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Gorenstein transpose with respect to a semidualizing bimodule

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Abstract: Let S and R be rings and ${}_SC_R$ be a semidualizing bimodule. We first give the definitions of C-transpose and n-C-torsionfree and give a criterion for a module A to be G_C-projective by some property of the C-transpose of A. Then we introduce the notion of C-Gorenstein transpose of a module over two-sided Noetherian rings. We prove that a module M in $\operatorname{mod} R^{op}$ is a C-Gorenstein transpose of a module $A \in \operatorname{mod} S$ if and only if M can be embedded into a C-transpose of A with the cokernel G_C-projective. Finally we investigate some homological properties of the C-Gorenstein transpose of a given module.

Key words: Semidualizing bimodule, G_C-projective, C-transpose, n-C-torsionfree, C-Gorenstein transpose

1. Introduction

The notion of the transpose of a finitely generated module, which was introduced by Asulander and Bridger in [1] to investigate the n-torsionfree modules over two-sided Noetherian rings, plays an important role in the study of the representation theory of algebra. We know that the transpose of a given module M is obtained from a projective presentation of M. Replacing the projective presentation by Gorenstein projective presentation, Huang and Huang [6] introduced the notion of Gorenstein transpose. Although Gorenstein transpose of a module M may be dependent on the choice of the Gorenstein projective presentation of M, any different two Gorenstein transposes of the same module share some common homological properties; see [6, Proposition 3.4]. Moreover, the relations between the Gorenstein transpose of a given module M and the transpose of M were investigated, see [6, Theorem 3.1].

Recently, the research of semidualizing modules has caught many authors' attention. For example, Holm and Jørgensen in [4] introduced and investigated the so-called C-Gorenstein projective (injective, flat) dimension with respect to a semidualizing module C, while Sather-Wagstaff, Sharif and White in [10] investigated Tate cohomology of modules over a commutative Noetherian ring with respect to semidualizing modules. In fact, semidualizing modules were first defined over commutative Noetherian rings, while Holm and White [5] extended the definition of semidualizing modules to a pair of arbitrary associative rings.

In this paper, we extend the notions of transpose, Gorenstein transpose and n-torsionfree modules to the semidualizing setting, that is, C-transpose, C-Gorenstein transpose and n-C-torsionfree modules with respect to a semidualizing module C. In fact, Huang in [7] introduced ω-transpose and n-ω-torsionfree, where ${}_S\omega_R$ is

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a faithfully balanced and selforthogonal bimodule over two-sided Noetherian rings. These two notions coincide with C-transpose and n-C-torsionfree studied in our paper.

This paper is organized as follows.

Section 2 is devoted to some preliminary works.

In section 3, for a semidualizing bimodule ${}_SC_R$ over two-sided Noetherian rings, we study C-transpose and n-C-torsionfree modules, which was studied by Huang in [7] under different names. We give a new characterization of n-C-torsionfree modules (see Proposition 3.3) and, in particular, we give a criterion for a module to be G_C-projective; see Theorem 3.6.

In section 4, for a semidualizing bimodule ${}_SC_R$ and a module $A \in \operatorname{mod} S$, we introduce the C-Gorenstein transpose of A. We first get some interesting exact sequences with respect to C-Gorenstein transpose, and then we show the tight relation between the C-transpose and the C-Gorenstein transpose of a same module in Theorem 4.6, which extend the result given in [6, Theorem 3.1]. Finally, we investigate some homological properties of C-Gorenstein transpose, which also extend the corresponding results given in [6].

2. Preliminaries

In this section, S and R are associative rings with identities and all modules are unitary. We use $\operatorname{Mod} S$ (resp. $\operatorname{Mod} R^{op}$) to denote the class of left S-modules (resp. right R-modules).

At the beginning of this section we recall some notions.

A *degreewise finite projective resolution* of a module M is a projective resolution $\mathbf{P}$ of M such that each P_i is a finitely generated projective module.

Definition 2.1 *([5, Definition 2.1]) An (S,R)-bimodule $C = {}_SC_R$ is semidualizing if*

$(a1)$ ${}_SC$ admits a degreewise finite S-projective resolution.

$(a2)$ C_R admits a degreewise finite R^{op}-projective resolution.

