Structure theorems for rings under certain coactions of a Hopf algebra

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Abstract: Let \{D₁, \ldots, Dₙ\} be a system of derivations of a \(k\)-algebra \(A\), \(k\) a field of characteristic \(p > 0\), defined by a coaction \(δ\) of the Hopf algebra \(H_c = k[X₁, \ldots, Xₙ]/(X₁², \ldots, Xₙ^n)\), \(c \in \{0, 1\}\), the Lie Hopf algebra of the additive group and the multiplicative group on \(A\), respectively. If there exist \(x₁, \ldots, xₙ \in A\), with the Jacobian matrix \((D_i(x_j))\) invertible, \([D_i, D_j] = 0\), \(D_i^p = cD_i\), \(c \in \{0, 1\}\), \(1 \leq i, j \leq n\), we obtain elements \(y₁, \ldots, yₙ \in A\), such that \(D_i(y_j) = δ_{ij}(1 + cy_j)\), using properties of \(H_c\)-Galois extensions. A concrete structure theorem for a commutative \(k\)-algebra \(A\), as a free module on the subring \(A^p\) of \(A\) consisting of the coinvariant elements with respect to \(δ\), is proved in the additive case.

Key words: Hopf algebras, derivations, Jacobian criterion

1. Introduction
A series of articles in commutative algebra ([5], [6], [7], [8]) have focused on the following problem:

(P): Let \{D₁, \ldots, Dₙ\} be a system of derivations of a \(k\)-algebra \(A\), \(k\) field of characteristic \(p > 0\), such that there exist \(x₁, \ldots, xₙ \in A\), with the Jacobian matrix \((D_i(x_j))\) invertible, \([D_i, D_j] = 0\), \(D_i^p = cD_i\), \(c \in k\), \(1 \leq i, j \leq n\). Do elements \(y₁, \ldots, yₙ \in A\) exist such that \(D_i(y_j) = (1 + cy_j)δ_{ij}\) ?

If a positive answer is given, structure theorems for \(A\) follow in terms of the subring of constants of \(A\) with respect to the derivations \(D₁, \ldots, Dₙ\), the main one of which is contained in [5]. We recall that a finite dimensional Hopf algebra over \(k\) is a \(k\)-algebra, with comultiplication \(Δ : H \rightarrow H \otimes_k H\), antipode \(S : H \rightarrow H\) and counity \(ε : H \rightarrow k\) and a coaction of \(H\) on a \(k\)-algebra \(A\) (or an \(H\)-comodule algebra structure on \(A\)) is a morphism of algebras \(δ : A \rightarrow A \otimes H\) such that \((1 \otimes ε)δ \equiv 1\) and \((1 \otimes Δ)δ = (δ \otimes 1)δ\). Given such a coaction, the subalgebra \(\{a \in A : δ(a) = a \otimes 1\}\) of \(A\) is called the algebra of coinvariant elements of \(δ\) and it is denoted by \(A^δ = A^{coH}\).

In [6], surprisingly, for a local commutative algebra \(A\), the authors prove that the jacobian condition (which states that there are elements \(y₁, \ldots, yₙ \in A\) such that for all \(1 \leq m \leq n\) the \(m \times m\) matrix \((D_i(y_j))_{1 \leq i, j \leq m}\) over \(A\) is invertible) is equivalent to the property for \(A\) to be an \(H\)-Galois extension over the subring \(A^p\) of the coinvariant elements of \(A\) with respect to a coaction \(δ : A \rightarrow A \otimes H\), where \(H\) is a (co)commutative Hopf algebra with underlying algebra

\[\begin{array}{c}
H = k[X₁, \ldots, Xₙ]/(X₁^{p₁}, \ldots, Xₙ^{pₙ}), \\
n \geq 1, \ s₁ \geq \cdots \geq sₙ \geq 1.
\end{array}\]

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For the Lie Hopf algebra $H$ of the additive group, from the strong jacobian condition (which states that there are elements $y_i, y_n \in A$ such that $D_i(y_j) = \delta_{ij}$) an important structure theorem follows for $A$ (not necessarily commutative), precisely $A$ has an $A^\delta$-basis as a left $A^\delta$-module, consisting of the monomials $y_1^{\alpha_1} \cdots y_n^{\alpha_n}, \alpha_i \in \mathbb{N}, 0 \leq \alpha_i < p^s_i, 1 \leq i \leq n, (6)$, Theorem 3.1).

