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Half inverse problem for Sturm-Liouville operators with boundary conditions dependent on the spectral parameter

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Abstract: In this paper, we discuss the half inverse problem for the Sturm-Liouville operator with boundary conditions dependent on the spectral parameter and show that if $q(x)$ is prescribed on $[\frac{\pi}{2},\pi]$, then one spectrum is sufficient to determine the potential $q(x)$ on the whole interval $[0,\pi]$ and coefficient function $\frac{a_1\lambda+b_1}{c_1\lambda+d_1}$ of the boundary condition.

Key words: Half inverse problem, Sturm-Liouville operator, boundary condition dependent on the spectral parameter

1. Introduction

Consider the following Sturm-Liouville operator L defined by

$$Ly=-y''+q(x)y=\lambda y, (x\in[0,\pi]) \tag{1.1}$$

with boundary conditions dependent on the spectral parameter

$$y'(0,\lambda)-hy(0,\lambda)=0 \tag{1.2}$$

or

$$(a_1\lambda+b_1)y(0,\lambda)-(c_1\lambda+d_1)y'(0,\lambda)=0, \tag{1.2'}$$

and

$$y'(\pi,\lambda)+Hy(\pi,\lambda)=0 \tag{1.3}$$

or

$$(a_2\lambda+b_2)y(\pi,\lambda)-(c_2\lambda+d_2)y'(\pi,\lambda)=0, \tag{1.3'}$$

where $h,H,a_k,b_k,c_k,d_k\in\mathbf{R}, c_1c_2\neq 0$ such that

$$(-1)^k\delta_k=a_kd_k-b_kc_k>0 (k=1,2)$$

and q is a real-valued function and $q\in L^1[0,\pi]$.

Let $B(q,h,H)$, $B(q,h)$ and $B(q)$ be the Sturm-Liouville problem (1.1)–(1.3), the Sturm-Liouville problem (1.1), (1.2), (1.3′) and the Sturm-Liouville problem (1.1), (1.2′), (1.3′), respectively.

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Sturm-Liouville operators with boundary conditions dependent on the spectral parameter were originated from engineering, physics and mathematics and have received substantial attention (see [1–3, 6, 8, 9, 11]). Fulton [8] discussed the Sturm-Liouville problem $B(q,h)$ and obtained the eigenfunction expansion and asymptotic estimates of eigenvalues. Binding, Browne and Seddighi [2] considered the Sturm-Liouville operator l satisfying

$$ly = -(py')' + qy = \lambda ry$$

and boundary conditions $(1.2')$ and $(1.3')$. They got oscillation and comparison results as well as the asymptotic estimates of eigenvalues, which can be considered as extension of Fulton's results. Browne and Sleeman [3] discussed the inverse nodal problem for the Sturm-Liouville problem $B(q,h)$ and showed that a dense set of nodal points of eigenfunctions for the Sturm-Liouville problem $B(q,h)$ is sufficient to determine the potential q and coefficient h of the boundary condition. Guliyev [11] considered the regularized trace problem for the Sturm-Liouville equation with spectral parameter in the boundary conditions and obtained the trace formula.

Inverse problem for Sturm-Liouville operators consists of reconstruction of the operator by its spectral data (see [3, 4, 10, 12–14, 17–23]). Half inverse problem for Sturm-Liouville operators is to determine a differential operator by one spectrum and half of its potential (see [4, 10, 13, 14, 17–19]). Hochstadt and Lieberman [13] firstly considered the half inverse problem for the Sturm-Liouville problem $B(q,h,H)$ and showed that if $q(x)$ is prescribed on $[\frac{\pi}{2},\pi]$, then the potential $q(x)$ on the interval $[0,\pi]$ can be uniquely determined by one spectrum. Castillo [4] also discussed the half inverse problem for the Sturm-Liouville problem $B(q,h,H)$. By an example, Castillo showed that the necessity of the boundary condition (1.3) is given. Koyunbakan and Panakhov [14] considered the half inverse problem for diffusion operators on the finite interval $[0,\pi]$ and showed that if $p(x)$ is known on the $[0,\pi]$, and $q(x)$ is prescribed on $[\frac{\pi}{2},\pi]$, then only one spectrum is sufficient to determine the potential $q(x)$ on the interval $[0,\frac{\pi}{2})$ for the diffusion operator on the finite interval $[0,\pi]$. Hryniv and Mykytyuk [17] studied the half inverse spectral problems for Sturm-Liouville operators with singular potentials. Sakhnovich [18] proved the existence of the solution for the half inverse problem of Sturm-Liouville problems and gave a method of reconstructing this solution under some conditions. Using Weyl m-function techniques, Gesztesy and Simon [10] established a uniqueness theorem (see [10, Theorem 1.3]) by partial spectra and information on the potential, which is a generalization of Hochstadt and Lieberman's theorem [13]. In 2009, Wei and Xu [19] considered the half inverse problem missing one eigenvalue and obtained a uniqueness theorem on the potential q.

In this paper, we discuss the half inverse problem for the Sturm-Liouville problem $B(q)$ and show that if $q(x)$ is prescribed on $[\frac{\pi}{2},\pi]$, then the potential $q(x)$ on the whole interval $[0,\pi]$ and coefficient function $\frac{a_1\lambda+b_1}{c_1\lambda+d_1}$ of the boundary condition can be uniquely determined by one spectrum. Although the technique is based on the Hochstadt and Lieberman's methods, our model is different from that in the reference [13].

The following two Lemmas are important to prove our results.