$(b1)$ The homothety map ${}_SS_S \longrightarrow \operatorname{Hom}_{R^{op}}(C,C)$ is an isomorphism.

$(b2)$ The homothety map ${}_RR_R \longrightarrow \operatorname{Hom}_S(C,C)$ is an isomorphism.

$(c1)$ $\operatorname{Ext}^i_S(C,C) = 0$ for any $i \geq 1$.

$(c2)$ $\operatorname{Ext}^i_{R^{op}}(C,C) = 0$ for any $i \geq 1$.

Assume that ${}_SC_R$ is a semidualizing bimodule.

Definition 2.2 *([5, Definition 5.1]) A module in $\operatorname{Mod} S$ is called C-projective if it is isomorphic to a module of the form $C \otimes_R P$ for some projective module $P \in \operatorname{Mod} R$.*

$$\mathcal{P}_C(S) = \text{ the class of } C\text{-projective modules in } \operatorname{Mod} S.$$

Let $M \in \operatorname{Mod} S$. We denote by $\operatorname{Add}_S M$ (resp. $\operatorname{add}_S M$) the subclass of $\operatorname{Mod} S$ (resp. $\operatorname{mod} S$) consisting of all modules isomorphic to direct summands of direct sums (resp. finite direct sums) of copies of M.

Remark 2.3 *By [3, Theorem 3.1], we know that $\operatorname{Add}_S C$ is just the class of C-projective modules in $\operatorname{Mod} S$. Recall that for a module $M \in \operatorname{Mod} S$, the $\operatorname{Add}_S C$-dimension of M, denoted by $\operatorname{Add}_S C\text{-}\dim_S M$, is defined as $\inf\{n \mid$ there exists an exact sequence $0 \to C_n \to \cdots \to C_1 \to C_0 \to M \to 0$ in $\operatorname{Mod} S$ with all $C_i \in \operatorname{Add}_S C\}$. We set $\operatorname{Add}_S C\text{-}\dim_S M = \infty$ if no such integer exists.*

Let $\mathcal{C}$ be a subclass of $\operatorname{Mod} S$*. Recall that a sequence* $\mathbf{L}: \cdots \to L_1 \to L_0 \to L_{-1} \to \cdots$ *with* $L_i \in \operatorname{Mod} S$ *is called* $\operatorname{Hom}_S(-, \mathcal{C})$*-exact if the sequence* $\operatorname{Hom}_S(\mathbf{L}, C')$ *is exact for any* $C' \in \mathcal{C}$*. The following notions were introduced by Holm and Jørgensen in [4] and White in [12] for commutative rings. In the non-commutative case, the definition can be given in a similar way.*

Definition 2.4 *A complete* $\mathcal{PP}_C$*-resolution is a* $\operatorname{Hom}_S(-, \operatorname{Add}_S C)$*-exact exact sequence:*

$$\mathbf{X} = \cdots \to P_1 \to P_0 \to C^0 \to C^1 \to \cdots \tag{2.1}$$

in $\operatorname{Mod} S$ *with all* P_i *projective and* $C^i \in \operatorname{Add}_S C$*. A module* $M \in \operatorname{Mod} S$ *is called* G_C*-projective if there exists a complete* $\mathcal{PP}_C$*-resolution as in (2.1) with* $M \cong \operatorname{Im}(P_0 \to C^0)$*. Set*

$$\mathcal{GP}_C(S) = \text{ the class of } G_C - \text{projective modules in } \operatorname{Mod} S.$$

Definition 2.5 *([12]) For a module* $M \in \operatorname{Mod} S$*, the* G_C*-projective dimension of* M*, denoted by* $G_C\text{-pd}_S M$*, is defined as* $\inf\{n \mid$ *there exists an exact sequence* $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ *in* $\operatorname{Mod} S$ *with all* G_i G_C*-projective*$\}$*. Since projective modules are always* G_C*-projective, we have* $G_C\text{-pd}_S M \geq 0$ *and we set* $G_C\text{-pd}_S M = \infty$ *if no such integer exists.*

Remark 2.6 *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence in* $\operatorname{Mod} S$*. If* $L \neq 0$ *and* N *is* G_C*-projective, then* $G_C\text{-pd}_S L = G_C\text{-pd}_S M$*.*

Proof It is easy to get the assertions by [12, Propositions 2.12 and 2.14]. □

The following Proposition generalizes [2, Lemma 2.17].

