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Structure theorems for rings under certain coactions of a Hopf algebra

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Abstract: Let $\{D_1,\dots,D_n\}$ be a system of derivations of a k-algebra A, k a field of characteristic $p>0$, defined by a coaction δ of the Hopf algebra $H_c = k[X_1,\dots,X_n]/(X_1^p,\dots,X_n^p)$, $c\in\{0,1\}$, the Lie Hopf algebra of the additive group and the multiplicative group on A, respectively. If there exist $x_1,\dots,x_n\in A$, with the Jacobian matrix $(D_i(x_j))$ invertible, $[D_i,D_j]=0$, $D_i^p = cD_i$, $c\in\{0,1\}$, $1\le i,j\le n$, we obtain elements $y_1,\dots,y_n\in A$, such that $D_i(y_j)=\delta_{ij}(1+cy_i)$, using properties of H_c-Galois extensions. A concrete structure theorem for a commutative k-algebra A, as a free module on the subring A^δ of A consisting of the coinvariant elements with respect to δ, is proved in the additive case.

Key words: Hopf algebras, derivations, Jacobian criterion

1. Introduction

A series of articles in commutative algebra ([5], [6], [7], [8] have focused on the following problem:

(P): *Let $\{D_1,\dots,D_n\}$ be a system of derivations of a k-algebra A, k field of characteristic $p>0$, such that there exist $x_1,\dots,x_n\in A$, with the Jacobian matrix $(D_i(x_j))$ invertible, $[D_i,D_j]=0$, $D_i^p = c_i^{p-1}D_i$, $c_i\in k$, $1\le i,j\le n$. Do elements $y_1,\dots,y_n\in A$ exist such that $D_i(y_j)=(1+c_jy_j)\delta_{ij}$?*

If a positive answer is given, structure theorems for A follow in terms of the subring of constants of A with respect to the derivations $D_1,\dots,D_n$, the main one of which is contained in [5]. We recall that a finite dimensional Hopf algebra over k is a k-algebra, with comultiplication $\Delta : H \longrightarrow H\otimes_k H$, antipode $S : H\longrightarrow H$ and counity $\varepsilon : H\longrightarrow k$ and a coaction of H on a k-algebra A (or an H-comodule algebra structure on A) is a morphism of algebras $\delta : A\longrightarrow A\otimes H$ such that $(1\otimes\varepsilon)\delta\cong 1$ and $(1\otimes\Delta)\delta = (\delta\otimes 1)\delta$. Given such a coaction, the subalgebra $\{a\in A : \delta(a)=a\otimes 1\}$ of A is called the algebra of coinvariant elements of δ and it is denoted by $A^\delta = A^{\mathrm{coH}}$.

In [6], surprisingly, for a local commutative algebra A, the authors prove that the jacobian condition (which states that there are elements $y_1,\dots,y_n\in A$ such that for all $1\le m\le n$ the $m\times m$ matrix $(D_i(y_j))_{1\le i,j\le m}$ over A is invertible) is equivalent to the property for A to be an H-Galois extension over the subring A^δ of the coinvariant elements of A with respect to a coaction $\delta : A\longrightarrow A\otimes H$, where H is a (co)commutative Hopf algebra with underlying algebra

$$H = k[X_1,\dots,X_n]/(X_1^{p^{s_1}},\dots,X_n^{p^{s_n}}),\ n\ge 1,\ s_1\ge\cdots\ge s_n\ge 1.$$

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For the Lie Hopf algebra H of the additive group, from the strong jacobian condition (which states that there are elements $y_1,\dots,y_n \in A$ such that $D_i(y_j)_{1\le i,j\le n} = \delta_{ij}$) an important structure theorem follows for A (not necessarily commutative), precisely A has an A^δ-basis as a left A^δ-module, consisting of the monomials $y_1^{\alpha_1}\dots y_n^{\alpha_n}$, $\alpha_i \in \mathbb{N}$, $0 \le \alpha_i < p^{s_i}$, $1 \le i \le n$, ([6], Theorem 3.1).

In this paper we consider Hopf algebras that "live" on the truncated algebra $H_{\underline{s}} = k[X_1,\dots,X_n]/(X_1^{p^{s_1}},\dots,X_n^{p^{s_n}})$ $\underline{s} = (s_1,\dots,s_n)$. According to ([11], 14.4), the assumption is not too restrictive because any finite-dimensional, commutative and local algebra over a perfect field has this structure. Using the notion just mentioned, we formulate a more general theorem where we postulate the existence of the elements $y_1,\dots,y_n \in A$ with the strong jacobian condition in the Lie algebra case of the additive group for $H = H_0 = k[X_1,\dots,X_n]/(X_1^p,\dots,X_n^p)$, with $c_i = 0$ in **(P)**, $i = 1,\dots,n$. The same result is given in the Lie algebra case of the multiplicative group for $H = H_1$ with $c_i = 1$ in **(P)**, $i = 1,\dots,n$, under the hypotheses A local and $A = A^\delta + m$, where m is the maximal ideal of A. More precisely, the main result of section 1 concerns a positive answer to the previous question that can be deduced from the following theorem.