Lemma 1.1 ([**2, Theorem 4.2**]) *Let $\sigma(L) = \{\lambda_n\}_{n=0}^{\infty}$ be the spectrum of the Sturm-Liouville problem $B(q)$, then $\lambda_n (n = 0,1,2,\cdots)$ is real and simple and satisfies the relations*

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \to +\infty$$

and

$$\sqrt{\lambda}_n = n[1 + O(\frac{1}{n})]. \tag{1.4}$$

Suppose $\varphi(x), \theta(x)$ are the two fundamental solutions of the equation (1.1) and satisfy

$$\varphi(0) = 1,\ \varphi'(0) = 0$$

and

$$\theta(0) = 0,\ \theta'(0) = 1,$$

respectively, then the solution of the equation (1.1) with boundary condition $(1.2')$ is

$$y(x, \lambda) = (c_1\lambda + d_1)\varphi(x) + (a_1\lambda + b_1)\theta(x). \tag{1.5}$$

By the transformations (see [15, p. 144–145 or 7, p. 20–22]), we have

$$\begin{array}{l} \varphi(x) = \cos\sqrt{\lambda}x + \int_0^x A(x,t)\cos\sqrt{\lambda}tdt \\ \theta(x) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}}\int_0^x B(x,t)\sin\sqrt{\lambda}tdt, \end{array} \tag{1.6}$$

where the kernels $A(x,t)$ and $B(x,t)$ satisfy

$$\frac{\partial^2 K(x,t)}{\partial x^2} - q(x)K(x,t) = \frac{\partial^2 K(x,t)}{\partial t^2},$$

where $q(x) = 2\frac{d}{dx}A(x,x) = 2\frac{d}{dx}B(x,x)$, $\frac{\partial A(x,t)}{\partial t}|_{t=0} = B(x,0) = 0$.

By virtue of (1.5) and (1.6), this yields the following lemma.

Lemma 1.2 ([7, 15]) *The solution of the equation (1.1) with boundary condition* $(1.2')$ *can be written as*

$$\begin{array}{rl} y(x,\lambda) = & (c_1\lambda + d_1)[\cos\sqrt{\lambda}x + \int_0^x A(x,t)\cos\sqrt{\lambda}tdt] \\ & +(a_1\lambda + b_1)[\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}}\int_0^x B(x,t)\sin\sqrt{\lambda}tdt]. \end{array} \tag{1.7}$$

2. Main results and Proofs

Consider another Sturm-Liouville operator $\tilde{L}$ satisfying

$$\tilde{L}y = -y'' + \tilde{q}(x)y = \lambda y \tag{2.1}$$

with boundary conditions dependent on the spectral parameter

$$(\tilde{a}_1\lambda + \tilde{b}_1)y(0,\lambda) - (\tilde{c}_1\lambda + \tilde{d}_1)y'(0,\lambda) = 0 \tag{2.2}$$

and

$$(a_2\lambda + b_2)y(\pi,\lambda) - (c_2\lambda + d_2)y'(\pi,\lambda) = 0, \tag{2.3}$$

where $\tilde{a}_1, \tilde{b}_1, \tilde{c}_1, \tilde{d}_1, a_2, b_2, c_2, d_2 \in \mathbf{R}, \tilde{c}_1 c_2 \neq 0$ such that

$$\tilde{\delta}_1 = \tilde{a}_1\tilde{d}_1 - \tilde{b}_1\tilde{c}_1 < 0, \delta_2 = a_2 d_2 - b_2 c_2 > 0$$

and $\tilde{q}$ is a real-valued function and $\tilde{q} \in L^1[0,\pi]$.

Denote the Sturm-Liouville problem (2.1)-(2.3) by $B(\tilde{q})$. We have the following main results.

Theorem 2.1 *Let $\{\lambda_n\}_{n=0}^{\infty}$ be real and simple spectrum of the Sturm-Liouville problem $B(q)$ and Let $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ be real and simple spectrum of the Sturm-Liouville problem $B(\tilde{q})$, respectively. If $\lambda_n = \tilde{\lambda}_n (n = 0, 1, 2, \cdots)$, $q(x) = \tilde{q}(x)$ on $[\frac{\pi}{2}, \pi]$ and $q, \tilde{q} \in W_2^6[0, \pi]$, then*

$$q(x) = \tilde{q}(x) \quad a.e. \quad on \ [0, \pi],$$

and

$$\frac{\tilde{a}_1\lambda + \tilde{b}_1}{\tilde{c}_1\lambda + \tilde{d}_1} = \frac{a_1\lambda + b_1}{c_1\lambda + d_1}, \ \forall \lambda \in \mathbf{C},$$

where q, $\tilde{q}$ are real-valued functions.

Remark 2.2 *We use the condition $q, \tilde{q} \in W_2^6[0, \pi]$ in the proof of Theorem 2.1 (see below).*

Proof According to Lemma 1.2, the solutions of the equation (1.1) with boundary condition (1.2′) and the equation (2.1) with boundary condition (2.2) can be expressed in the integral forms

$$\begin{aligned} y(x,\lambda) = \ & (c_1\lambda + d_1)[\cos\sqrt{\lambda}x + \textstyle\int_0^x A(x,t)\cos\sqrt{\lambda}tdt] \\ & +(a_1\lambda + b_1)[\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}}\textstyle\int_0^x B(x,t)\sin\sqrt{\lambda}tdt] \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \tilde{y}(x,\lambda) = \ & (\tilde{c}_1\lambda + \tilde{d}_1)[\cos\sqrt{\lambda}x + \textstyle\int_0^x \tilde{A}(x,t)\cos\sqrt{\lambda}tdt] \\ & +(\tilde{a}_1\lambda + \tilde{b}_1)[\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}}\textstyle\int_0^x \tilde{B}(x,t)\sin\sqrt{\lambda}tdt], \end{aligned} \tag{2.5}$$