Proposition 2.7 *Let* $M \in \operatorname{Mod} S$ *with* $G_C\text{-pd}_S M = n$*. Then there exists an exact sequence* $0 \to M \to N \to G \to 0$ *in* $\operatorname{Mod} S$ *with* $\operatorname{Add}_S C\text{-dim}_S N = n$ *and* G G_C*-projective.*

Proof Since $G_C\text{-pd}_S M = n$, we have an exact sequence $0 \to L \to G' \to M \to 0$ with $\operatorname{Add}_S C\text{-dim}_S L \leq n-1$ and G' G_C-projective by [12, Theorem 3.6]. Thus we have an exact sequence $0 \to G' \to C' \to G \to 0$ with $C' \in \operatorname{Add}_S C$ and G G_C-projective by [12, Proposition 2.9]. Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & G' & \to & M & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & C' & \to & N & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

So we have the exact sequence $0 \to M \to N \to G \to 0$ in $\operatorname{Mod} S$ with $\operatorname{Add}_S C\text{-dim}_S N \leq n$ and G G_C-projective. By Lemma 2.6, $G_C\text{-pd}_S N = n$, and thus $\operatorname{Add}_S C\text{-dim}_S N = n$. □

3. C-transpose and n-C-torsionfree module

Assume that S is a left Noetherian ring and R is a right Noetherian ring, $\mathrm{mod}\, S$ (resp. $\mathrm{mod}\, R^{op}$) is the category of finitely generated left S-modules (resp. right R-modules).

Huang in [7] introduced ω-n-torsionfree modules with respect to a faithfully balanced and selforthogonal bimodule ${}_S\omega_R$ and characterized these modules by the notion of ω-transpose $\mathrm{Tr}_\omega A$ of a given module A. In this section, we first introduce the notions of C-transpose and n-C-torsionfree, which, in fact, is given by replacing ω with the semidualizing bimodule ${}_SC_R$. Then we give some characterizations of n-C-torsionfree modules, which generalize [7, Theorem 1]. Finally, for a given module $A \in \mathrm{mod}\, S$, we give a criterion for A to be G_C-projective by the vanishing of Ext with respect to C, A and the C-transpose of A.

Definition 3.1 *(1) For any $A \in \mathrm{mod}\, S$, there is an exact sequence $\varepsilon :\ P_1 \xrightarrow{f} P_0 \to A \to 0$ in $\mathrm{mod}\, S$ with P_0 and P_1 projective. Then we have an exact sequence $0 \to A^\dagger \to P_0^\dagger \xrightarrow{f^\dagger} P_1^\dagger \to X \to 0$, where $(\ )^\dagger = \mathrm{Hom}_S(\ , C)$ and $X = \mathrm{Coker}\, f^\dagger$ which we call a C-transpose of A and denote it by $\mathrm{Tr}_C^\varepsilon A$.*

(2) (cf. [7, Definition 2]) Let A and $\mathrm{Tr}_C^\varepsilon A$ be as above. A is called a n-C-torsionfree module if $\mathrm{Ext}^i_{R^{op}}(\mathrm{Tr}_C^\varepsilon A, C) = 0$ for any $1 \le i \le n$.

(3) We say that A is a ∞-C-torsionfree module if it is n-C-torsionfree for any $n \ge 1$.

Remark 3.2 *(1) Masiek in [11] proved that the transpose of a given finitely generated module M over a commutative Noetherian ring is unique up to projective equivalence. Following his arguments in the proof of [11, Proposition 4], for a given module $A \in \mathrm{mod}\, S$ and any two C-transposes $\mathrm{Tr}_C^{\varepsilon_1} A$ and $\mathrm{Tr}_C^{\varepsilon_2} A$ of A, we have a C-transpose $\mathrm{Tr}_C^{\varepsilon_3} A$ and two exact sequences: $0 \to \mathrm{Tr}_C^{\varepsilon_1} A \to \mathrm{Tr}_C^{\varepsilon_3} A \to K_1 \to 0$ and $0 \to \mathrm{Tr}_C^{\varepsilon_2} A \to \mathrm{Tr}_C^{\varepsilon_3} A \to K_2 \to 0$ with $K_i \in \mathrm{add}_S C$. Thus, any two C-transposes of A have the same G_C-projective dimension by Lemma 2.6.*

(2) If R is a two-sided Noetherian ring and ${}_SC_R = {}_RR_R$, then n-C-torsionfree is the same as n-torsionfree.