In this paper we consider Hopf algebras that “live” on the truncated algebra $H_n = k[X_1, \ldots, X_n]/(X_1^{p^{s_1}}, \ldots, X_n^{p^{s_n}})$ for $(y_1, \ldots, y_n)$. According to ([11], 14.4), the assumption is not too restrictive because any finite-dimensional, commutative and local algebra over a perfect field has this structure. Using the notion just mentioned, we formulate a more general theorem where we postulate the existence of the restrictive because any finite-dimensional, commutative and local algebra over a perfect field has this structure.

Theorem Let $H_c$ be the Hopf algebra defined as before, $c \in \{0, 1\}, A$ a right $H_c$-comodule algebra with structure map $\delta : A \to A \otimes H_c$. If there are $y_1, \ldots, y_n \in A$ with $\delta(y_i) = y_i \otimes 1 + (1 + cy_i) \otimes x_i$, for all $1 \leq i \leq n$, then the map

$$\gamma : A^\delta \otimes H_c \to A, r \otimes x^\alpha \mapsto ry^\alpha, r \in A^\delta, \alpha \in \mathbb{A}, x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{A}, \mathbb{A}$ the set of all multindices $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $0 \leq \alpha_i < p, 1 \leq i \leq n$, is a left $A^\delta$-linear and right $H_c$-colinear isomorphism. In particular, the elements $y^\alpha, \alpha \in \mathbb{A}$, form an $A^\delta$-basis of $A$ as a left $A^\delta$-module.

By using the previous theorem we are able to prove Theorem 2.5, where the property of $H_c$-Galois extension permits, starting from the strong jacobian condition on $n - 1$ elements $y_1, \ldots, y_{n-1}$ of $A$, to have the strong jacobian condition on $n$ elements of $A$, assuming there exists $y \in A$ such that $D_n(y) = 1 + cy$, $c \in \{0, 1\}$.

In section 2 we use Theorem 2.5 in the additive case and for a commutative $k$-algebra $A$, to give “explicitly” $y_1, \ldots, y_n \in A$, the special elements that verify the strong condition $D_i(y_j) = \delta_{ij}$ of derivability, $1 \leq i, j \leq n$. Some consequences are discussed in section 3, where we consider the structure of $A$ as an $A^\delta = A^{(D_1, \ldots, D_n)}$-algebra, $A^{(D_1, \ldots, D_n)}$ the constant subring of $A$ with respect to the derivations $D_1, \ldots, D_n$.

2. Coactions of a Hopf algebra $H$ and $H$-Galois type extensions

Throughout the paper, $k$ is an arbitrary field of characteristic $p > 0$. All vector spaces, algebras, coalgebras are over $k$ and maps between them are at least $k$-linear. We refer to the books by Montgomery [4] and Sweedler [10] for general Hopf algebra theory and to the book by Schauenburg and Schneider [9] for Galois type extensions of Hopf algebras. In this section we recall some definitions and theorems and we establish a structure theorem for the Hopf algebra of the multiplicative group. For $H = H_0$ the result is known [6]. Let $H$ be a Hopf algebra over the field $k$, with comultiplication $\Delta : H \to H \otimes H$, counit $\varepsilon : H \to k$, antipode $S : H \to H$. The augmentation ideal of $H$ will be denoted by $H^+ = \text{ker} \, \varepsilon$. If $A$ is a right $H$-comodule algebra, with structure map $\delta : A \to A \otimes H$, then

$$A^{\text{co}H} = A^\delta := \{ a \in A | \delta(a) = a \otimes 1 \}$$
is the algebra of coinvariant elements of $A$. We are interested in algebra extensions $B \subseteq A$ in a Hopf algebraic context. Precisely, $A^{coH} \subseteq A$. In fact, by definition, the sequence

$$A^{coH} \subseteq A \xrightarrow{\delta} A \xrightarrow{i_1} A \otimes H$$

is exact, that is $A^{coH} \subseteq A$ is the difference kernel of the maps $\delta$ and $i_1: A \rightarrow A \otimes H$, $a \mapsto a \otimes 1$.

**Definition 2.1** [2] Let $A$ be a right $H$-comodule algebra with structure map $\delta: A \rightarrow A \otimes H$. Then the extension $A^{coH} \subseteq A$ is a right $H$-Galois extension if the canonical map $A \otimes_{A^{coH}} A \rightarrow A \otimes_k H$ given by $a \otimes b \mapsto (a \otimes 1)\delta(b) = ab_{(0)} \otimes b_{(1)}$ is bijective.

