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right| ds \\
    & \leq \xi_1 - \xi_2 + p \int_t^\infty \left| \mathbf{Y}_s^1 - \mathbf{Y}_s^2 \right|^{p-2} \left| \mathbf{Z}_s^1 - \mathbf{Z}_s^2 \right|^2 ds.
\end{align*}
\]
Denote
\[ M_t := \int_0^t \left| Y_s - Y_s^2 \right|^{p-1} \frac{Y_s^2 - Y_s^2}{Y_s - Y_s} 1_{(Y_s - Y_s^2 \neq 0)} \left( Z_s - Z_s^2 \right) dW_s. \]

By the Burkholder-Davis-Gundy inequality (for example, see Sect. 3 of Chap. VII of Dellacherie and Meyer [12]) and Young’s inequality (i.e. \( ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \ a \geq 0, \ b \geq 0, \ p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1, \) see, e.g., Kuang [18, page 136]), we have
\[
E[|M_t|] \leq E \left[ \left( \int_0^\infty \left| Y_s - Y_s^2 \right|^{2p-2} \left| Z_s - Z_s^2 \right|^2 ds \right)^{\frac{1}{2}} \right] 
\leq E \left[ \sup_{s \geq 0} \left| Y_s - Y_s^2 \right|^{p-1} \left( \int_0^\infty \left| Z_s - Z_s^2 \right|^2 ds \right)^{\frac{1}{2}} \right] 
\leq \frac{p-1}{p} E \left[ \sup_{s \geq 0} \left| Y_s - Y_s^2 \right|^p \right] \]
\[
\leq \frac{p-1}{p} E \left[ \sup_{s \geq 0} \left| Y_s - Y_s^2 \right|^p \right] + \frac{1}{p} E \left[ \left( \int_0^\infty \left| Z_s - Z_s^2 \right|^2 ds \right)^\frac{2}{p} \right] < \infty.
\]

It then follows that \( \{M_t\}_{t \geq 0} \) is a martingale. For notational simplification, let
\[
X := \int_0^\infty \left| Y_s - Y_s^2 \right|^{p-2} 1_{(Y_s - Y_s^2 \neq 0)} \left| Z_s - Z_s^2 \right|^2 ds.
\]

Coming back to inequality (14), we get both
\[
\frac{p(p-1)}{4} E[X] \leq E \left[ \left| \xi_1 - \xi_2 \right|^p \right] + p E \left[ \int_0^\infty \left| Y_s - Y_s^2 \right|^{p-1} e^{\int_0^s \theta(r) dr} \left| \phi_1(s) - \phi_2(s) \right| ds \right] \tag{15}
\]
and
\[
E \left[ \sup_{s \geq 0} \left| Y_s - Y_s^2 \right|^p \right] \leq E \left[ \left| \xi_1 - \xi_2 \right|^p \right] + p \int_0^\infty \left| Y_s - Y_s^2 \right|^{p-1} e^{\int_0^s \theta(r) dr} \left| \phi_1(s) - \phi_2(s) \right| ds \tag{16}
\]
where \( D_p \) is a positive constant depending only on \( p. \) Applying the Burkholder-Davis-Gundy inequality (for example, see Sect. 3 of Chap. VII of Dellacherie and Meyer [12]) again, we have
\[
D_p E[|M_\infty|] \leq D_p E \left[ \left( \int_0^\infty \left| Y_s - Y_s^2 \right|^{2p-2} \left| Z_s - Z_s^2 \right|^2 ds \right)^{\frac{1}{2}} \right] 
\leq D_p E \left[ \sup_{s \geq 0} \left| Y_s - Y_s^2 \right| \left( \int_0^\infty \left| Y_s - Y_s^2 \right|^{p-2} \left| Z_s - Z_s^2 \right|^{2p} ds \right)^{\frac{1}{2}} \right] 
\leq \frac{1}{2} E \left[ \sup_{s \geq 0} \left| Y_s - Y_s^2 \right|^p \right] + \frac{D_p^2}{2} E[X].
\]