(3) The definition of n-C-torsionfree modules above is well-defined by [7, Proposition 3], that is, it does not depend on the choice of a projective resolution of the given module.

In the following, some characterizations of n-C-torsionfree modules are given, which generalize [7, Theorem 1]. For the definition of left approximations we refer the reader to [7, Definition 1]. For any $M \in \mathrm{mod}\, S$ and $n \ge 1$, we denote $\mathrm{Ext}^n_S(M, \mathrm{add}_S C) = \{\mathrm{Ext}^n_S(M, C') \mid C' \in \mathrm{add}_S C\}$.

Definition 3.3 *Let $A \in \mathrm{mod}\, S$ and n be a positive integer. The following statements are equivalent.*

(1) A is an n-C-torsionfree module.

(2) There is an exact sequence $0 \to A \xrightarrow{f_1} C^{m_1} \xrightarrow{f_2} \cdots \xrightarrow{f_n} C^{m_n}$ such that each $\mathrm{Im}\, f_i \to C^{m_i}$ is a left $\mathrm{add}_S C$-approximation of $\mathrm{Im}\, f_i$ for $1 \le i \le n$.

(3) There is an exact sequence $0 \to A \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$ such that each $\mathrm{Im}\, f_i \to X_i$ is a left $\mathrm{add}_S C$-approximation of $\mathrm{Im}\, f_i$ for $1 \le i \le n$.

(4) There is an exact sequence $0 \to A \xrightarrow{f_1} G_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} G_n$ with G_i G_C-projective, which is $\mathrm{Hom}_S(-, \mathrm{add}_S C)$-exact.

Proof The equivalences among (1), (2) and (3) are from [7, Theorem 1] and (3) implies (4) by [12, Proposition 2.6]. We only have to show that (4) implies (3).

Assume that there is an exact sequence $0 \to A \xrightarrow{f_1} G_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} G_n$ with G_i G_C-projective, which is $\mathrm{Hom}_S(-,\ \mathrm{add}_S C)$-exact. Putting $\mathrm{Im}\, f_i = K_i$, we have $\mathrm{Ext}^1_S(K_i, \mathrm{add}_S C) = 0$ for any $2 \le i \le n$ and $\mathrm{Hom}_S(-,\ \mathrm{add}_S C)$-exact exact sequences $0 \to K_i \to G_i \to K_{i+1} \to 0$. Since all the $G_i \in \mathcal{GP}_C(S)$, for any G_i we have an $\mathrm{Hom}_S(-, \mathrm{add}_S C)$-exact exact sequence $0 \to G_i \xrightarrow{g_i^0} C_i^0 \xrightarrow{g_i^1} C_i^1 \xrightarrow{g_i^2} \cdots$ with all the $C_i^j \in \mathrm{add}_S C$. Setting $\mathrm{Im}\, g_i^j = B_i^j$, we have $\mathrm{Ext}^1_S(B_i^j, \mathrm{add}_S C) = 0$ for any $1 \le i \le n$ and $j \ge 0$. In the pushout diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A & = & A & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & G_1 & \to & C_1^0 & \to & B_1^1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & K_2 & \to & D_1 & \to & B_1^1 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0, & & & &
\end{array}
$$

we have $\mathrm{Ext}^1_S(D_1, \mathrm{add}_S C) = 0$, and the middle column is a $\mathrm{Hom}_S(-, \mathrm{add}_S C)$-exact exact sequence.

Similar arguments to K_2 show that there exists an exact sequence $0 \to K_2 \to C_2^0 \to D_1^{'} \to 0$ with $\mathrm{Ext}^1_S(D_1^{'}, \mathrm{add}_S C) = 0$. Since the bottom row of the above diagram is a $\mathrm{Hom}_S(-, \mathrm{add}_S C)$-exact exact sequence, we have the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_2 & \to & D_1 & \to & B_1^1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_2^0 & \to & C_2^0 \oplus C_1^1 & \to & C_1^1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & D_1^{'} & \to & D_2 & \to & B_1^2 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}
$$

And also we have $\mathrm{Ext}^1_S(D_2, \mathrm{add}_S C) = 0$ and the middle column is a $\mathrm{Hom}_S(-, \mathrm{add}_S C)$-exact exact sequence.