In the following we will consider commutative Hopf algebras with underlying algebra:

$$H = k[X_1, \ldots, X_n]/\left(X_1^{p_1}, \ldots, X_n^{p_n}\right), \quad n \geq 1, \quad s_1 \geq \cdots \geq s_n \geq 1.$$  

We denote by $\mathbb{A}$ the set of all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $0 \leq \alpha_i < p^i, 1 \leq i \leq n$. For $\beta = (\beta_1, \ldots, \beta_n), \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$ we define

$$\beta + \gamma = (\beta_1 + \gamma_1, \ldots, \beta_n + \gamma_n), \quad |\beta| = \beta_1 + \cdots + \beta_n.$$ 

If we denote by $x_i$ the residue class of $X_i$ in $H$, for all $i$, then the elements $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \alpha \in \mathbb{A}$ form a $k$-basis of $H$. Let $A$ be an algebra, $\delta: A \rightarrow A \otimes H$ be an algebra map and a right $H$-comodule algebra structure on $A$. We will write

$$\delta(a) = \sum_{\alpha \in \mathbb{A}} D_\alpha(a) \otimes x^\alpha, \quad \text{for all } a \in A.$$ 

Thus for all $\alpha \in \mathbb{A}$ and $a, b \in A$,

$$D_\alpha(ab) = \sum_{\beta+\gamma=\alpha, \beta, \gamma \in \mathbb{A}} D_\beta(a)D_\gamma(b), \quad \text{and } D_{(0, \ldots, 0)} = \text{id.}$$ 

For all $i$, let $\delta_i = (\delta_{ij})_{1 \leq j \leq n} \in \mathbb{A}$, where $\delta_{ij} = 1$, if $j = i$, and $\delta_{ij} = 0$, otherwise. We put $D_i = D_{\delta_i}, 1 \leq i \leq n$. Thus the linear maps $D_i: A \rightarrow A$ are derivations of the algebra $A$, and for all $a \in A$ we have

$$\delta(a) = a \otimes 1 + \sum_{1 \leq i \leq n} D_i(a) \otimes x_i + \sum_{|\alpha| \geq 2} D_\alpha(a) \otimes x^\alpha. \quad (1)$$

From now we will consider the Hopf algebra $H_a$ of the additive group, that is

$$H_a = k[X_1, \ldots, X_n]/\left(X_1^{p_1}, \ldots, X_n^{p_n}\right), \quad n \geq 1, \quad s_1 \geq \cdots \geq s_n \geq 1,$$  

with comultiplication

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i, \quad 1 \leq i \leq n \quad (3)$$
and the Hopf algebra of the multiplicative group, that is
\[ H_m = k[X_1, \ldots, X_n]/\left( X_1^{p_1}, \ldots, X_n^{p_n} \right) \quad n \geq 1, \quad s_1 \geq \cdots \geq s_n \geq 1, \]
with comultiplication
\[ \Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i + x_i \otimes x_i, \quad 1 \leq i \leq n \]
We call these algebras \( H_c, \ c \in \{0, 1\} \), respectively. In the Lie algebra case of the additive group, that is
\[ H_0 = k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p), \]
coactions have a special form. Precisely they are derivations \( D_1, \ldots, D_n \in Der(A) \) with \( D_i D_j = D_j D_i, D_i^p = 0 \) and
\[ D_\alpha = \frac{D_1^{\alpha_1} \cdots D_n^{\alpha_n}}{\alpha_1! \cdots \alpha_n!}, \quad \alpha = (\alpha_1, \ldots, \alpha_n), \quad 0 \leq \alpha_i < p, \ 1 \leq i \leq n. \]
In the Lie algebra case of the multiplicative group, that is
\[ H_1 = k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p), \]
coactions are derivations \( D_1, \ldots, D_n \in Der(A) \) with \( D_i D_j = D_j D_i, D_i^p = D_i \) and
\[ D_\alpha = \frac{\prod_{j=0}^{\alpha_i-1} (D_i - j_i) \prod_{j=0}^{\alpha_2-1} (D_2 - j_2) \cdots \prod_{j=0}^{\alpha_n-1} (D_n - j_n)}{\alpha_1! \cdots \alpha_n!} \]
with \( \alpha = (\alpha_1, \ldots, \alpha_n), \ 0 \leq \alpha_i < p, \ 1 \leq i \leq n \) and \( \alpha! = \alpha_1! \alpha_2! \cdots \alpha_n! \) (see [1], Theorem 3.3).