Theorem *Let H_c be the Hopf algebra defined as before, $c \in \{0,1\}$, A a right H_c-comodule algebra with structure map $\delta : A \longrightarrow A\otimes H_c$. If there are $y_1,\dots,y_n \in A$ with $\delta(y_i) = y_i \otimes 1 + (1+cy_i)\otimes x_i$, for all $1 \le i \le n$, then the map*

$$\gamma : A^\delta \otimes H_c \longrightarrow A,\ r\otimes x^\alpha \mapsto ry^\alpha,\ r\in A^\delta,\ \alpha\in\mathbb{A}, x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}, y^\alpha = y_1^{\alpha_1}\cdots y_n^{\alpha_n},$$

where $\alpha = (\alpha_1,\dots,\alpha_n)\in\mathbb{A}$, $\mathbb{A}$ the set of all multiindices $\alpha = (\alpha_1\dots,\alpha_n)$, with $0 \le \alpha_i < p$, $1 \le i \le n$, is a left A^δ-linear and right H_c-colinear isomorphism. In particular, the elements y^α, $\alpha\in\mathbb{A}$, form an A^δ-basis of A as a left A^δ-module.

By using the previous theorem we are able to prove Theorem 2.5, where the property of H_c-Galois extension permits, starting from the strong jacobian condition on $n-1$ elements $y_1,\dots,y_{n-1}$ of A, to have the strong jacobian condition on n elements of A, assuming there exists $y \in A$ such that $D_n(y) = 1 + cy$, $c \in \{0,1\}$. In section 2 we use Theorem 2.5 in the additive case and for a commutative k-algebra A, to give "explicitly" $y_1,\dots,y_n \in A$, the special elements that verify the strong condition $D_i(y_j) = \delta_{ij}$ of derivability, $1 \le i,j \le n$. Some consequences are discussed in section 3, where we consider the structure of A as an $A^\delta = A^{\{D_1,\dots,D_n\}}$-algebra, $A^{\{D_1,\dots,D_n\}}$ the constant subring of A with respect to the derivations $D_1,\dots,D_n$.

2. Coactions of a Hopf algebra H and H-Galois type extensions

Throughout the paper, k is an arbitrary field of characteristic $p > 0$. All vector spaces, algebras, coalgebras are over k and maps between them are at least k-linear. We refer to the books by Montgomery [4] and Sweedler [10] for general Hopf algebra theory and to the book by Schauenburg and Schneider [9] for Galois type extensions of Hopf algebras. In this section we recall some definitions and theorems and we establish a structure theorem for the Hopf algebra of the multiplicative group. For $H = H_0$ the result is known [6]. Let H be a Hopf algebra over the field k, with comultiplication $\Delta : H \longrightarrow H\otimes H$, counit $\varepsilon : H \longrightarrow k$, antipode $S : H \longrightarrow H$. The augmentation ideal of H will be denoted by $H^+ = \ker\varepsilon$. If A is a right H-comodule algebra, with structure map $\delta : A \longrightarrow A\otimes H$, then

$$A^{\mathrm{coH}} = A^\delta := \{a \in A | \delta(a) = a\otimes 1\}$$

is the algebra of H-coinvariant elements of A. We are interested in algebra extensions $B \subseteq A$ in a Hopf algebraic context. Precisely, $A^{\mathrm{coH}} \subseteq A$. In fact, by definition, the sequence

$$A^{\mathrm{coH}} \stackrel{\subseteq}{\longrightarrow} A \underset{i_1}{\overset{\delta}{\rightrightarrows}} A \otimes H$$

is exact, that is $A^{\mathrm{co}H} \subseteq A$ is the difference kernel of the maps δ and $i_1 : A \longrightarrow A \otimes H$, $a \mapsto a \otimes 1$.

Definition 2.1 [2] Let A be a right H-comodule algebra with structure map $\delta : A \longrightarrow A \otimes H$. Then the extension $A^{\mathrm{co}H} \subseteq A$ is a *right H-Galois extension* if the canonical map $\mathrm{can} : A \otimes_{A^{\mathrm{co}H}} A \longrightarrow A \otimes_k H$ given by $a \otimes b \mapsto (a \otimes 1)\delta(b) = ab_{(0)} \otimes b_{(1)}$ is bijective.