The similar arguments to $D_1^{'}$ show that there exists an exact sequence $0 \to D_1^{'} \to C_3^0 \oplus C_2^1 \to D_2^{'} \to 0$ with $\mathrm{Ext}^1_S(D_2^{'}, \mathrm{add}_S C) = 0$. Since the bottom row of the above diagram is $\mathrm{Hom}_S(-, \mathrm{add}_S C)$-exact, we have the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_1' & \longrightarrow & D_2 & \longrightarrow & B_1^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_3^0 \oplus C_2^1 & \longrightarrow & C_3^0 \oplus C_2^1 \oplus C_1^3 & \longrightarrow & C_1^3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_2' & \longrightarrow & D_3 & \longrightarrow & B_1^3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with $\operatorname{Ext}_S^1(D_3, \operatorname{add}_S C) = 0$, and the middle column is $\operatorname{Hom}_S(-, \operatorname{add}_S C)$-exact. Iterating this procedure, we eventually obtain an $\operatorname{Hom}_S(-, \operatorname{add}_S C)$-exact exact sequence:

$$0 \to A \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$$

such that each $\operatorname{Im} f_i \to X_i$ is a left $\operatorname{add}_S C$-approximation of $\operatorname{Im} f_i$ for $1 \leq i \leq n$. □

For any $A \in \operatorname{mod} S$, let $\sigma_A : A \to A^{\dagger\dagger}$ via $\sigma_A(x)(f) = f(x)$ for any $x \in A$ and $f \in A^{\dagger}$ be the canonical evaluation homomorphism. A is called a *C-torsionless* module if σ_A is a monomorphism; and A is called a *C-reflexive* module if σ_A is an isomorphism. By [7, Lemma 4], A is C-torsionless (resp. C-reflexive) if and only if A is 1-C-torsionfree (resp. 2-C-torsionfree). Note that this can also be obtained from Lemma 4.3 in the following section.

Recall from [9, Definition 3.1], we know that a module A in $\operatorname{mod} S$ is said to have *generalized Gorenstein dimension zero with respect to C* if the following conditions hold:

(1) A is C-reflexive.

(2) $\operatorname{Ext}_S^i(A, C) = 0 = \operatorname{Ext}_{R^{op}}^i(A^{\dagger}, C)$ for any $i \geq 1$.

Remark 3.4 *It is easy to verify that a module A in $\operatorname{mod} S$ has generalized Gorenstein dimension zero with respect to C if and only if it is G_C-projective over two-sided Noetherian rings by [12, Theorem 4.4].*

Lemma 3.5 ([8, Lemma 2.9]) *Let $n \geq 3$. Then a C-reflexive module A in $\operatorname{mod} S$ is n-C-torsionfree if and only if $\operatorname{Ext}_{R^{op}}^i(A^{\dagger}, C) = 0$ for any $1 \leq i \leq n-2$.*

Now we can give a criterion for a module $A \in \operatorname{mod} S$ to be G_C-projective.

Theorem 3.6 *Let $A \in \operatorname{mod} S$. Then A is G_C-projective if and only if $\operatorname{Ext}_S^i(A, C) = 0 = \operatorname{Ext}_{R^{op}}^i(\operatorname{Tr}_C^{\varepsilon} A, C)$ for any C-transpose of A and any $i \geq 1$.*

Proof Let $A \in \operatorname{mod} S$. If A is G_C-projective, then we have that A is C-reflexive and $\operatorname{Ext}_S^i(A, C) = 0 = \operatorname{Ext}_{R^{op}}^i(A^{\dagger}, C)$ for any $i \geq 1$. Thus A is ∞-C-torsionfree by Lemma 3.5. Hence $\operatorname{Ext}_S^i(A, C) = 0 = \operatorname{Ext}_{R^{op}}^i(\operatorname{Tr}_C^{\varepsilon} A, C)$ for any C-transpose of A and any $i \geq 1$.