It then follows from (15) and (16) that
\[
E \left[ \sup_{s \geq 0} \left| Y_s - Y_s^2 \right|^p \right] \leq K_p E \left[ \left| \xi_1 - \xi_2 \right|^p \right] + p \int_0^\infty \left| Y_s - Y_s^2 \right|^{p-1} e^{\int_0^s \theta(r) dr} \left| \phi_1(s) - \phi_2(s) \right| ds \tag{17}
\]

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where $K_p$ is a positive constant depending only on $p$. Applying once again Young’s inequality, we get
\[
pK_p E \left[ \int_0^\infty |Y_s^1 - Y_s^2|^{p-1} e^{\int_0^s \theta(r) dr} |\phi_1(s) - \phi_2(s)| ds \right] \\
\leq pK_p E \sup_{s \geq 0} |Y_s^1 - Y_s^2|^{p-1} \int_0^\infty e^{\int_0^s \theta(r) dr} |\phi_1(s) - \phi_2(s)| ds \\
\leq \frac{1}{2} E \left[ \sup_{s \geq 0} |Y_s^1 - Y_s^2|^p \right] + M_p E \left[ \left( \int_0^\infty e^{\int_0^s \theta(r) dr} |\phi_1(s) - \phi_2(s)| ds \right)^p \right] \\
\leq \frac{1}{2} E \left[ \sup_{s \geq 0} |Y_s^1 - Y_s^2|^p \right] + M_p \left( e^{\int_0^\infty \theta(s) ds} \right)^p E \left[ \left( \int_0^\infty |\phi_1(s) - \phi_2(s)| ds \right)^p \right],
\]
where $M_p$ is a positive constant depending only on $p$. From this, we deduce that
\[
E \left[ \sup_{s \geq 0} |Y_s^1 - Y_s^2|^p \right] \leq C' E \left[ |\xi_1 - \xi_2|^p + \left( \int_0^\infty |\phi_1(s) - \phi_2(s)| ds \right)^p \right],
\]
where $C'$ is a positive constant depending only on $p$.

Combining (9) with (18), we get
\[
E \left[ \sup_{s \geq 0} |Y_s^1 - Y_s^2|^p + \left( \int_0^\infty |Z_s^1 - Z_s^2|^2 ds \right)^{\frac{p}{2}} \right] \\
\leq C_p E \left[ |\xi_1 - \xi_2|^p + \left( \int_0^\infty |\phi_1(s) - \phi_2(s)| ds \right)^p \right],
\]
where $C_p$ is a positive constant depending only on $p$. The proof of Lemma 3.1 is complete. \qed

**Lemma 3.2** ([5]) Let $\xi \in L^2(\Omega, \mathcal{F}, P)$ be given. Suppose that (A.1) and (A.2) hold for $g$, then BSDE
\[
Y_t = \xi + \int_t^\infty g(s, Y_s, Z_s) ds - \int_t^\infty Z_s dW_s
\]
has a unique solution $(Y, Z) \in S^2(\mathbb{R}) \times L^2(\mathbb{R})$.

**Proof of Theorem 3.1.** We prove this theorem in two steps.

**Step 1.** We prove the existence and uniqueness to BSDE (19). Let $\xi^n := (\xi \wedge n) \vee (-n)$ and $g_n(t, y, z) := g(t, y, z) - g(t, 0, 0) + f_n(g(t, 0, 0))$, where $f_n(g(t, 0, 0)) := \frac{g(t, 0, 0) - g(t, 0, 0)}{|g(t, 0, 0)|}$, if $t \leq T$; $f_n(g(t, 0, 0)) = g(t, 0, 0)$, if $t > T$. It is easy to check that for each $n$, the function $g_n$ satisfies (A.1) and (A.2). Then by Lemma 3.2, BSDE
\[
Y_t^n = \xi^n + \int_t^\infty g_n(s, Y_s^n, Z_s^n) ds - \int_t^\infty Z_s^n dW_s
\]
has a unique solution $(Y^n, Z^n) \in S^2(\mathbb{R}) \times L^2(\mathbb{R})$. Using Lemma 3.1, we have
\[
E \left[ \sup_{t \geq 0} |Y_t^{n+m} - Y_t^n|^p + \left( \int_0^\infty |Z_s^{n+m} - Z_s^n|^2 ds \right)^{\frac{p}{2}} \right] \\
\leq C_p E \left[ |\xi^{n+m} - \xi^n|^p + \left( \int_0^\infty |f_n+m(g(s, 0, 0)) - f_n(g(s, 0, 0))| ds \right)^p \right].
\]
The right-hand side of the above inequality clearly tends to 0, as \( n \to \infty \), uniformly in \( m \), so we have a Cauchy sequence and the limit is a solution to BSDE (19). Let us consider \((Y,Z)\) and \((Y',Z')\) to be two solutions to BSDE (19). Using Lemma 3.1 again, we get immediately \((Y,Z) = (Y',Z')\).