In the following we will consider commutative Hopf algebras with underlying algebra:

$$H = k[X_1, \ldots, X_n]/\left(X_1^{p^{s_1}}, \ldots, X_n^{p^{s_n}}\right), \; n \geq 1, \quad s_1 \geq \cdots \geq s_n \geq 1.$$

We denote by $\mathbb{A}$ the set of all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $0 \leq \alpha_i < p^{s_i}, 1 \leq i \leq n$. For $\beta = (\beta_1, \ldots, \beta_n), \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$ we define

$$\beta + \gamma = (\beta_1 + \gamma_1, \ldots, \beta_n + \gamma_n), \text{ and } |\beta| = \beta_1 + \cdots + \beta_n.$$

If we denote by x_i the residue class of X_i in H, for all i, then the elements $x^\alpha := x_i^{\alpha_1} \ldots x_n^{\alpha_n}$, $\alpha \in \mathbb{A}$ form a k-basis of H. Let A be an algebra, $\delta : A \to A \otimes H$ be an algebra map and a right H-comodule algebra structure on A. We will write

$$\delta(a) = \sum_{\alpha \in \mathbb{A}} D_\alpha(a) \otimes x^\alpha, \text{ for all } a \in A.$$

Thus for all $\alpha \in \mathbb{A}$ and $a, b \in A$,

$$D_\alpha(ab) = \sum_{\substack{\beta+\gamma=\alpha \\ \beta,\gamma \in \mathbb{A}}} D_\beta(a) D_\gamma(b), \text{ and } D_{(0,\ldots,0)} = \mathrm{id}.$$

For all i, let $\delta_i = (\delta_{ij})_{1 \leq j \leq n} \in \mathbb{A}$, where $\delta_{ij} = 1$, if $j = i$, and $\delta_{ij} = 0$, otherwise. We put $D_i = D_{\delta_i}, 1 \leq i \leq n$. Thus the linear maps $D_i : A \to A$ are derivations of the algebra A, and for all $a \in A$ we have

$$\delta(a) = a \otimes 1 + \sum_{1 \leq i \leq n} D_i(a) \otimes x_i + \sum_{\substack{\alpha \in \mathbb{A} \\ |\alpha| \geq 2}} D_\alpha(a) \otimes x^\alpha. \tag{1}$$

From now we will consider the Hopf algebra H_a of the additive group, that is

$$H_a = k[X_1, \ldots, X_n]/\left(X_1^{p^{s_1}}, \ldots, X_n^{p^{s_n}}\right) \; n \geq 1, \quad s_1 \geq \cdots \geq s_n \geq 1, \tag{2}$$

with comultiplication

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i, \;\; 1 \leq i \leq n \tag{3}$$

and the Hopf algebra of the multiplicative group, that is

$$H_m = k[X_1, \dots, X_n]/\left(X_1^{p^{s_1}}, \dots, X_n^{p^{s_n}}\right) \;\; n \geq 1, \;\; s_1 \geq \cdots \geq s_n \geq 1, \tag{4}$$

with comultiplication

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i + x_i \otimes x_i, \;\; 1 \leq i \leq n \tag{5}$$

We call these algebras H_c, $c \in \{0, 1\}$, respectively. In the Lie algebra case of the additive group, that is

$$H_0 = k[X_1, \dots, X_n]/(X_1^p, \dots, X_n^p), \tag{6}$$

coactions have a special form. Precisely they are derivations $D_1, \dots, D_n \in Der(A)$ with $D_iD_j = D_jD_i$, $D_i^p = 0$ and

$$D_\alpha = \frac{D_1^{\alpha_1}}{\alpha_1!} \cdots \frac{D_n^{\alpha_n}}{\alpha_n!}, \qquad \alpha = (\alpha_1, \dots, \alpha_n), \qquad 0 \leq \alpha_i < p, \; 1 \leq i \leq n.$$

In the Lie algebra case of the multiplicative group, that is

$$H_1 = k[X_1, \dots, X_n]/(X_1^p, \dots, X_n^p), \tag{7}$$

coactions are derivations $D_1, \dots, D_n \in Der(A)$ with $D_iD_j = D_jD_i$, $D_i^p = D_i$ and

$$D_\alpha = \frac{\prod_{j_1=0}^{\alpha_1-1}(D_1 - j_1)}{\alpha_1!} \frac{\prod_{j_2=0}^{\alpha_2-1}(D_2 - j_2)}{\alpha_2!} \cdots \frac{\prod_{j_n=0}^{\alpha_n-1}(D_n - j_n)}{\alpha_n!} =$$

$$= \frac{\prod_{t=1}^{n}\prod_{j_t=0}^{\alpha_t-1}(D_t - j_t)}{\alpha!}$$

with $\alpha = (\alpha_1, \dots, \alpha_n)$, $0 \leq \alpha_i < p$, $1 \leq i \leq n$ and $\alpha! = \alpha_1!\alpha_2! \dots \alpha_n!$ (see [1], Theorem 3.3).