If A satisfies $\mathrm{Ext}^i_{R^{op}}(\mathrm{Tr}^{\varepsilon}_C A, C) = 0$ for any C-transpose of A and any $i \geq 1$, then A is ∞-C-torsionfree by definition. Thus A is C-reflexive, and $\mathrm{Ext}^i_{R^{op}}(A^{\dagger}, C) = 0$ for any $i \geq 1$ by Lemma 3.5. The proof is finished. □

Remark 3.7 *By Lemma 3.5 and Theorem 3.6, it is not difficult to see that if $A \in \mathrm{mod}\, S$ is G_C-projective, then so is $A^{\dagger}$.*

4. C-Gorenstein transpose

Chonghui Huang and Zhaoyong Huang in [6] introduced Gorenstein transpose of a module and investigated the relations between the Gorenstein transpose and the transpose of the same module. In this section, we extend the notion of Gorenstein transpose to C-Gorenstein transpose as follows.

Let $A \in \mathrm{mod}\, S$. Then there exists a G_C-projective presentation of A in $\mathrm{mod}\, S$

$$\pi : X_1 \xrightarrow{g} X_0 \to A \to 0.$$

Then we get an exact sequence:

$$0 \to A^{\dagger} \to X_0^{\dagger} \xrightarrow{g^{\dagger}} X_1^{\dagger} \to \mathrm{Coker}\, g^{\dagger} \to 0,$$

in $\mathrm{mod}\, R^{op}$.

Definition 4.1 *Let A and $\mathrm{Coker}\, g^{\dagger}$ as above. We call $\mathrm{Coker}\, g^{\dagger}$ a C-Gorenstein transpose of A and denote it by $\mathrm{Tr}^{\pi}_{G_C} A$.*

It is trivial that a C-transpose of A is a C-Gorenstein transpose of A, but the converse does not hold true in general.

In the following, we will establish a relation between a C-Gorenstein transpose and a C-transpose of the same module. First, we show that any C-Gorenstein transpose of a given module A can be embedded into a C-transpose of the same module.

Proposition 4.2 *Let $A \in \mathrm{mod}\, S$. For any C-Gorenstein transpose $\mathrm{Tr}^{\pi}_{G_C} A$, there exists an exact sequence $0 \to \mathrm{Tr}^{\pi}_{G_C} A \to \mathrm{Tr}^{\varepsilon}_C A \to G \to 0$ in $\mathrm{mod}\, R^{op}$ for some C-transpose $\mathrm{Tr}^{\varepsilon}_C A$ of A and some G_C-projective module G. In particular, for any $A \in \mathrm{mod}\, S$ and any $\mathrm{Tr}^{\pi}_{G_C} A$ and any $\mathrm{Tr}^{\varepsilon}_C A$, there exists an isomorphism $\mathrm{Ext}^i_{R^{op}}(\mathrm{Tr}^{\pi}_{G_C} A, C) \cong \mathrm{Ext}^i_{R^{op}}(\mathrm{Tr}^{\varepsilon}_C A, C)$ for any $i \geq 1$.*

Proof Let $A \in \mathrm{mod}\, S$. For a C-Gorenstein transpose $\mathrm{Tr}^{\pi}_{G_C} A$, there exists an exact sequence $\pi : X_1 \xrightarrow{g} X_0 \to A \to 0$ in $\mathrm{mod}\, S$ with X_0 and X_1 G_C-projective such that $\mathrm{Tr}^{\pi}_{G_C} A = \mathrm{Coker}\, g^{\dagger}$. Then there exists an exact sequence $0 \to G_1' \to P_0' \to X_0 \to 0$ in $\mathrm{mod}\, S$ with P_0' projective and G_1' G_C-projective. Let $K_1 = \mathrm{Im}\, g$ and $g = i\alpha$ be the natural epic-monic decomposition of g. Then we have the following pull-back diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_1' & = & G_1' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K_1' & \longrightarrow & P_0' & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & K_1 & \xrightarrow{i} & X_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