respectively. Let $\lambda = s^2$, from (2.4) and (2.5), we have

$$\begin{aligned} y\tilde{y} = \ & (c_1s^2 + d_1)(\tilde{c}_1s^2 + \tilde{d}_1)(\cos sx + \textstyle\int_0^x A(x,t)\cos stdt) \\ & \cdot(\cos sx + \textstyle\int_0^x \tilde{A}(x,t)\cos stdt) + \frac{(a_1s^2+b_1)(\tilde{a}_1s^2+\tilde{b}_1)}{s^2} \\ & \cdot(\sin sx + \textstyle\int_0^x B(x,t)\sin stdt)(\sin sx + \int_0^x \tilde{B}(x,t)\sin stdt) \\ & +\frac{1}{s}(c_1s^2 + d_1)(\tilde{a}_1s^2 + \tilde{b}_1)\cdot(\cos sx + \textstyle\int_0^x A(x,t)\cos stdt) \\ & \cdot(\sin sx + \textstyle\int_0^x \tilde{B}(x,t)\sin stdt) + \frac{1}{s}(\tilde{c}_1s^2 + \tilde{d}_1)(a_1s^2 + b_1) \\ & \cdot(\cos sx + \textstyle\int_0^x \tilde{A}(x,t)\cos stdt)(\sin sx + \int_0^x B(x,t)\sin stdt). \end{aligned} \tag{2.6}$$

Using the trigonometric addition formulae, extending the range of $A(x,t)$, $\tilde{A}(x,t)$, $B(x,t)$ and $\tilde{B}(x,t)$ with respect to the second argument, respectively. i.e., $A(x,-t) = A(x,t), \tilde{A}(x,-t) = \tilde{A}(x,t)$, $B(x,-t) = -B(x,t)$ and $\tilde{B}(x,-t) = -\tilde{B}(x,t)$, we can find that

$$\begin{aligned} & (\cos sx + \textstyle\int_0^x A(x,t)\cos stdt)(\cos sx + \int_0^x \tilde{A}(x,t)\cos stdt) \\ & = \frac{1}{2}[1 + \cos 2sx + \textstyle\int_0^x k(x,\tau)\cos 2s\tau d\tau], \end{aligned} \tag{2.7}$$

$$\begin{aligned} & (\sin sx + \textstyle\int_0^x B(x,t)\sin stdt)(\sin sx + \int_0^x \tilde{B}(x,t)\sin stdt) \\ & = \frac{1}{2}[1 - \cos 2sx + \textstyle\int_0^x h(x,\tau)\cos 2s\tau d\tau], \end{aligned} \tag{2.8}$$

$$\begin{aligned} & (\cos sx + \textstyle\int_0^x A(x,t)\cos stdt)(\sin sx + \int_0^x \tilde{B}(x,t)\sin stdt) \\ & = \frac{1}{2}[\sin 2sx + \textstyle\int_0^x l(x,\tau)\sin 2s\tau d\tau] \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} & (\cos sx + \textstyle\int_0^x \tilde{A}(x,t)\cos stdt)(\sin sx + \int_0^x B(x,t)\sin stdt) \\ & = \frac{1}{2}[\sin 2sx + \textstyle\int_0^x m(x,\tau)\sin 2s\tau d\tau], \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} k(x,\tau) &= A(x,x-2\tau)+\tilde{A}(x,x-2\tau)+A(x,x+2\tau)+\tilde{A}(x,x+2\tau) \\ &\quad +2[\textstyle\int_{-x+2\tau}^{x} A(x,v)\tilde{A}(x,v-2\tau)ds+\int_{-x}^{x-2\tau} A(x,v)\tilde{A}(x,v+2\tau)dv], \\ h(x,\tau) &= B(x,x+2\tau)+\tilde{B}(x,x+2\tau)-B(x,x-2\tau)-\tilde{B}(x,x-2\tau) \\ &\quad +2[\textstyle\int_{-x+2\tau}^{x} B(x,v)\tilde{B}(x,v-2\tau)dv+\int_{-x}^{x-2\tau} B(x,v)\tilde{B}(x,v+2\tau)dv], \\ l(x,\tau) &= 2[A(x,x-2\tau)+\tilde{B}(x,x-2\tau)] \\ &\quad +2[\textstyle\int_{-x+2\tau}^{x} A(x,v)\tilde{B}(x,v-2\tau)ds+\int_{-x}^{x-2\tau} A(x,v)\tilde{B}(x,v+2\tau)dv] \end{aligned}$$

and

$$\begin{aligned} m(x,\tau) &= 2[B(x,x-2\tau)+\tilde{A}(x,x-2\tau)] \\ &\quad +2[\textstyle\int_{-x+2\tau}^{x} B(x,v)\tilde{A}(x,v-2\tau)ds+\int_{-x}^{x-2\tau} B(x,v)\tilde{A}(x,v+2\tau)dv]. \end{aligned}$$