Theorem 2.2 *Let H_c, $c \in \{0, 1\}$, be the Hopf algebra in the Lie cases, defined as before and A a right H_c-comodule algebra with structure map $\delta : A \longrightarrow A \otimes H_c$. Let $R = A^{coH_c}$. Assume, for $c = 1$, A is a commutative local algebra with maximal ideal m and $R + m = A$.*

(a) The following are equivalent:

(i) $R \subset A$ is a faithfully flat H_c-Galois extension.

(ii) There are $y_1, \dots, y_n \in A$ with $\delta(y_i) = y_i \otimes 1 + (1 + y_i) \otimes x_i$, for all $1 \leq i \leq n$

(b) Suppose (ii) holds. Then

$$R \otimes H_1 \longrightarrow A, \; r \otimes x^\alpha \mapsto ry^\alpha, \; r \in R, \; \alpha \in \mathbb{A}, \; y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}, \; \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{A}$$

is a left R-linear and right H_c-colinear isomorphism.

In particular, the elements y^α, $\alpha \in \mathbb{A}$, form an R-basis of A as a left R-module.

Proof For $c=0$, see [6], Theorem 3.1.

For $c=1$, (a) is proved in [1], Proposition 4.2. To prove (b) we observe that the coradical C of H_1 is the k-subalgebra of H_1:

$$C = k \oplus kx_1 \oplus \cdots \oplus kx_n, \qquad x_i = X_i + (X_1^p, \ldots, X_n^p).$$

For this, it is sufficient to prove for $i=1$ that $C = k \oplus kx$, $H_1 = k[X]/(X^p) = k[x]$.

$$\Delta(1+x) = \Delta(1) + \Delta(x) = 1 \otimes 1 + 1 \otimes x + x \otimes 1 = (1+x) \otimes (1+x) \in C \otimes C.$$

Moreover, the vector subspaces of H_1, k and kx, are the only simple coalgebras of H_1. Hence the assertion.

Suppose (ii) of (a) holds. Then we define a k-linear map $\gamma : H_1 \to A$ by $\gamma : x^\alpha \mapsto ry^\alpha$ for all $\alpha \in \mathbb{A}$. Since Δ and δ are algebra maps and, for all i,

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i + x_i \otimes x_i, \qquad \delta(y_i) = y_i \otimes 1 + (1+y_i) \otimes x_i,$$

γ is right H_1-colinear. If we prove that the map γ is convolution invertible, the H_1-extension $R \subset A$ is H_1-cleft, hence H_1-Galois and

$$R \otimes H_1 \to A, \quad r \otimes x^\alpha \mapsto ry^\alpha, \quad r \in R, \alpha \in \mathbb{A}$$

is bijective ([9], 8.2.4, 7.2.3). To prove that $\gamma \in \mathrm{Hom}(A, A)$ is invertible with respect to the convolution $*$, it is sufficient to prove that $\gamma_{/C}$ is invertible as an element of $\mathrm{Hom}(C, A)$. For $f \in \mathrm{Hom}(C, A)$, $i = 1, \ldots, n$, it results in

$$\begin{aligned} f * \gamma(1+x_i) &= m(f \otimes \gamma)(\Delta(1+x_i)) = m(f \otimes \delta(1 \otimes x_i + x_i \otimes 1 + x_i \otimes x_i)) \\ &= m(f(1) \otimes \gamma(x_i) + f(x_i) \otimes \delta(1) + f(x_i) \otimes \delta(x_i)) = 1_A y_i + f(x_i) + f(x_i) y_i \\ &= y_i + f(x_i)(1+y_i) \end{aligned}$$

and

$$u\varepsilon(1+x_i) = u(\varepsilon(1) + \varepsilon(x_i)) = u(1) = 1_A,$$

with $m : H_1 \otimes H_1 \longrightarrow H_1$ and $u : k \to H_1$ being the multiplication and the unit maps of H_1, respectively. If we put $f(x_i) = \frac{1-y_i}{1+y_i}$, we have $y_i + f(x_i)(1+y_i) = 1$, γ is left invertible and its inverse map is f. Hence the conclusion follows. □

Remark 2.3 The result contained in Theorem 2.2, (b) can be deduced from (ii), under the hypotheses that the elements $1 + y_i, 1 < i < n$, are invertible, A not necessarily local.