**Step 2.** Let \( \hat{\xi} := \xi + V_\infty \) and \( \hat{Y}_t := Y_t + V_t \), then BSDE (2) can be rewritten as

\[
\hat{Y}_t = \hat{\xi} + \int_t^\infty \hat{g}(s, \hat{Y}_s, \hat{Z}_s) \, ds - \int_t^\infty \hat{Z}_s \, dW_s,
\]

where \( \hat{g}(t, y, z) := g(t, y - V_t, z) \). It is easy to check that \( \hat{g}(t, y, z) \) satisfies (A.2), (A.3) and \( \hat{\xi} \in L^p(\Omega, \mathcal{F}, P) \).

By Step 1, there exists a unique pair \((\hat{Y}, \hat{Z})\) of adapted processes in \( S^p(\mathbb{R}) \times L^p(\mathbb{R}) \) solving BSDE (20). Using the fact \(|Y_t|^p \leq 2^p(|\hat{Y}_t|^p + |V_t|^p)\), we have \((Y, Z) \in S^p(\mathbb{R}) \times L^p(\mathbb{R})\). The proof of Theorem 3.1 is complete.

**Remark 3.1** If \( g(t, 0, 0) \equiv 0 \), then by Theorem 3.1, we have: Under assumptions (A.2) and (A.4), for each given \( \xi \in L(\Omega, \mathcal{F}, P) \), BSDE (2) has a unique solution \((Y, Z) \in S(\mathbb{R}) \times L(\mathbb{R})\).

**Example 3.1** Suppose that \( 1 < p < 2 \). Consider the BSDE:

\[
Y_t = \exp\left(\frac{W_t^2}{2p} - W_t\right) 1_{\{W_t \geq p\}} + \int_t^\infty \frac{1}{(1+s)^2} (Y_s + Z_s) \, ds - \int_t^\infty Z_s \, dW_s.
\]

For notational simplification, let \( \xi := \exp\left(\frac{W_t^2}{2p} - W_t\right) 1_{\{W_t \geq p\}} \), \( g(t, y, z) := \frac{1}{(1+t)^2}(y + z) \), \( \alpha(t) := \frac{1}{(1+t)^2} \), \( \beta(t) := \frac{1}{(1+t)^p} \). Obviously, \( g \) satisfies (A.2) and (A.3). On the other hand,

\[
E[|\xi|^p] = \int_p^\infty \exp\left(\frac{x^2}{2} - px\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx = \frac{1}{\sqrt{2\pi p}} e^{-p^2} < \infty,
\]

and

\[
E[|\xi|^2] = \infty.
\]

Hence, \( \xi \in L^p(\Omega, \mathcal{F}, P) \), \( \xi \notin L^2(\Omega, \mathcal{F}, P) \). But by Theorem 3.1, we have: BSDE (21) has a unique solution \((Y, Z) \in S^p(\mathbb{R}) \times L^p(\mathbb{R})\).

The following comparison theorem is very useful. Since the proof is very similar to that of Theorem 2.2 in [13], we omit it.