From (2.7), (2.8), (2.9) and (2.10), we obtain

$$\begin{aligned} y\tilde{y} &= \tfrac{(c_1s^2+d_1)(\tilde{c}_1s^2+\tilde{d}_1)}{2}(1+\cos 2sx+\textstyle\int_0^x k(x,\tau)\cos 2s\tau d\tau) \\ &\quad +\tfrac{(a_1s^2+b_1)(\tilde{a}_1s^2+\tilde{b}_1)}{2s^2}(1-\cos 2sx+\textstyle\int_0^x h(x,\tau)\cos 2s\tau d\tau) \\ &\quad +\tfrac{1}{2s}(c_1s^2+d_1)(\tilde{a}_1s^2+\tilde{b}_1)(\sin 2sx+\textstyle\int_0^x l(x,\tau)\sin 2s\tau d\tau) \\ &\quad +\tfrac{1}{2s}(a_1s^2+b_1)(\tilde{c}_1s^2+\tilde{d}_1)(\sin 2sx+\textstyle\int_0^x m(x,\tau)\sin 2s\tau d\tau), \end{aligned} \tag{2.11}$$

Define the function $\omega(\lambda)$ by

$$\omega(\lambda)=(a_2\lambda+b_2)y(\pi,\lambda)-(c_2\lambda+d_2)y'(\pi,\lambda).$$

From (2.4), we have the following asymptotic forms

$$y(\pi,\lambda)=(c_1\lambda+d_1)\cos\sqrt{\lambda}\pi+O(|\sqrt{\lambda}|e^{|Im\sqrt{\lambda}|\pi})]$$

and

$$y'(\pi,\lambda)=-(c_1\lambda+d_1)\sqrt{\lambda}\sin\sqrt{\lambda}\pi+O(|\lambda|e^{|Im\sqrt{\lambda}|\pi})].$$

Hence,

$$\omega(\lambda)=(c_1\lambda+d_1)(c_2\lambda+d_2)\sqrt{\lambda}\sin\sqrt{\lambda}\pi+O(|\lambda|^2e^{|Im\sqrt{\lambda}|\pi}). \tag{2.12}$$

Zeros of $\omega(\lambda)$ are the eigenvalues of the Sturm-Liouville problem $B(q)$. $\omega(\lambda)$ is an entire function of order $\frac{1}{2}$ of λ. Multiplying (2.1) by y, (1.1) by $\tilde{y}$ and subtracting and integrating from 0 to π, we obtain

$$(\tilde{y}y'-y\tilde{y}')|_0^\pi+\int_0^\pi(\tilde{q}-q)y\tilde{y}dx=0.$$

Using $y(0,\lambda)=c_1\lambda+d_1, \tilde{y}(0,\lambda)=\tilde{c}_1\lambda+\tilde{d}_1$, $y'(0,\lambda)=a_1\lambda+b_1, \tilde{y}'(0,\lambda)=\tilde{a}_1\lambda+\tilde{b}_1$ and $\tilde{q}(x)=q(x), x\in[\frac{\pi}{2},\pi]$, we get

$$\begin{aligned} &[\tilde{y}(\pi,\lambda)y'(\pi,\lambda)-y(\pi,\lambda)\tilde{y}'(\pi,\lambda)]+(a_1\lambda+b_1)(\tilde{c}_1\lambda+\tilde{d}_1) \\ &-(c_1\lambda+d_1)(\tilde{a}_1\lambda+\tilde{b}_1)+\textstyle\int_0^{\frac{\pi}{2}}(\tilde{q}(x)-q(x))y\tilde{y}dx=0. \end{aligned} \tag{2.13}$$

Let

$$Q(x)=\tilde{q}(x)-q(x)$$

and

$$\begin{aligned} K(\lambda) &= (a_1\tilde{c}_1 - \tilde{a}_1 c_1)\lambda^2 + (a_1\tilde{d}_1 + b_1\tilde{c}_1 - \tilde{a}_1 d_1 - \tilde{b}_1 c_1)\lambda \\ &+ (b_1\tilde{d}_1 - \tilde{b}_1 d_1) + \int_0^{\frac{\pi}{2}} Q(x)y(x,\lambda)\tilde{y}(x,\lambda)dx. \end{aligned} \tag{2.14}$$

Obviously, the function $K(\lambda)$ is an entire function. Since the first term of equation (2.13) for $\lambda = \lambda_n$ is zero, then

$$K(\lambda_n) = 0.$$

In addition, using (2.4) and (2.5) for $0 \le x \le \frac{\pi}{2}$, we have

$$|K(\lambda)| \le M|\lambda|^2 e^{|Im\sqrt{\lambda}|\pi}, \tag{2.15}$$

where M is constant. Let

$$\psi(\lambda) = \frac{K(\lambda)}{\omega(\lambda)}. \tag{2.16}$$

Then, $\psi(\lambda)$ is an entire function. Using (2.12) and (2.16), we obtain

$$|\psi(\lambda)| = O(\frac{1}{\sqrt{|\lambda|}}).$$

By the Liouville theorem, we get

$$\psi(\lambda) = 0, \ \forall\lambda \in \mathbf{C}.$$

Hence

$$K(\lambda) = 0, \ \forall\lambda \in \mathbf{C}.$$

i.e.,

$$\begin{aligned} &(a_1\tilde{c}_1 - \tilde{a}_1 c_1)\lambda^2 + (a_1\tilde{d}_1 + b_1\tilde{c}_1 - \tilde{a}_1 d_1 - \tilde{b}_1 c_1)\lambda \\ &+ (b_1\tilde{d}_1 - \tilde{b}_1 d_1) + \int_0^{\frac{\pi}{2}} Q(x)y(x,\lambda)\tilde{y}(x,\lambda)dx = 0. \end{aligned} \tag{2.17}$$