In the following, for $c = 1$, we will suppose that A is commutative, local and $A = R + m$, where R is the coinvariant subring of A with respect to the coaction δ and m is the maximal ideal of A.

Corollary 2.4 *Let H_c be the Hopf Lie algebra of the group H_c, A an algebra and $\delta : A \longrightarrow A \otimes H_c$ a coaction. Put $D_1, \ldots, D_n$ the derivations defined by (1) and $R := A^{coH_c}$. The following are equivalent:*

(1) $R \subset A$ is a faithfully flat H_c-Galois extension.

(2) There are $y_1, \ldots, y_n \in A$ with $D_i(y_j) = \delta_{ij}(1 + cy_i)$, for all $1 \le i, j \le n$.

(3) If A is local there are $y_1, \dots, y_n \in A$ such that for all $1 \le m \le n$, the $m \times m$ matrix $(D_i(y_j))_{1 \le i,j \le m}$ over A is invertible.

Proof For $c = 0$ the result is in [6], Corollary 3.3 and Theorem 4.1.

For $c = 1$, (1) $\iff$ (2) by Theorem 1.8(a), (1) $\iff$ (3) by Theorem 4.1 in [6]. □

Recall that an H-Galois extension $R \subset A$ is faithfully flat if A is faithfully flat over R as a left (or equivalently right) module over R. Recently Schauenburg and Schneider ([9], Theorem 4.5.1) have proved a theorem which allows one to reduce questions about faithfully flat Hopf Galois extensions for H to the case of Hopf subalgebras and quotient algebras of H. We use it to prove the following:

Theorem 2.5 *Let A be a k-algebra, k a field of characteristic $p > 0$ and let $\{D_1, \dots, D_n\} \subset Der_k(A)$ such that $D_iD_j = D_jD_i$, $D_i^p = cD_i$, $c \in \{0, 1\}$, for all $i, j, 1 \le i, j \le n$. Suppose that*

1) There exist $z_1, \dots, z_{n-1} \in A$ such that $D_i(z_j) = \delta_{ij}(1 + cz_i)$, $1 \le i, j \le n - 1$.

2) There exists $y \in A$ such that $D_n(y) = 1 + cy$.

Then $R := A^{coH_c} \subset A$ is a faithfully flat H_c-Galois extension and, consequently, there are $y_1, \dots, y_n \in A$ with $D_i(y_j) = \delta_{ij}(1 + cy_i)$ for all $1 \le i, j \le n$.

Proof The set of derivations comes from a comodule structure of A on H_c, $H = k[x_1, \dots, x_n]$, $x_i^p = 0$, given by $\delta : A \longrightarrow A \otimes H_c$,

$$\delta(a) = a \otimes 1 + \sum_{1 \le i \le n} D_i(a) \otimes x_i + \sum_{\substack{\alpha \in \mathbb{A} \\ |\alpha| \ge 2}} D_\alpha(a) \otimes x^\alpha. \tag{8}$$

$\alpha \in \mathbb{N}^n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let $R = A^{\mathrm{co}H_c}$ be the coinvariant subring of A with respect to δ and let $\overline{H_c} = k[x_n]$, $x_n^p = 0$, $B = A^{\mathrm{co}\overline{H}_c}$ the coinvariant subring of A with respect to $\overline{\delta} : A \longrightarrow A \otimes \overline{H}_c$, $\overline{H}_c = H_c/K^+H_c$, $K = k[x_1, \dots, x_{n-1}]$, $x_i^p = 0$, $i = 1, \dots, n-1$, $K^+ = (x_1, \dots, x_{n-1})$. Consider the extension $R \subset B \subset A$.

$B \subset A$ is $\overline{H}$-Galois extension (Corollary 2.4). By hypothesis 2) and by Corollary 2.4, $R \subset B$ is a K-Galois extension. By Theorem 4.5.1 [9], $R \subset A$ is a faithfully flat H_c-Galois extension and, by Corollary 2.4, there exist $y_1, \dots, y_n \in A$ with $D_i(y_j) = \delta_{ij}(1 + cy_i)$ for all $1 \le i, j \le n$. By 2) the assertion follows. □

3. A constructive theorem

We will describe, in the additive case, the special elements $y_1, \dots, y_n$ that appear in Theorem 2.5 and satisfy a strong condition on the derivability. Following the same direction of research contained in the papers by Matsumura, Restuccia and Utano [5], [8], where the elements are computed, we obtain the result contained in [8] without the hypotheses that A is local, regular and k a separably closed field, but requiring that the last derivation evaluates to one on an element $t \in U(A)$.