**Theorem 3.2 (Comparison Theorem)** We make the same assumptions as in Theorem 3.1. Let \((\overline{Y}, \overline{Z})\) be the solution of the BSDE

\[
\overline{Y}_t = \overline{\xi} + \int_t^\infty \overline{g}(s, \overline{Y}_s, \overline{Z}_s) \, ds + \overline{V}_\infty - \overline{V}_t - \int_t^\infty \overline{Z}_s \, dW_s,
\]

where \( \overline{g}(t, y, z) \) satisfies (A.2) and (A.3), \( \overline{V}_t \) satisfies (A.4) and \( \overline{\xi} \in L^p(\Omega, \mathcal{F}, P) \). If we suppose that

\[
\dot{\xi} := \xi - \overline{\xi} \geq 0, \quad \dot{g}_t := g(t, \overline{Y}_t, \overline{Z}_t) - \overline{g}(t, \overline{Y}_t, \overline{Z}_t) \geq 0, \quad a.s.,
\]

\( V_t := V_t - \overline{V}_t \) is an RCLL increasing process,
then
\[ Y_t \geq \mathbf{Y}_t, \quad \text{a.s., \ \forall t \in [0, \infty)}. \]
Moreover, if \( P(\xi > 0) > 0 \), then \( P(Y_t > \mathbf{Y}_t) > 0 \), for all \( t \geq 0 \). In particular, \( Y_0 > \mathbf{Y}_0 \).

4. Generalized \( g \)-expectation and generalized \( g \)-martingale

In this section, we make an additional assumption on the function \( g \):

(A.5) \( g(\cdot, y, 0) \equiv 0, \ \forall y \in \mathbb{R} \).

For any given \( g \), the solution \((Y, Z)\) of BSDE (19) depends on terminal value \( \xi \). Referring to Definition 36.1 in [27] or Definition 3.1 in [14], now we introduce the definitions of generalized \( g \)-expectation and generalized conditional \( g \)-expectation via the solution of BSDE (19).

**Definition 4.1 (Generalized \( g \)-expectation)** Suppose \( g \) satisfies (A.2) and (A.5). For any \( \xi \in L(\Omega, \mathcal{F}, P) \), let \((Y, Z)\) be the solution of BSDE (19). Consider the mapping \( E_g[\cdot] : L(\Omega, \mathcal{F}, P) \mapsto \mathbb{R} \) denoted by \( E_g[\xi] := Y_0 \). We call \( E_g[\xi] \) generalized \( g \)-expectation of \( \xi \).

**Definition 4.2 (Generalized conditional \( g \)-expectation)** Suppose \( g \) satisfies (A.2) and (A.5). Generalized conditional \( g \)-expectation of \( \xi \) with respect to \( \mathcal{F}_t \) is defined by
\[ E_g[\xi|\mathcal{F}_t] := Y_t. \]

Generalized \( g \)-expectation has the following property.

**Proposition 4.1** \( E_g[\xi|\mathcal{F}_t] \) is the unique random variable \( \eta \) in \( L(\Omega, \mathcal{F}_t, P) \) such that
\[ E_g[1_A \xi] = E_g[1_A \eta], \quad \forall A \in \mathcal{F}_t. \]

By Theorem 3.2 and (A.5), we can prove Proposition 4.1 by using the same method as that of Proposition 36.4 in [27], so we omit the proof.

The following proposition will tell us that generalized conditional \( g \)-expectations that we introduced meet some basic properties of Peng’s conditional \( g \)-expectations.

**Proposition 4.2** Suppose \( \xi, \xi_1, \xi_2 \in L(\Omega, \mathcal{F}, P) \), then

(i) If \( \xi \) is \( \mathcal{F}_t \)-measurable, then \( E_g[\xi|\mathcal{F}_t] = \xi \);

(ii) For all stopping times \( \tau \) and \( \sigma \), \( E_g[\xi|\mathcal{F}_t]\mathcal{F}_\sigma = E_g[\xi|\mathcal{F}_{\tau \wedge \sigma}] \);

(iii) If \( \xi_1 \geq \xi_2 \) a.s., then \( E_g[\xi_1|\mathcal{F}_t] \geq E_g[\xi_2|\mathcal{F}_t] \); if, moreover, \( P(\xi_1 > \xi_2) > 0 \), then
\[ P(E_g[\xi_1|\mathcal{F}_t] > E_g[\xi_2|\mathcal{F}_t]) > 0; \]