**Theorem 2.2** Let \( H_c, \ c \in \{0, 1\} \), be the Hopf algebra in the Lie cases, defined as before and \( A \) a right \( H_c \)-comodule algebra with structure map \( \delta : A \rightarrow A \otimes H_c \). Let \( R = A^{c \otimes H_c} \). Assume, for \( c = 1 \), \( A \) is a commutative local algebra with maximal ideal \( m \) and \( R + m = A \).

(a) The following are equivalent:

(i) \( R \subset A \) is a faithfully flat \( H_c \)-Galois extension.

(ii) There are \( y_1, \ldots, y_n \in A \) with \( \delta(y_i) = y_i \otimes 1 + (1 + y_i) \otimes x_i \), for all \( 1 \leq i \leq n \)

(b) Suppose (ii) holds. Then
\[ R \otimes H_1 \rightarrow A, \ r \otimes x^\alpha \mapsto ry^\alpha, \ r \in R, \ \alpha \in \mathbb{A}, \ y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}, \ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{A} \]
is a left \( R \)-linear and right \( H_c \)-colinear isomorphism.
In particular, the elements \( y^\alpha, \ \alpha \in \mathbb{A} \), form an \( R \)-basis of \( A \) as a left \( R \)-module.
Results in conclusion follows.

For $c = 0$, see [6], Theorem 3.1.

For $c = 1$, (a) is proved in [1], Proposition 4.2. To prove (b) we observe that the coradical $C$ of $H_1$ is the $k$-subalgebra of $H_1$:

$$C = k \oplus kx_1 \oplus \cdots \oplus kx_n, \quad x_i = X_i + (X^P_1, \ldots, X^P_n).$$

For this, it is sufficient to prove for $i = 1$ that $C = k \oplus kx$, $H_1 = k[X]/(X^p) = k[x]$.

$$\Delta(1 + x) = \Delta(1) + \Delta(x) = 1 \otimes 1 + 1 \otimes x + x \otimes 1 = (1 + x) \otimes (1 + x) \in C \otimes C.$$ Moreover, the vector subspaces of $H_1$, $k$ and $kx$, are the only simple coalgebras of $H_1$. Hence the assertion.

Suppose (ii) of (a) holds. Then we define a $k$-linear map $\gamma : H_1 \rightarrow A$ by $\gamma : x^a \mapsto ry^a$ for all $a \in A$. Since $\Delta$ and $\delta$ are algebra maps and, for all $i$,

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i + x_i \otimes x_i, \quad \delta(y_i) = y_i \otimes 1 + (1 + y_i) \otimes x_i,$$

$\gamma$ is right $H_1$-colinear. If we prove that the map $\gamma$ is convolution invertible, the $H_1$-extension $R \subset A$ is $H_1$-cleft, hence $H_1$-Galois and

$$R \otimes H_1 \rightarrow A, \quad r \otimes x^a \mapsto ry^a, \quad r \in R, a \in A$$

is bijective ([9], 8.2.4, 7.2.3). To prove that $\gamma \in \text{Hom}(A, A)$ is invertible with respect to the convolution $*$, it is sufficient to prove that $\gamma / c$ is invertible as an element of $\text{Hom}(C, A)$. For $f \in \text{Hom}(C, A)$, $i = 1, \ldots, n$, it results in

$$f \ast \gamma(1 + x_i) = m(f \otimes \gamma)(\Delta(1 + x_i)) = m(f \otimes \delta(1 \otimes x_i + x_i \otimes 1 + x_i \otimes x_i))$$

$$= m(f(1) \otimes \gamma(x_i) + f(x_i) \otimes \delta(1) + f(x_i) \otimes \delta(x_i)) = 1_A y_i + f(x_i) + f(x_i)y_i$$

and

$$u(1 + x_i) = u(1) + u(x_i) = u(1) = 1_A,$$

with $m : H_1 \otimes H_1 \rightarrow H_1$ and $u : k \rightarrow H_1$ being the multiplication and the unit maps of $H_1$, respectively. If we put $f(x_i) = \frac{1 - y_i}{1 + y_i}$, we have $y_i + f(x_i)(1 + y_i) = 1$, $\gamma$ is left invertible and its inverse map is $f$. Hence the conclusion follows.

\[\Box\]

Remark 2.3 The result contained in Theorem 2.2, (b) can be deduced from (ii), under the hypotheses that the elements $1 + y_i, 1 < i < n$, are invertible, $A$ not necessarily local.