Theorem 3.1 *Let A be a commutative k-algebra, k a field of characteristic $p > 0$ and let $\{D_1, \dots, D_n\} \subset Der_k(A)$ such that $D_iD_j = D_jD_i$, $D_i^p = 0$ for all $i, j, 1 \le i, j \le n$. Suppose that*

1) There exist $z_1, \dots, z_{n-1} \in A$ such that $D_i(z_j) = \delta_{ij}$, $1 \le i, j \le n - 1$.

2) There exists $y \in A$ such that $D_n(y) = 1$.

Then there exists $t \in A$ such that $D_n(t) = 1$ and $D_i(t) = 0$, for all $i = 1, \ldots, n-1$.

Proof The set of derivations comes from a comodule structure of A on H, $H = k[x_1, \ldots, x_n]$, $x_i^p = 0$, x_i primitive, given by $\delta : A \longrightarrow A \otimes H$,

$$\delta(a) = a \otimes 1 + \sum_{1 \le i \le n} D_i(a) \otimes x_i + \sum_{\substack{\alpha \in \mathbb{A} \\ |\alpha| \ge 2}} D_\alpha(a) \otimes x^\alpha. \tag{9}$$

$\alpha \in \mathbb{N}^n$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Let $R = A^{\mathrm{co}H}$ be the coinvariant subring of A with respect to δ and let $\overline{H} = k[x_n]$, $x_n^p = 0$, $B = A^{\mathrm{co}\overline{H}}$ the coinvariant subring of A with respect to $\overline{\delta} : A \longrightarrow A \otimes \overline{H}$, $\overline{H} = H/K^+H$, $K = k[x_1, \ldots, x_{n-1}]$, $x_i^p = 0$, $i = 1, \ldots, n-1$. Consider the extensions $R \subset B \subset A$. By 2), $B \subset A$ is $\overline{H}$-Galois and $1, y, y^2, \ldots, y^{p-1}$ is a basis of A on $B = A^{D_n}$. By 1), $R \subset B$ is K-Galois and the monomials $z_1^{j_1} \cdots z_{n-1}^{j_{n-1}}$, $1 \le j_i \le p-1$, $i = 1, \ldots, n-1$, are a basis of B on R. We want to find $t \in A$ such that $D_n(t) = 1$ and $D_i(t) = 0$ for all $i = 1, \ldots, n-1$. Put $t = \sum_{i=0}^{p-1} b_i y^i$. Then $D_n(t) = 1 = \sum_{i=0}^{p-1} b_i i y^{i-1}$ implies $b_1 = 1$ and $b_i = 0$, for all $i > 1$. We can rewrite $t = b_0 + y$ as $t = y - b$, $b \in B$. Then we need an element $b \in B$ such that $D_i(y) = D_i(b)$, $i = 1, \ldots, n-1$. Moreover for $i = 1, \ldots, n-1$, $D_i(y) \in B$, since $D_n(D_i(y)) = D_i(D_n(y)) = D_i(1) = 0$, for all $i = 1, \ldots, n-1$. Then we can write:

$$D_j(y) = \sum_{0 \le i_j \le p-1} s_{j,i_1,\ldots,i_{n-1}} z_1^{i_1} \ldots z_{n-1}^{i_{n-1}}, \quad j = 1, \ldots, n-1, s_{j,i_1,\ldots,i_{n-1}} \in R.$$

Since $D_j^p = 0$, for all $j = 1, \ldots, n-1$, we have:

$$\begin{cases} D_1^{p-1}(D_1(y)) & = & 0 & = & \displaystyle\sum_{0 \le i_j \le p-1} s_{1,i_1,\ldots,i_{n-1}} D_1^{p-1}\left(z_1^{i_1}\right) \ldots z_{n-1}^{i_{n-1}}, \\ \ldots & & \ldots & & \ldots \\ D_{n-1}^{p-1}(D_{n-1}(y)) & = & 0 & = & \displaystyle\sum_{0 \le i_j \le p-1} s_{n-1,i_1,\ldots,i_{n-1}} z_1^{i_1} \ldots D_{n-1}^{p-1}\left(z_{n-1}^{i_{n-1}}\right). \end{cases}$$