(iv) For each \( B \in \mathcal{F}_t \), \( E_g[1_B \xi|\mathcal{F}_t] = 1_B E_g[\xi|\mathcal{F}_t] \);

(v) If \( g \) does not depend on \( y \), then for any \((\xi, \eta) \in L(\Omega, \mathcal{F}, P) \times L(\Omega, \mathcal{F}_t, P)\),
\[ E_g[\xi + \eta|\mathcal{F}_t] = E_g[\xi|\mathcal{F}_t] + \eta. \]

By Theorem 3.2 and using the similar arguments as that of Lemma 36.6 in [27] and Lemma 4.2 in [2], we can prove Proposition 4.2.

Now we shall prove the stability theorem of generalized \( g \)-expectations.
Theorem 4.1 (Stability Theorem) Suppose \( g \) satisfies (A.2) and (A.5). For \( \xi, \eta_n \in \mathcal{L}(\Omega, \mathcal{F}, P) \), where \( n = 1, 2, \ldots \), if \( E[|\xi - \eta_n|^p|\mathcal{F}_t] \to 0 \), a.s., \( t \in [0, \infty) \), then

\[
\lim_{n \to \infty} E_g[\eta_n|\mathcal{F}_t] = E_g[\xi|\mathcal{F}_t], \quad \text{a.s., \ } t \in [0, \infty).
\]

Proof From Theorem 3.1, we know that

\[
E_g[\eta_n|\mathcal{F}_t] = \eta_n + \int_0^t g(s, E_g[\eta_n|\mathcal{F}_s], Z^n_s) \, ds - \int_0^t Z^n_s \, dW_s, \quad n = 1, 2, \ldots,
\]

Then

\[
E_g[\xi|\mathcal{F}_t] - E_g[\eta_n|\mathcal{F}_t] = \xi - \eta_n + \int_0^t \left[ a_s (E_g[\xi|\mathcal{F}_s] - E_g[\eta_n|\mathcal{F}_s]) + b_s (Z_s - Z^n_s) \right] \, ds
\]

where

\[
a_s := \frac{g(s, E_g[\xi|\mathcal{F}_s], Z_s) - g(s, E_g[\eta_n|\mathcal{F}_s], Z^n_s)}{(E_g[\xi|\mathcal{F}_s] - E_g[\eta_n|\mathcal{F}_s] \neq 0)},
\]

\[
b_s := \frac{g(s, E_g[\eta_n|\mathcal{F}_s], Z^n_s) - g(s, E_g[\xi|\mathcal{F}_s], Z_s)}{(Z_s - Z^n_s \neq 0)},
\]

which imply \( |a_s| \leq a(t), \ |b_s| \leq b(t) \).

Relation (22) can be rewritten as follows:

\[
E_g[\xi|\mathcal{F}_t] - E_g[\eta_n|\mathcal{F}_t] = \xi - \eta_n + \int_t^\infty a_s (E_g[\xi|\mathcal{F}_s] - E_g[\eta_n|\mathcal{F}_s]) \, ds - \int_t^\infty (Z_s - Z^n_s) \, d\overline{W}_s,
\]

where \( \overline{W}_t = W_t - \int_0^t b_s \, ds \). By the Girsanov theorem, we know that \( (\overline{W}_t)_{t \geq 0} \) is \( Q^b \)-Brownian motion, where

\[
\frac{dQ^b}{dP} = e^{-\int_0^\infty |b_s|^2 \, ds + \int_0^\infty b_s \, dW_s}.
\]

Solving (23), we obtain

\[
E_g[\xi|\mathcal{F}_t] - E_g[\eta_n|\mathcal{F}_t] = (\xi - \eta_n) e^{\int_t^\infty a_s \, ds} - \int_t^\infty (Z_s - Z^n_s) e^{\int_s^\infty a_r \, dr} \, d\overline{W}_s.
\]