Substituting (2.11) into (2.17), we have

$$\begin{aligned} &2(a_1\tilde{c}_1 - \tilde{a}_1 c_1)s^6 + 2(a_1\tilde{d}_1 + b_1\tilde{c}_1 - \tilde{a}_1 d_1 - \tilde{b}_1 c_1)s^4 + 2(b_1\tilde{d}_1 - \tilde{b}_1 d_1)s^2 \\ &+ \int_0^{\frac{\pi}{2}} Q(x)\{[c_1\tilde{c}_1 s^6 + (c_1\tilde{d}_1 + \tilde{c}_1 d_1)s^4 + d_1\tilde{d}_1 s^2] \\ &\cdot(1 + \cos 2sx + \int_0^x k(x,\tau)\cos 2s\tau d\tau) \\ &+ [a_1\tilde{a}_1 s^4 + (a_1\tilde{b}_1 + \tilde{a}_1 b_1)s^2 + b_1\tilde{b}_1](1 - \cos 2sx + \int_0^x h(x,\tau)\cos 2s\tau d\tau) \\ &+ s^5[(\tilde{a}_1 c_1 + a_1\tilde{c}_1)\sin 2sx + c_1\tilde{c}_1 \int_0^x k_5(x,\tau)\sin 2s\tau d\tau)] \\ &+ s^3[(\tilde{a}_1 d_1 + a_1\tilde{d}_1 + \tilde{b}_1 c_1 + b_1\tilde{c}_1)\sin 2sx + c_1\tilde{c}_1 \int_0^x k_3(x,\tau)\sin 2s\tau d\tau)] \\ &+ s[(\tilde{b}_1 d_1 + b_1\tilde{d}_1)\sin 2sx + c_1\tilde{c}_1 \int_0^x k_1(x,\tau)\sin 2s\tau d\tau]\}dx = 0, \end{aligned} \tag{2.18}$$

where

$$k_1(x,\tau) = \frac{1}{c_1\tilde{c}_1}(\tilde{b}_1 d_1 l(x,\tau) + b_1\tilde{d}_1 m(x,\tau)),$$

$$k_5(x,\tau) = \frac{1}{c_1\tilde{c}_1}(\tilde{a}_1 c_1 l(x,\tau) + a_1\tilde{c}_1 m(x,\tau))$$

and

$$k_3(x,\tau) = \frac{1}{c_1\tilde{c}_1}(\tilde{a}_1 d_1 l(x,\tau) + a_1\tilde{d}_1 m(x,\tau) + \tilde{b}_1 c_1 l(x,\tau) + b_1\tilde{c}_1 m(x,\tau)).$$

Changing the integral variables, it can be written as

$$\begin{aligned}
&2(a_1\tilde{c}_1-\tilde{a}_1c_1)s^6+2(a_1\tilde{d}_1+b_1\tilde{c}_1-\tilde{a}_1d_1-\tilde{b}_1c_1)s^4+2(b_1\tilde{d}_1-\tilde{b}_1d_1)s^2\\
&+[c_1\tilde{c}_1s^6+(c_1\tilde{d}_1+\tilde{c}_1d_1)s^4+d_1\tilde{d}_1s^2]\int_0^{\frac{\pi}{2}}Q(\tau)\{(1+\cos 2s\tau\\
&+\cos 2s\tau\int_\tau^{\frac{\pi}{2}}Q(x)k(x,\tau)dx)+[a_1\tilde{a}_1s^4+(a_1\tilde{b}_1+\tilde{a}_1b_1)s^2+b_1\tilde{b}_1]\\
&\cdot[(1-\cos 2s\tau+\cos 2s\tau\int_\tau^{\frac{\pi}{2}}Q(x)h(x,\tau)dx]\\
&+s^5[(\tilde{a}_1c_1+a_1\tilde{c}_1)\sin 2s\tau+c_1\tilde{c}_1\sin 2s\tau\int_0^{\frac{\pi}{2}}Q(x)k_5(x,\tau)dx)]\\
&+s^3[(\tilde{a}_1d_1+a_1\tilde{d}_1+\tilde{b}_1c_1+b_1\tilde{c}_1)\sin 2s\tau+c_1\tilde{c}_1\sin 2s\tau\int_0^{\frac{\pi}{2}}Q(x)k_3(x,\tau)dx)]\\
&+s[(\tilde{b}_1d_1+b_1\tilde{d}_1)\sin 2s\tau+c_1\tilde{c}_1\sin 2s\tau\int_0^{\frac{\pi}{2}}Q(x)k_1(x,\tau)]dx\}d\tau=0.
\end{aligned}\tag{2.19}$$

Divided by s^6 in (2.19) and letting $s=\sqrt{\lambda}\to+\infty(\lambda\in\mathbf{R})$, from the Riemann-Lebesgue lemma, we obtain

$$2(a_1\tilde{c}_1-\tilde{a}_1c_1)+c_1\tilde{c}_1\int_0^{\frac{\pi}{2}}Q(x)dx=0,\tag{2.20}$$