Hence we get the relations

$$\begin{cases} 0 & = & \displaystyle\sum_{\substack{0 \le i_j \le p-1 \\ j \ne 1}} s_{1,p-1,i_2,\ldots,i_{n-1}} (p-1)! z_2^{i_2} \ldots z_{n-1}^{i_{n-1}}, \\ \ldots & \ldots & \ldots \\ 0 & = & \displaystyle\sum_{\substack{0 \le i_j \le p-1 \\ j \ne n-1}} s_{n-1,i_1,\ldots,p-1} (p-1)! z_1^{i_1} \ldots z_{n-2}^{i_{n-2}}, \end{cases}$$

and

$$\begin{cases} s_{1,p-1,i_2,\ldots,i_{n-1}} & = & 0 & 0 \le i_2, \ldots, i_{n-1} \le p-1 \\ \ldots & & \ldots & \ldots \\ s_{n-1,i_1,\ldots,i_{n-2},p-1} & = & 0 & 0 \le i_1, \ldots, i_{n-2} \le p-1. \end{cases}$$

Writing

$$b = \sum_{0 \le j_i \le p-1} t_{j_1,\ldots,j_{n-1}} z_1^{j_1} \ldots z_{n-1}^{j_{n-1}},$$

b is uniquely determined by coefficients $t_{j_1,\dots,j_{n-1}}$, $0 \le j_i \le p-1$. By derivation, we obtain

$$\begin{cases} D_1(b) & = \displaystyle\sum_{0\le j_i\le p-1} t_{j_1,\dots,j_{n-1}} j_1 z_1^{j_1-1}\dots z_{n-1}^{j_{n-1}}, \\ & = \displaystyle\sum_{\substack{0\le j_1\le p-2\\ 0\le j_i\le p-1, i\neq 1}} t_{j_1+1,j_2,\dots,j_{n-1}}(j_1+1) z_1^{j_1}\dots z_{n-1}^{j_{n-1}}. \\ \dots & \dots \quad \dots \\ D_{n-1}(b) & = \displaystyle\sum_{0\le j_i\le p-1} t_{j_1,j_2,\dots,j_{n-1}} j_{n-1} z_1^{j_1}\dots z_{n-1}^{j_{n-1}-1} \\ & = \displaystyle\sum_{\substack{0\le j_{n-1}\le p-2\\ 0\le j_i\le p-1, i\neq n-1}} t_{j_1,\dots,j_{n-1}+1}(j_{n-1}+1) z_1^{j_1}\dots z_{n-1}^{j_{n-1}}. \end{cases}$$

From $D_i(y) = D_i(b)$, for $i = 1, \dots, n-1$, it follows

$$\begin{cases} \displaystyle\sum_{\substack{0\le j_1\le p-2\\ 0\le j_i\le p-1, i\neq 1}} t_{j_1+1,j_2,\dots,j_{n-1}}(j_1+1) z_1^{j_1}\dots z_{n-1}^{j_{n-1}} & = \displaystyle\sum_{\substack{0\le j_1\le p-2\\ 0\le j_i\le p-1, i\neq 1}} s_{1,j_1,\dots,j_{n-1}} z_1^{j_1}\dots z_{n-1}^{j_{n-1}}, \\ \dots & \dots \quad \dots \\ \displaystyle\sum_{\substack{0\le j_{n-1}\le p-2\\ 0\le j_i\le p-1, i\neq n-1}} t_{j_1,j_2,\dots,j_{n-1}+1}(j_{n-1}+1) z_1^{j_1}\dots z_{n-1}^{j_{n-1}} & = \displaystyle\sum_{\substack{0\le j_{n-1}\le p-2\\ 0\le j_i\le p-1, i\neq n-1}} s_{n-1,j_1,\dots,j_{n-1}} z_1^{j_1}\dots z_{n-1}^{j_{n-1}}. \end{cases}$$

Hence we get the relations

$$\begin{cases} t_{j_1+1,j_2,\dots,j_{n-1}}(j_1+1) & = \quad s_{1,j_1,\dots,j_{n-1}}\ 0\le j_1\le p-2, 0\le j_i\le p-1, i\neq 1, \\ \dots & \dots \quad \dots \\ t_{j_1,j_2,\dots,j_{n-1}+1}(j_{n-1}+1) & = \quad s_{n-1,j_1,\dots,j_{n-1}}\ 0\le j_{n-1}\le p-2, 0\le j_i\le p-1, i\neq n-1 \end{cases} \tag{10}$$