Now consider the following pull-back diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G_1' & = & G_1' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2 & \longrightarrow & G & \longrightarrow & K_1' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2 & \longrightarrow & X_1 & \xrightarrow{\alpha} & K_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}
$$

where $K_2 = \operatorname{Ker} g$. Since both X_1 and G_1' are G_C-projective, G is G_C-projective by [12, Theorem 2.8]. So there exists an exact sequence $0 \to G_1 \to P_0 \to G \to 0$ in $\operatorname{mod} S$ with P_0 projective and G_1 G_C-projective. Consider the following pull-back diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G_1 & = & G_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K_2' & \longrightarrow & P_0 & \longrightarrow & K_1' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow & & \| & & \\
0 & \longrightarrow & K_2 & \longrightarrow & G & \longrightarrow & K_1' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}
$$

So we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_2' & \longrightarrow & P_0 & \longrightarrow & K_1' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow & & \Vert & & \\
0 & \longrightarrow & K_2 & \longrightarrow & G & \longrightarrow & K_1' & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2 & \longrightarrow & X_1 & \xrightarrow{\alpha} & K_1 & \longrightarrow & 0.
\end{array}$$

It yields the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & G_1 & & H_1 & & G_1' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2' & \longrightarrow & P_0 & \longrightarrow & K_1' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2 & \longrightarrow & X_1 & \xrightarrow{\alpha} & K_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}$$

where $H_1 = \mathrm{Ker}(P_0 \to X_1)$ and $G_1' = \mathrm{Ker}(K_1' \to K_1)$. By the Snake Lemma, we get an exact sequence $0 \to G_1 \to H_1 \to G_1' \to 0$ with H_1 G_C-projective. Combining the above diagram with the first one in this proof, we get the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & G_1 & & H_1 & & G_1' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2' & \longrightarrow & P_0 & \longrightarrow & K_1' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2 & \longrightarrow & X_1 & \xrightarrow{\alpha} & K_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}$$

By applying the functor $(\ )^{\dagger}$ to the above diagram, we get the following commutative diagram with exact columns and rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A^{\dagger} & = & A^{\dagger} & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_0^{\dagger} & \longrightarrow & P_0^{'\dagger} & \longrightarrow & G_1^{'\dagger} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g^{\dagger}} & & \downarrow & & \downarrow{\scriptstyle h^{\dagger}} & & \\
0 & \longrightarrow & X_1^{\dagger} & \longrightarrow & P_0^{\dagger} & \longrightarrow & H_1^{\dagger} & \longrightarrow & 0.
\end{array}$$

By the Snake Lemma, we get an exact sequence:

$$0 \to \operatorname{Tr}^{\pi}_{G_C} A (= \operatorname{Coker} g^{\dagger}) \to \operatorname{Tr}^{\varepsilon}_{C} A \to \operatorname{Coker} h^{\dagger} \to 0$$

in $\operatorname{mod} R^{op}$ with $\operatorname{Coker} h^{\dagger} = G_1^{\dagger}$ G_C-projective.

So $\operatorname{Ext}^i_{R^{op}}(\operatorname{Coker} h^{\dagger}, C) = 0$ for any $i \geq 1$ and hence $\operatorname{Ext}^i_{R^{op}}(\operatorname{Tr}^{\pi}_{G_C} A, C) \cong \operatorname{Ext}^i_{R^{op}}(\operatorname{Tr}^{\varepsilon}_{C} A, C)$ for any $i \geq 1$. □

Lemma 4.3 *([9, Lemma 2.1]) Let $A \in \operatorname{mod} S$ and $\operatorname{Tr}^{\varepsilon}_{C} A$ be a C-transpose of A. Then we have the following exact sequences:*

$$(*) \quad 0 \to \operatorname{Ext}^1_{R^{op}}(\operatorname{Tr}^{\varepsilon}_{C} A, C) \to A \xrightarrow{\sigma_A} A^{\dagger\dagger} \to \operatorname{Ext}^2_{R^{op}}(\operatorname{Tr}^{\varepsilon}_{C} A, C) \to 0.$$

$$0 \to \operatorname{Ext}^1_S(A, C) \to \operatorname{Tr}^{\varepsilon}_{C} A \xrightarrow{\sigma_{\operatorname{Tr}^{\varepsilon}_{C} A}} (\operatorname{Tr}^{\varepsilon}_{C} A)^{\dagger\dagger} \to \operatorname{Ext}^2_S(A, C) \to 0.$$