In the following, for $c = 1$, we will suppose that $A$ is commutative, local and $A = R + m$, where $R$ is the coinvariant subring of $A$ with respect to the coaction $\delta$ and $m$ is the maximal ideal of $A$.

Corollary 2.4 Let $H_c$ be the Hopf Lie algebra of the group $H_c$, $A$ an algebra and $\delta : A \rightarrow A \otimes H_c$ a coaction. Put $D_1, \ldots, D_n$ the derivations defined by (1) and $R := A^m H_c$. The following are equivalent:

1. $R \subset A$ is a faithfully flat $H_c$-Galois extension.

2. There are $y_1, \ldots, y_n \in A$ with $D_i(y_j) = \delta_{ij}(1 + cy_j)$, for all $1 \leq i, j \leq n$. 431
The set of derivations comes from a comodule structure of $H$. We use it to prove the following:

**Theorem 2.5** Let $A$ be a $k$-algebra, $k$ a field of characteristic $p > 0$ and let $\{D_1, \ldots, D_n\} \subset \text{Der}_k(A)$ such that $D_iD_j = D_jD_i$, $D_i^p = cD_i$, $c \in \{0, 1\}$, for all $i, j, 1 \leq i, j \leq n$. Suppose that

1) There exist $z_1, \ldots, z_{n-1} \in A$ such that $D_i(z_j) = \delta_{ij}(1 + cz_i)$, $1 \leq i, j \leq n - 1$.

2) There exists $y \in A$ such that $D_n(y) = 1 + cy$.

Then $R := A^{coH_c} \subset A$ is a faithfully flat $H_c$-Galois extension and, consequently, there are $y_1, \ldots, y_n \in A$ with $D_i(y_j) = \delta_{ij}(1 + cy_i)$ for all $1 \leq i, j \leq n$.

**Proof** The set of derivations comes from a comodule structure of $A$ on $H_c$, $H = k[x_1, \ldots, x_n]$, $x_i^p = 0$, given by $\delta : A \longrightarrow A \otimes H_c$,

$$\delta(a) = a \otimes 1 + \sum_{1 \leq i \leq n} D_i(a) \otimes x_i + \sum_{\alpha \in \mathbb{N}^n, \sum_{i \geq 2} |\alpha_i| \geq 2} D_\alpha(a) \otimes x^\alpha. \tag{8}$$

Then $R := A^{coH_c} \subset A$ is a faithfully flat $H_c$-Galois extension and, consequently, there are $y_1, \ldots, y_n \in A$ with $D_i(y_j) = \delta_{ij}(1 + cy_i)$ for all $1 \leq i, j \leq n$. By 2) the assertion follows. $\square$

### 3. A constructive theorem

We will describe, in the additive case, the special elements $y_1, \ldots, y_n$ that appear in Theorem 2.5 and satisfy a strong condition on the derivability. Following the same direction of research contained in the papers by Matsumura, Restuccia and Utano [5], [8], where the elements are computed, we obtain the result contained in [8] without the hypotheses that $A$ is local, regular and $k$ a separably closed field, but requiring that the last derivation evaluates to one on an element $t \in U(A)$.

**Theorem 3.1** Let $A$ be a commutative $k$-algebra, $k$ a field of characteristic $p > 0$ and let $\{D_1, \ldots, D_n\} \subset \text{Der}_k(A)$ such that $D_iD_j = D_jD_i$, $D_i^p = 0$ for all $i, j, 1 \leq i, j \leq n$. Suppose that

1) There exist $z_1, \ldots, z_{n-1} \in A$ such that $D_i(z_j) = \delta_{ij}$, $1 \leq i, j \leq n - 1$.

2) There exists $y \in A$ such that $D_n(y) = 1$. 

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Then there exists $t \in A$ such that $D_n(t) = 1$ and $D_i(t) = 0$, for all $i = 1, \ldots, n - 1$.