From the conditions $D_k D_\ell = D_\ell D_k$ for $1 \le \ell < k \le n-1$ we obtain the compatibility relations

$$j_k s_{\ell,j_1,j_2,\dots,j_\ell,\dots,j_k,\dots,j_{n-1}} = (j_\ell+1) s_{k,j_1,j_2,\dots,j_\ell+1,\dots,j_k-1,\dots,j_{n-1}} \tag{11}$$

with $0 \le j_\ell, \le p-2, \quad 1 \le j_k \le p-1, 0 \le j_i \le p-1, i \neq \ell, k, \quad 1 \le \ell < k \le n-1$. The first two relations of (10) give, for $\ell = 1, k = 2$

$$t_{j_1+1,j_2,\dots,j_{n-1}}(j_1+1) = s_{1,j_1,\dots,j_{n-1}} \qquad 0\le j_i\le p-1, i\neq 1, 0\le j_1\le p-2, \tag{12}$$

$$t_{j_1,j_2+1,\dots,j_{n-1}}(j_2+1) = s_{2,j_1,\dots,j_{n-1}} \qquad 0\le j_i\le p-1, i\neq 2, 0\le j_2\le p-2. \tag{13}$$

We rewrite the relations (12) and (13)

$$t_{j_1,j_2,\dots,j_{n-1}} j_1 = s_{1,j_1-1,j_2,\dots,j_{n-1}} \qquad 0\le j_i\le p-1, i\neq 1, 1\le j_1\le p-2,$$

$$t_{j_1,j_2,\dots,j_{n-1}} j_2 = s_{2,j_1,j_2-1,\dots,j_{n-1}}, 0\le j_i\le p-1, i\neq 2, 1\le j_2\le p-2,$$

obtaining

$$j_1 j_2 t_{j_1,j_2,\dots,j_{n-1}} = j_2 s_{1,j_1-1,j_2,\dots j_{n-1}} = j_1 s_{2,j_1,j_2-1,\dots,j_{n-1}}.$$

Likewise, we can deduce

$$j_1\dots j_{n-1} t_{j_1,j_2,\dots,j_{n-1}} = j_2\dots j_{n-1} s_{1,j_1-1,j_2,\dots,j_{n-1}} = j_1 j_3\dots j_{n-1} s_{2,j_1,j_2-1,\dots,j_{n-1}} =$$

$$\cdots = j_1 j_2 \ldots j_{n-2} s_{n-1,j_1,j_2,\ldots,j_{n-1}+1}, \quad \text{for} \quad 0 \leq j_i \leq p-1.$$

Hence, the elements $t_{j_1,j_2,\ldots,j_{n-1}}$ are determined and, as a consequence, the element b is obtained. □

Corollary 3.2 *Let A be a k-algebra, k a field of characteristic $p > 0$ and let $\{D_1, \ldots, D_n\} \subset Der_k(A)$ such that $D_i D_j = D_j D_i$, $D_i^p = 0$ for all $i, j, 1 \leq i, j \leq n$.*

Suppose that

1) *There exist $z_1, \ldots, z_{n-1} \in A$ such that $D_i(z_j) = \delta_{ij}$, $1 \leq i, j \leq n-1$.*

2) *There exists $y \in A$ such that $D_n(y) = 1$.*

Then there exist $z_1, \ldots, z_{n-1}, z_n$ such that $D_i(z_j) = \delta_{ij}$.

Proof Follows from Theorem 3.1, with $z_n = t$. □

Corollary 3.3 *Let A be a k-algebra, k a field of characteristic $p > 0$ and let $\{D_1, \ldots, D_n\} \subset Der_k(A)$ such that $D_i D_j = D_j D_i$, $D_i^p = 0$ for all $i, j, 1 \leq i, j \leq n$.*

Suppose that:

1) *There exist $z_1, \ldots, z_{n-1} \in A$ such that $D_i(z_j) = \delta_{ij}$, $1 \leq i, j \leq n-1$.*

2) *There exists $y \in A$ such that $D_n(y) = 1$.*

Then the set $\{z_1, \ldots, z_{n-1}\}$ of p-independent elements of A on the subring of constants $A^{\{D_1,\ldots,D_n\}}$ can be completed to a p-basis B of n elements of A on $A^{\{D_1,\ldots,D_n\}}$ and $A = A^{\{D_1,\ldots,D_n\}}[B]$.

Proof It is easy to prove that $z_1, \ldots, z_{n-1}, z_n$, with z_n as in Corollary 3.2, verify $D_i(z_j) = \delta_{ij}, 1 \leq i \leq j \leq n, D_i^p = 0, [D_i, D_j] = 0$ and form a p-basis of A on $A^{\{D_1,\ldots,D_n\}}$. The structure of A follows by definition of p-basis [3]). □

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