By the Burkholder-Davis-Gundy inequality (for example, see Sect. 3 of Chap. VII of Dellacherie and Meyer [12]), Hölder’s inequality and noting the fact that

\[
E \left[ e^{-\frac{1}{2} \int_0^\infty |b_s|^2 \, ds + \int_0^\infty b_s \, dW_s} \right] = 1
\]

and

\[
E \left[ e^{-\frac{1}{2} \int_0^\infty |a_s|^2 \, ds + \int_0^\infty a_s \, d\overline{W}_s} \right] = 1,
\]

we have

\[
E_{Q^b} \left[ \int_0^t (Z_s - Z^n_s) e^{\int_s^t a_r \, dr} \, d\overline{W}_s \right] \leq e^{\int_0^t a(t) \, dt} E\left[ \left( \int_0^\infty (Z_s - Z^n_s)^2 \, ds \right)^{\frac{1}{2}} \right]
\]

\[
\leq e^{\frac{1}{2} \int_0^t a(t) \, dt} \left( E \left[ \left( \int_0^\infty (Z_s - Z^n_s)^2 \, ds \right)^{\frac{1}{2}} \right] \right)^{\frac{1}{2}} \left( E \left[ \left( \int_0^\infty (Z_s - Z^n_s)^2 \, ds \right)^{\frac{1}{2}} \right] \right)^{\frac{1}{2}}
\]

\[
\leq e^{\left( \frac{1}{2} (q-1) \int_0^t b^2(t) \, dt + \int_0^t a(t) \, dt \right)} \left( E \left[ \left( \int_0^\infty (Z_s - Z^n_s)^2 \, ds \right)^{\frac{1}{2}} \right] \right)^{\frac{1}{2}}
\]

\[
< \infty.
\]
By Hölder’s inequality, we obtain

\[ E_{Q^b} \left[ \int_0^t (Z_s - Z^n_s) e^{\int_0^s a_s \, dr} \, dW_r \right]_{t \geq 0} = 0. \]

Taking conditional expectation \( E_{Q^b} \cdot [\mathcal{F}_t] \) on both sides of (24), we have

\[ E_g [\xi | \mathcal{F}_t] - E_g [\eta_n | \mathcal{F}_t] = E_{Q^b} \left[ (\xi - \eta_n) e^{\int_0^t a_s \, ds} | \mathcal{F}_t \right]. \]

Note that \( |a_t| \leq \alpha(t) \) and hence

\[ |E_g [\xi | \mathcal{F}_t] - E_g [\eta_n | \mathcal{F}_t]| \leq e^{\int_0^t \alpha(t) \, dt} E_{Q^b} [|\xi - \eta_n| | \mathcal{F}_t]. \]

By Hölder’s inequality, we obtain

\[ E_{Q^b} [|\xi - \eta_n| | \mathcal{F}_t] = \frac{E \left[ |\xi - \eta_n| \frac{dQ^b}{dP} | \mathcal{F}_t \right]}{E \left[ \frac{dQ^b}{dP} | \mathcal{F}_t \right]} \leq \frac{\left( E \left[ |\xi - \eta_n|^p | \mathcal{F}_t \right] \right)^{\frac{1}{p}}}{E \left[ \frac{dQ^b}{dP} | \mathcal{F}_t \right]} \leq \frac{\left( E \left[ \frac{dQ^b}{dP} | \mathcal{F}_t \right] \right)^{\frac{1}{q}}}{E \left[ \frac{dQ^b}{dP} | \mathcal{F}_t \right]}. \]

Since \( \left( e^{-\frac{1}{2} \int_0^t |b_s|^2 \, ds + \int_0^t b_s \, dW_s} \right)_{t \geq 0} \) and \( \left( e^{-\frac{1}{2} \int_0^t |q_s|^2 \, ds + \int_0^t q_s \, dW_s} \right)_{t \geq 0} \) are both martingales with respect to \( (\mathcal{F}_t)_{t \geq 0} \), hence

\[ \frac{\left( E \left[ \frac{dQ^b}{dP} | \mathcal{F}_t \right] \right)^{\frac{1}{q}}}{E \left[ \frac{dQ^b}{dP} | \mathcal{F}_t \right]} \leq \frac{\left( E \left[ (\frac{dQ^b}{dP})^q | \mathcal{F}_t \right] \right)^{\frac{1}{q}}}{E \left[ \frac{dQ^b}{dP} | \mathcal{F}_t \right]} \leq \frac{\left( E \left[ (\frac{dQ^b}{dP})^q | \mathcal{F}_t \right] \right)^{\frac{1}{q}}}{E \left[ \frac{dQ^b}{dP} | \mathcal{F}_t \right]}. \]