**Proof** The set of derivations comes from a comodule structure of $A$ on $H$, $H = k[x_1, \ldots, x_n]$, $x_i^p = 0$, $x_i$ primitive, given by $\delta : A \to A \otimes H$,

$$\delta(a) = a \otimes 1 + \sum_{1 \leq i \leq n} D_i(a) \otimes x_i + \sum_{\alpha \in \mathbb{A}} \frac{\delta(\alpha)}{|\alpha| \geq 2} D_\alpha(a) \otimes x^\alpha. \quad (9)$$

$\alpha \in \mathbb{N}^n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Let $R = A^{coH}$ be the coinvariant subring of $A$ with respect to $\delta$ and let $\overline{A} = k[x_1]$, $x_n^p = 0$, $B = A^{co\overline{A}}$ the coinvariant subring of $A$ with respect to $\overline{\delta} : A \to A \otimes \overline{A}$, $\overline{A} = H/K^+H$, $K = k[x_1, \ldots, x_{n-1}]$, $x_i^p = 0$, $i = 1, \ldots, n - 1$. Consider the extensions $R \subset B \subset A$. By 2), $B \subset A$ is $\overline{A}$-Galois and $1, y, y^2, \ldots, y^{p-1}$ is a basis of $A$ on $B = A^{D_n}$. By 1), $R \subset B$ is $K$-Galois and the monomials $z_1^{i_1} \cdots z_{n-1}^{i_{n-1}}$, $1 \leq j_i \leq p - 1$, $i = 1, \ldots, n - 1$, are a basis of $B$ on $R$. We want to find $t \in A$ such that $D_n(t) = 1$ and $D_i(t) = 0$ for all $i = 1, \ldots, n - 1$. Put $t = \sum_{0 \leq i \leq p-1} b_i y^i$. Then $D_n(t) = 1 = \sum_{0 \leq i \leq p-1} b_i y^{i-1}$ implies $b_1 = 1$ and $b_i = 0$, for all $i > 1$. We can rewrite $t = b_0 + y$ as $t = y - b$, $b \in B$. Then we need an element $b \in B$ such that $D_i(y) = D_i(b)$, $i = 1, \ldots, n - 1$. Moreover for $i = 1, \ldots, n - 1$, $D_i(y) \in B$, since $D_n(D_i(y)) = D_i(D_n(y)) = D_i(1) = 0$, for all $i = 1, \ldots, n - 1$. Then we can write:

$$D_j(y) = \sum_{0 \leq i_1 \leq p-1} s_{j,i_1,\ldots,i_{n-1}} z_1^{i_1} \cdots z_{n-1}^{i_{n-1}}, \quad j = 1, \ldots, n - 1, s_{j,i_1,\ldots,i_{n-1}} \in R.$$ 

Since $D_j^p = 0$, for all $j = 1, \ldots, n - 1$, we have:

$$\begin{cases} 
D_j^{p-1}(D_1(y)) = 0 = \sum_{0 \leq i_1 \leq p-1} s_{1,i_1,\ldots,i_{n-1}} z_1^{i_1} \cdots z_{n-1}^{i_{n-1}}, \\
\vdots \\
D_{n-1}^{p-1}(D_{n-1}(y)) = 0 = \sum_{0 \leq i_1 \leq p-1} s_{n-1,i_1,\ldots,i_{n-1}} z_1^{i_1} \cdots z_{n-1}^{i_{n-1}}.
\end{cases}$$

Hence we get the relations

$$\begin{cases} 
0 = \sum_{0 \leq i_1 \leq p-1, j \neq 1} s_{1,i_1,\ldots,i_{n-1}}(p-1)!z_2^{i_2} \cdots z_{n-1}^{i_{n-1}}, \\
\vdots \\
0 = \sum_{0 \leq i_1 \leq p-1, j \neq n-1} s_{n-1,i_1,\ldots,p-1}(p-1)!z_1^{i_1} \cdots z_{n-2}^{i_{n-2}},
\end{cases}$$

and

$$\begin{cases} 
s_{1,\ldots,i_{n-1}} = 0 & \quad 0 \leq i_2, \ldots, i_{n-1} \leq p-1 \\
\vdots \\
s_{n-1,\ldots,i_{n-2},p-1} = 0 & \quad 0 \leq i_1, \ldots, i_{n-2} \leq p-1.
\end{cases}$$