and

$$\begin{aligned}
&\tfrac{1}{c_1\tilde{c}_1}\{[2(a_1\tilde{d}_1+b_1\tilde{c}_1-\tilde{a}_1d_1-\tilde{b}_1c_1)+a_{04}\int_0^{\frac{\pi}{2}}Q(x)dx]s^4\\
&+[2(b_1\tilde{d}_1-\tilde{b}_1d_1)+a_{02}\int_0^{\frac{\pi}{2}}Q(x)dx]s^2\}\\
&+\int_0^{\frac{\pi}{2}}\{s^6[Q(\tau)\cos 2s\tau+\cos 2s\tau\int_\tau^{\frac{\pi}{2}}Q(x)k(x,\tau)dx]d\tau\\
&+s^5[a_{15}Q(\tau)\sin 2s\tau+\sin 2s\tau\int_0^{\frac{\pi}{2}}Q(x)k_5(x,\tau)dx]\\
&+s^4[a_{14}Q(\tau)\cos 2s\tau+\cos 2s\tau\int_\tau^{\frac{\pi}{2}}Q(x)k_4(x,\tau)dx]\\
&+s^3[a_{13}Q(\tau)\sin 2s\tau+\sin 2s\tau\int_0^{\frac{\pi}{2}}Q(x)k_3(x,\tau)dx]\\
&+s^2[a_{12}Q(\tau)\cos 2s\tau+\cos 2s\tau\int_\tau^{\frac{\pi}{2}}Q(x)k_2(x,\tau)dx]\\
&+s[a_{11}Q(\tau)\sin 2s\tau+\sin 2s\tau\int_0^{\frac{\pi}{2}}Q(x)k_1(x,\tau)dx]\\
&+[-a_{10}Q(\tau)\cos 2s\tau+a_{10}\cos 2s\tau\int_\tau^{\frac{\pi}{2}}Q(x)h(x,\tau)dx]\}d\tau=0,
\end{aligned}\tag{2.21}$$

where

$$\begin{aligned}
&a_{10}=\tfrac{1}{c_1\tilde{c}_1}b_1\tilde{b}_1, && a_{11}=\tfrac{1}{c_1\tilde{c}_1}(\tilde{b}_1d_1+b_1\tilde{d}_1),\\
&a_{02}=\tfrac{1}{c_1\tilde{c}_1}(d_1\tilde{d}_1+a_1\tilde{b}_1+\tilde{a}_1d_1), && a_{12}=\tfrac{1}{c_1\tilde{c}_1}(d_1\tilde{d}_1-a_1\tilde{b}_1-\tilde{a}_1d_1),\\
&a_{13}=\tfrac{1}{c_1\tilde{c}_1}(\tilde{a}_1d_1+a_1\tilde{d}_1+\tilde{b}_1c_1+b_1\tilde{c}_1),\\
&a_{04}=\tfrac{1}{c_1\tilde{c}_1}(c_1\tilde{d}_1+\tilde{c}_1d_1+a_1\tilde{a}_1), && a_{14}=\tfrac{1}{c_1\tilde{c}_1}(c_1\tilde{d}_1+\tilde{c}_1d_1-a_1\tilde{a}_1),\\
&a_{15}=\tfrac{1}{c_1\tilde{c}_1}(\tilde{a}_1c_1+a_1\tilde{c}_1),\\
&k_2(x,\tau)=\tfrac{1}{c_1\tilde{c}_1}[d_1\tilde{d}_1k(x,\tau)+(a_1\tilde{b}_1+\tilde{a}_1b_1)h(x,\tau)],\\
&k_4(x,\tau)=\tfrac{1}{c_1\tilde{c}_1}[(c_1\tilde{d}_1+\tilde{c}_1d_1)k(x,\tau)+a_1\tilde{a}_1h(x,\tau)].
\end{aligned}$$

By integration by parts, letting $s=\sqrt{\lambda}\to+\infty$, from the Riemann-Lebesgue lemma we get

$$Q(\frac{\pi}{2})=0$$

and

$$
\begin{array}{l}
\frac{1}{c_1\tilde{c}_1}\{[2(a_1\tilde{d}_1+b_1\tilde{c}_1-\tilde{a}_1d_1-\tilde{b}_1c_1)+a_{04}\int_0^{\frac{\pi}{2}}Q(x)dx]s^4 \\
+[2(b_1\tilde{d}_1-\tilde{b}_1d_1)+a_{02}\int_0^{\frac{\pi}{2}}Q(x)dx]s^2\}+\int_0^{\frac{\pi}{2}}\{-\frac{1}{2}s^5 \\
\cdot[(Q'(\tau)-Q(\tau)k(\tau,\tau))\sin 2s\tau+\sin 2s\tau\int_\tau^{\frac{\pi}{2}}Q(x)\frac{\partial k(x,\tau)}{\partial\tau}dx] \\
+s^5[a_{15}Q(\tau)\sin 2s\tau+\sin 2s\tau\int_0^{\frac{\pi}{2}}Q(x)k_5(x,\tau)dx] \\
+s^4[a_{14}Q(\tau)\cos 2s\tau+\cos 2s\tau\int_\tau^{\frac{\pi}{2}}Q(x)k_4(x,\tau)dx] \\
+s^3[a_{13}Q(\tau)\sin 2s\tau+\sin 2s\tau\int_0^{\frac{\pi}{2}}Q(x)k_3(x,\tau)dx] \\
+s^2[a_{12}Q(\tau)\cos 2s\tau+\cos 2s\tau\int_\tau^{\frac{\pi}{2}}Q(x)k_2(x,\tau)dx] \\
+s[a_{11}Q(\tau)\sin 2s\tau+\sin 2s\tau\int_0^{\frac{\pi}{2}}Q(x)k_1(x,\tau)dx] \\
+[-a_{10}Q(\tau)\cos 2s\tau+a_{10}\cos 2s\tau\int_\tau^{\frac{\pi}{2}}Q(x)h(x,\tau)dx]\}d\tau=0.
\end{array} \tag{2.22}
$$