Thus for all \( t \in [0, \infty), \)

\[ |E_g [\xi | \mathcal{F}_t] - E_g [\eta_n | \mathcal{F}_t]| \leq e^{\frac{1}{2} (q-1) \int_0^t \beta^2(s) \, ds + \int_0^t \beta^x(s) \, ds} \leq e^{\frac{1}{2} (q-1) \int_0^t \beta^2(s) \, ds}. \]

Noting that \( E[|\xi - \eta_n|^p | \mathcal{F}_t] \rightarrow 0 \), as \( n \rightarrow \infty, \ t \in [0, \infty) \), then

\[ |E_g [\xi | \mathcal{F}_t] - E_g [\eta_n | \mathcal{F}_t]| \rightarrow 0, \ \text{as} \ n \rightarrow \infty. \]

The proof of Theorem 4.1 is complete. \( \square \)

**Remark 4.1**

(i) In Theorem 4.1, if we replace (A.5) by (A.3), the following result \( \lim_{n \rightarrow \infty} Y^n_t = Y_t, \ a.s., \ t \in [0, \infty) \) holds.

(ii) For any \( \xi \in L(\Omega, \mathcal{F}, P) \), let \( \xi^n := (\xi \wedge n) \vee (-n), \ n = 1, 2, \ldots \), then by Theorem 4.1, we have:

\[ \lim_{n \rightarrow \infty} E_g [\xi^n | \mathcal{F}_t] = E_g [\xi | \mathcal{F}_t], \ \text{a.s.,} \ \forall t \in [0, \infty). \]

(iii) By the proof of Theorem 4.1, we have: if \( \xi \in L^p(\Omega, \mathcal{F}, P) \), then there exists a constant \( C > 0 \) such that \( E_g |\xi| | \mathcal{F}_t| \leq C (E[|\xi|^p | \mathcal{F}_t])^{\frac{1}{p}}, \ \forall t \in [0, \infty). \)

At the end of the paper, we introduce the definition of generalized \( g \)-martingale (resp. generalized \( g \)-supermartingale, generalized \( g \)-submartingale).
Definition 4.3 Suppose \( g \) satisfies (A.2) and (A.5). A process \((X_t)_{t \geq 0}\) satisfying that for each \( t \), \( X_t \in \mathcal{L}(\Omega, \mathcal{F}_t, P)\) is called a generalized \( g \)-martingale (resp. generalized \( g \)-supermartingale, generalized \( g \)-submartingale), if for any \( t \) and \( s \) satisfying \( t \leq s \),

\[
\mathcal{E}_g[X_s | \mathcal{F}_t] = X_t \quad (\text{resp. } \leq X_t, \geq X_t), \quad \text{a.s.}
\]

Example 4.1 Suppose that \( \xi \in \mathcal{L}(\Omega, \mathcal{F}, P)\) and \((A_t)_{t \geq 0}\) is an RCLL increasing process with \((A_t)_{t \geq 0} \in \mathcal{S}^2(\mathbb{R})\). Consider the BSDE:

\[
Y_t = \xi + \int_t^\infty \frac{1}{(1+s)^2} |Z_s| ds + A_\infty - A_t - \int_t^\infty Z_s dW_s. \quad (26)
\]

Let \( g(t, y, z) := \frac{1}{(1+t)^2} |z| \). Obviously, \( g \) satisfies (A.2) and (A.5). By Theorem 3.2, for any \( t \) and \( s \) satisfying \( t \leq s \), \( \mathcal{E}_g[Y_s | \mathcal{F}_t] \leq Y_t \), a.s.. Thus \((Y_t)_{t \geq 0}\) is a generalized \( g \)-supermartingale.

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