Writing

$$b = \sum_{0 \leq j \leq p-1} t_{j_1,\ldots,j_{n-1}} z_1^{j_1} \cdots z_{n-1}^{j_{n-1}},$$

we obtain

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Likewise, we can deduce

$$b \text{ is uniquely determined by coefficients } t_{j_1, \ldots, j_{n-1}}, \ 0 \leq j_i \leq p - 1. \text{ By derivation, we obtain}$$

$$D_1(b) = \sum_{0 \leq j_1 \leq p-1} \sum_{0 \leq j_2, \ldots, j_{n-1} \leq p-1} t_{j_1, \ldots, j_{n-1}} z_{j_1}^{j_1-1} \cdots z_{j_{n-1}}^{j_{n-1}-1},$$

$$= \sum_{0 \leq j_1 \leq p-1} t_{j_1+1, j_2, \ldots, j_{n-1}} (j_1 + 1) z_{j_1}^{j_1} \cdots z_{j_{n-1}}^{j_{n-1}}.$$  

$$\ldots \ldots \ldots$$

$$D_{n-1}(b) = \sum_{0 \leq j_1 \leq p-1} \sum_{0 \leq j_2, \ldots, j_{n-1} \leq p-1} t_{j_1, j_2, \ldots, j_{n-1}} z_{j_1}^{j_1} \cdots z_{j_{n-1}}^{j_{n-1}} - 1,$$

$$= \sum_{0 \leq j_1 \leq p-1, i \neq 1} t_{j_1, \ldots, j_{n-1}+1} (j_{n-1} + 1) z_{j_1}^{j_1} \cdots z_{j_{n-1}}^{j_{n-1}}.$$  

From $D_i(y) = D_i(b)$, for $i = 1, \ldots, n-1$, it follows

$$\left\{ \begin{array}{l}
\sum_{0 \leq j_1 \leq p-2} \sum_{0 \leq j_2, \ldots, j_{n-1} \leq p-1} t_{j_1, j_2, \ldots, j_{n-1}} (j_1 + 1) z_{j_1}^{j_1} \cdots z_{j_{n-1}}^{j_{n-1}} = \sum_{0 \leq j_1 \leq p-2} \sum_{0 \leq j_2, \ldots, j_{n-1} \leq p-1, i \neq 1} s_{j_1, j_2, \ldots, j_{n-1}} z_{j_1}^{j_1} \cdots z_{j_{n-1}}^{j_{n-1}}, \\
\sum_{0 \leq j_1 \leq p-2} \sum_{0 \leq j_2, \ldots, j_{n-1} \leq p-1} t_{j_1, j_2, \ldots, j_{n-1}+1} (j_{n-1} + 1) z_{j_1}^{j_1} \cdots z_{j_{n-1}}^{j_{n-1}} = \sum_{0 \leq j_1 \leq p-2} \sum_{0 \leq j_2, \ldots, j_{n-1} \leq p-1, i \neq n-1} s_{j_1, j_2, \ldots, j_{n-1}+1} z_{j_1}^{j_1} \cdots z_{j_{n-1}}^{j_{n-1}}.
\end{array} \right.$$  

Hence we get the relations

$$\left\{ \begin{array}{l}
t_{j_1+1, j_2, \ldots, j_{n-1}} (j_1 + 1) = s_{j_1, j_2, \ldots, j_{n-1}} 0 \leq j_1 \leq p - 2, 0 \leq j_i \leq p - 1, i \neq 1, \\
t_{j_1, j_2, \ldots, j_{n-1}+1} (j_{n-1} + 1) = s_{j_1, j_2, \ldots, j_{n-1}} 0 \leq j_1 \leq p - 2, 0 \leq j_i \leq p - 1, i \neq n - 1.
\end{array} \right. \quad (10)$$

From the conditions $D_k D_\ell = D_\ell D_k$ for $1 \leq \ell < k \leq n - 1$ we obtain the compatibility relations

$$jk s_{j_1, j_2, \ldots, j_{k-1}, j_k, \ldots, j_{n-1}} = (j_\ell + 1) s_{j_1, j_2, \ldots, j_{k-1}+1, \ldots, j_{n-1}} \quad (11)$$

with $0 \leq j_\ell \leq p - 1, 1 \leq j_k \leq p - 1, 0 \leq j_i \leq p - 1, i \neq \ell, k, \ 1 \leq \ell < k \leq n - 1$. The first two relations of (10) give, for $\ell = 1, k = 2$

$$t_{j_1+1, j_2, \ldots, j_{n-1}} (j_1 + 1) = s_{j_1, j_2, \ldots, j_{n-1}} 0 \leq j_1 \leq p - 1, i \neq 1, 0 \leq j_1 \leq p - 2, \quad (12)$$

$$t_{j_1, j_2+1, \ldots, j_{n-1}} (j_2 + 1) = s_{j_2, j_1, \ldots, j_{n-1}} 0 \leq j_1 \leq p - 1, i \neq 2, 0 \leq j_2 \leq p - 2. \quad (13)$$