Repeating the above arguments five times, we obtain

$$2(a_1\tilde{d}_1+b_1\tilde{c}_1-\tilde{a}_1d_1-\tilde{b}_1c_1)+a_{04}\int_0^{\frac{\pi}{2}}Q(x)dx=0, \tag{2.23}$$

$$2(b_1\tilde{d}_1-\tilde{b}_1d_1)+a_{02}\int_0^{\frac{\pi}{2}}Q(x)dx=0, \tag{2.24}$$

$$Q^{(i)}(\frac{\pi}{2})=0,(i=0,1,2,3,4,5) \tag{2.25}$$

and

$$
\begin{array}{l}
\int_0^{\frac{\pi}{2}}\cos 2s\tau\{Q^{(6)}(\tau)-k_{77}(\tau)Q^{(5)}(\tau)-k_{76}(\tau)Q^{(4)}(\tau)-k_{75}(\tau)Q'''(\tau) \\
-k_{74}(\tau)Q''(\tau)-k_{73}(\tau)Q'(\tau)-k_{72}(\tau)Q(\tau)+\int_\tau^{\frac{\pi}{2}}Q(x)k_{71}(x,\tau)dx\}d\tau=0,
\end{array} \tag{2.26}
$$

where

$$
\begin{array}{ll}
k_{77}(\tau)= & k(\tau,\tau)+2a_{15}, \\
k_{76}(\tau)= & (5\frac{dk(\tau,\tau)}{d\tau}+\frac{\partial k(x,\tau)}{\partial\tau}|_{x=\tau})-2k_5(\tau,\tau)+4a_{14}, \\
k_{75}(\tau)= & (10\frac{d^2k(\tau,\tau)}{d\tau^2}+4\frac{d(\frac{\partial k(x,\tau)}{\partial\tau}|_{x=\tau})}{d\tau}+\frac{\partial^2k(x,\tau)}{\partial\tau^2}|_{x=\tau}) \\
 & -2(4\frac{dk_5(\tau,\tau)}{d\tau}+\frac{\partial k_5(x,\tau)}{\partial\tau}|_{x=\tau})-4k_4(\tau,\tau)-8a_{13}, \\
k_{74}(\tau)= & (10\frac{d^3k(\tau,\tau)}{d\tau^3}+6\frac{d^2(\frac{\partial k(x,\tau)}{\partial\tau}|_{x=\tau})}{d\tau^2}+3\frac{d(\frac{\partial^2k(x,\tau)}{\partial\tau^2}|_{x=\tau})}{d\tau}+\frac{\partial^3k(x,\tau)}{\partial\tau^3}|_{x=\tau}) \\
 & -2(6\frac{d^2k_5(\tau,\tau)}{d\tau^2}+3\frac{d(\frac{\partial k_5(x,\tau)}{\partial\tau}|_{x=\tau})}{d\tau}+\frac{\partial^2k_5(x,\tau)}{\partial\tau^2}|_{x=\tau}) \\
 & -4(3\frac{dk_4(\tau,\tau)}{d\tau}+\frac{\partial k_4(x,\tau)}{\partial\tau}|_{x=\tau})+8k_3(\tau,\tau)-16a_{12}, \\
k_{73}(\tau)= & (5\frac{d^4k(\tau,\tau)}{d\tau^4}+4\frac{d^3(\frac{\partial k(x,\tau)}{\partial\tau}|_{x=\tau})}{d\tau^3}+3\frac{d^2(\frac{\partial^2k(x,\tau)}{\partial\tau^2}|_{x=\tau})}{d\tau^2}+2\frac{d(\frac{\partial^3k(x,\tau)}{\partial\tau^3}|_{x=\tau})}{d\tau} \\
 & +\frac{\partial^4k(x,\tau)}{\partial\tau^4}|_{x=\tau})-2(4\frac{d^3k_5(\tau,\tau)}{d\tau^3}+3\frac{d^2(\frac{\partial k_5(x,\tau)}{\partial\tau}|_{x=\tau})}{d\tau^2}+2\frac{d(\frac{\partial^2k_5(x,\tau)}{\partial\tau^2}|_{x=\tau})}{d\tau} \\
 & +\frac{\partial^3k_5(x,\tau)}{\partial\tau^3}|_{x=\tau})-4(3\frac{d^2k_3(\tau,\tau)}{d\tau^2}+2\frac{d(\frac{\partial k_3(x,\tau)}{\partial\tau}|_{x=\tau})}{d\tau}+\frac{\partial^2k_3(x,\tau)}{\partial\tau^2}|_{x=\tau}) \\
 & +8(2\frac{dk_2(\tau,\tau)}{d\tau}+\frac{\partial k_2(x,\tau)}{\partial\tau}|_{x=\tau})+16k_1(\tau,\tau)+32a_{11}, \\
k_{72}(\tau)= & (\frac{d^5k(\tau,\tau)}{d\tau^5}+\frac{d^4(\frac{\partial k(x,\tau)}{\partial\tau}|_{x=\tau})}{d\tau^4}+\frac{d^3(\frac{\partial^2k(x,\tau)}{\partial\tau^2}|_{x=\tau})}{d\tau^3}+\frac{d^2(\frac{\partial^3k(x,\tau)}{\partial\tau^3}|_{x=\tau})}{d\tau^2} \\
 & +\frac{d(\frac{\partial^4k(x,\tau)}{\partial\tau^4}|_{x=\tau})}{d\tau}+\frac{\partial^5k(x,\tau)}{\partial\tau^5}|_{x=\tau})-2(\frac{d^4k_5(\tau,\tau)}{d\tau^4}+\frac{d^3(\frac{\partial k_5(x,\tau)}{\partial\tau}|_{x=\tau})}{d\tau^3} \\
 & +\frac{d^2(\frac{\partial^2k_5(x,\tau)}{\partial\tau^2}|_{x=\tau})}{d\tau^2}+\frac{d(\frac{\partial^3k_5(x,\tau)}{\partial\tau^3}|_{x=\tau})}{d\tau}+\frac{\partial^4k_5(x,\tau)}{\partial\tau^4}|_{x=\tau}) \\
 & -4(\frac{d^3k_4(\tau,\tau)}{d\tau^3}+\frac{d^2(\frac{\partial k_4(x,\tau)}{\partial\tau}|_{x=\tau})}{d\tau^2}+\frac{d(\frac{\partial^2k_4(x,\tau)}{\partial\tau^2}|_{x=\tau})}{d\tau}+\frac{\partial^3k_4(x,\tau)}{\partial\tau^3}|_{x=\tau}) \\
 & +8(\frac{d^2k_3(\tau,\tau)}{d\tau^2}+\frac{d(\frac{\partial k_3(x,\tau)}{\partial\tau}|_{x=\tau})}{d\tau}+\frac{\partial^2k_3(x,\tau)}{\partial\tau^2}|_{x=\tau}) \\
 & +16(\frac{dk_2(\tau,\tau)}{d\tau}+\frac{\partial k_2(x,\tau)}{\partial\tau}|_{x=\tau})-32k_1(\tau,\tau)-64a_{10},
\end{array}
$$