We rewrite the relations (12) and (13)

$$t_{j_1, j_2, \ldots, j_{n-1}} j_1 = s_{j_1, j_1-1, j_2, \ldots, j_{n-1}} 0 \leq j_1 \leq p - 1, i \neq 1, 1 \leq j_i \leq p - 2,$$

$$t_{j_1, j_2, \ldots, j_{n-1}} j_2 = s_{j_2, j_1-1, \ldots, j_{n-1}} 0 \leq j_i \leq p - 1, i \neq 2, 1 \leq j_2 \leq p - 2,$$

obtaining

$$j_1 j_2 t_{j_1, j_2, \ldots, j_{n-1}} = j_2 s_{j_1, j_1-1, j_2, \ldots, j_{n-1}} = j_1 s_{j_2, j_1-1, \ldots, j_{n-1}}.$$  

Likewise, we can deduce

$$j_1 \ldots j_{n-1} t_{j_1, j_2, \ldots, j_{n-1}} = j_2 \ldots j_{n-1} s_{j_1, j_1-1, j_2, \ldots, j_{n-1}} = j_1 j_1 \ldots j_{n-1} s_{j_2, j_1-1, \ldots, j_{n-1}} =$$

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\[ \cdots = j_1 j_2 \cdots j_{n-2} s_{n-1}, j_1, j_2, \ldots, j_{n-1} + 1, \text{ for } 0 \leq j_i \leq p - 1. \]

Hence, the elements \( t_{j_1, j_2, \ldots, j_{n-1}} \) are determined and, as a consequence, the element \( b \) is obtained.

**Corollary 3.2** Let \( A \) be a \( k \)-algebra, \( k \) a field of characteristic \( p > 0 \) and let \( \{ D_1, \ldots, D_n \} \subset \text{Der}_k(A) \) such that \( D_i D_j = D_j D_i, \ D_i^p = 0 \) for all \( i, j, 1 \leq i, j \leq n \).

Suppose that

1) There exist \( z_1, \ldots, z_{n-1} \in A \) such that \( D_i(z_j) = \delta_{ij}, 1 \leq i, j \leq n - 1 \).

2) There exists \( y \in A \) such that \( D_n(y) = 1 \).

Then there exist \( z_1, \ldots, z_{n-1}, z_n \) such that \( D_i(z_j) = \delta_{ij} \).

**Proof** Follows from Theorem 3.1, with \( z_n = t \).

**Corollary 3.3** Let \( A \) be a \( k \)-algebra, \( k \) a field of characteristic \( p > 0 \) and let \( \{ D_1, \ldots, D_n \} \subset \text{Der}_k(A) \) such that \( D_i D_j = D_j D_i, \ D_i^p = 0 \) for all \( i, j, 1 \leq i, j \leq n \).

Suppose that:

1) There exist \( z_1, \ldots, z_{n-1} \in A \) such that \( D_i(z_j) = \delta_{ij}, 1 \leq i, j \leq n - 1 \).

2) There exists \( y \in A \) such that \( D_n(y) = 1 \).

Then the set \( \{ z_1, \ldots, z_{n-1} \} \) of \( p \)-independent elements of \( A \) on the subring of constants \( A^{\{D_1, \ldots, D_n\}} \) can be completed to a \( p \)-basis \( B \) of \( n \) elements of \( A \) on \( A^{\{D_1, \ldots, D_n\}} \) and \( A = A^{\{D_1, \ldots, D_n\}}[B] \).

**Proof** It is easy to prove that \( z_1, \ldots, z_{n-1}, z_n \), with \( z_n \) as in Corollary 3.2, verify \( D_i(z_j) = \delta_{ij}, 1 \leq i \leq j \leq n, D_i^p = 0, [D_i, D_j] = 0 \) and form a \( p \)-basis of \( A \) on \( A^{\{D_1, \ldots, D_n\}} \). The structure of \( A \) follows by definition of \( p \)-basis [3]).

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