$$\begin{aligned} k_{71}(x,\tau) = \quad & \tfrac{\partial^6 k(x,\tau)}{\partial \tau^6} - 2\tfrac{\partial^5 k_5(x,\tau)}{\partial \tau^5} - 4\tfrac{\partial^4 k_4(x,\tau)}{\partial \tau^4} + 8\tfrac{\partial^3 k_3(x,\tau)}{\partial \tau^3} \\ & +16\tfrac{\partial^2 k_2(x,\tau)}{\partial \tau^2} + 32\tfrac{\partial k_1(x,\tau)}{\partial \tau} - 64a_{10}h(x,\tau). \end{aligned}$$

On the other hand, by the completeness of the functions $\{\cos 2s\tau\}(s=\sqrt{\lambda}\in\mathbf{R})$, we see that $Q(x)$ satisfies

$$\begin{aligned} & Q^{(6)}(\tau) - k_{77}(\tau)Q^{(5)}(\tau) - k_{76}(\tau)Q^{(4)}(\tau) - k_{75}(\tau)Q'''(\tau) - k_{74}(\tau)Q''(\tau) \\ & -k_{73}(\tau)Q'(\tau) - k_{72}(\tau)Q(\tau) + \textstyle\int_\tau^{\frac{\pi}{2}} Q(x)k_{71}(x,\tau)dx = 0, \forall \tau \in [0,\frac{\pi}{2}]. \end{aligned} \tag{2.27}$$

From (2.25), we have the identities

$$Q^{(i)}(\tau) + \int_\tau^{\frac{\pi}{2}} Q^{(i+1)}(x)dx = 0, (i=0,1,2,3,4,5). \tag{2.28}$$

Using (2.28), (2.27) can be written as

$$\begin{aligned} & Q^{(6)}(\tau) + \textstyle\int_\tau^{\frac{\pi}{2}} [k_{77}(\tau)Q^{(6)}(x) + k_{76}(\tau)Q^{(5)}(x) + k_{75}(\tau)Q^{(4)}(x) + k_{74}(\tau)Q'''(x) \\ & +k_{73}(\tau)Q''(x) + k_{72}(\tau)Q'(x) + k_{71}(x,\tau)Q(x)]dx = 0, \forall \tau \in [0,\frac{\pi}{2}]. \end{aligned} \tag{2.29}$$

Let

$$\tilde{Q}(x) = (Q(x), Q'(x), \cdots, Q^{(6)}(x))^T.$$

From (2.28) and (2.29), we get a vectorial integral equation

$$\tilde{Q}(\tau) + \int_\tau^{\frac{\pi}{2}} \tilde{K}(x,\tau)\tilde{Q}(x)dx = 0, \forall \tau \in [0,\frac{\pi}{2}], \tag{2.30}$$

where $\tilde{K}(x,\tau) = (k_{ij}(x,\tau))_{7\times 7}$ is a 7×7 matrix-value function, where $k_{ij}(x,\tau) = 1(i=1,2,\cdots,6, j = i+1), k_{ij}(x,\tau) = 0(i=1,2,\cdots,6, j\neq i+1)$, $k_{7j}(x,\tau)(j=1,2,\cdots,7)$ are described above.

Since the equation (2.30) is a vectorial Volterra integral equation, it has only trivial solution (see [5, Theorem, p.1281, 16, Theorem 1.1.1, p.1]). Hence

$$\tilde{Q}(x) = (Q(x), Q'(x), \cdots, Q^{(6)}(x))^T = 0 \ a.e. \ on \ [0,\pi].$$

i.e.,

$$Q(x) = \tilde{q}(x) - q(x) = 0 \ a.e. \ on \ [0,\pi]. \tag{2.31}$$

From (2.20), (2.23) and (2.24), we have

$$a_1\tilde{c}_1 - \tilde{a}_1 c_1 = 0, \tag{2.32}$$

$$a_1\tilde{d}_1 + b_1\tilde{c}_1 - \tilde{a}_1 d_1 - \tilde{b}_1 c_1 = 0 \tag{2.33}$$

and

$$b_1\tilde{d}_1 - \tilde{b}_1 d_1 = 0. \tag{2.34}$$

By virtue of (2.32)-(2.34), this yields

$$\frac{\tilde{a}_1\lambda + \tilde{b}_1}{\tilde{c}_1\lambda + \tilde{d}_1} = \frac{a_1\lambda + b_1}{c_1\lambda + d_1}, \forall \lambda \in \mathbf{C}.$$

Therefore, this completes the proof of Theorem 2.1. □

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