Let $A \in \operatorname{mod} S$. By Proposition 4.2, we get C-Gorenstein version of the above lemma:

For any C-Gorenstein transpose $\operatorname{Tr}^{\pi}_{G_C} A$ of A, we have the following exact sequence:

$$(**) \quad 0 \to \operatorname{Ext}^1_{R^{op}}(\operatorname{Tr}^{\pi}_{G_C} A, C) \to A \xrightarrow{\sigma_A} A^{\dagger\dagger} \to \operatorname{Ext}^2_{R^{op}}(\operatorname{Tr}^{\pi}_{G_C} A, C) \to 0.$$

We claim that A is a C-Gorenstein transpose of $\operatorname{Tr}^{\pi}_{G_C} A$. In fact, let $\operatorname{Tr}^{\pi}_{G_C} A$ be any C-Gorenstein transpose of A. Then we have an exact sequence $G_1 \xrightarrow{g} G_0 \to A \to 0$ with G_0, G_1 G_C-projective and $\operatorname{Coker} g^{\dagger} = \operatorname{Tr}^{\pi}_{G_C} A$. Thus we get an exact sequence $0 \to A^{\dagger} \to G_0^{\dagger} \to G_1^{\dagger} \to \operatorname{Tr}^{\pi}_{G_C} A \to 0$. Since both G_0 and G_1 are C-reflexive, we get an exact sequence $0 \to (\operatorname{Tr}^{\pi}_{G_C} A)^{\dagger} \to G_1^{\dagger\dagger} \to G_0^{\dagger\dagger} \to A \to 0$. Thus A is a C-Gorenstein transpose of any C-Gorenstein transpose of A. Therefore we get the following exact sequence:

$$0 \to \operatorname{Ext}^1_S(A, C) \to \operatorname{Tr}^{\pi}_{G_C} A \xrightarrow{\sigma_{\operatorname{Tr}^{\pi}_{G_C} A}} (\operatorname{Tr}^{\pi}_{G_C} A)^{\dagger\dagger} \to \operatorname{Ext}^2_S(A, C) \to 0.$$

Moreover, we have the following corollary which generalizes [9, Theorem 2.2] and Lemma 4.3.

Corollary 4.4 *Let $G_n \xrightarrow{d_n} G_{n-1} \xrightarrow{d_{n-1}} \cdots \to G_1 \xrightarrow{d_1} G_0 \to A \to 0$ be an exact sequence in $\operatorname{mod} S$ with all G_i G_C-projective. If $\operatorname{Ext}^i_S(A, C) = 0$ for any $1 \leq i \leq n-1$, then we have the following exact sequence:*

$$0 \to \operatorname{Ext}^n_{R^{op}}(X, C) \to A \xrightarrow{\sigma_A} A^{\dagger\dagger} \to \operatorname{Ext}^{n+1}_{R^{op}}(X, C) \to 0$$

where $X = \operatorname{Coker} d_n^\dagger$.

Proof The case for $n = 1$ follows from $(**)$. Now suppose $n \geq 2$. Consider the given exact sequence

$$G_n \xrightarrow{d_n} G_{n-1} \xrightarrow{d_{n-1}} \cdots \to G_1 \xrightarrow{d_1} G_0 \to A \to 0$$

with all G_i G_C-projective. Since $\operatorname{Ext}^i_S(A, C) = 0$ for any $1 \leq i \leq n-1$, we have the following exact sequence:

$$0 \to A^\dagger \to G_0^\dagger \xrightarrow{d_1^\dagger} G_1^\dagger \to \cdots \to G_{n-1}^\dagger \xrightarrow{d_n^\dagger} G_n^\dagger \to X \to 0$$

where $X = \operatorname{Coker} d_n^\dagger$.

By $(**)$, there is an exact sequence

$$0 \to \operatorname{Ext}^1_{R^{op}}(Y, C) \to A \xrightarrow{\sigma_A} A^{\dagger\dagger} \to \operatorname{Ext}^2_{R^{op}}(Y, C) \to 0$$

where $Y = \operatorname{Coker} d_1^\dagger$. Since $G_i^\dagger$ is G_C-projective for $1 \leq i \leq n$, we have $\operatorname{Ext}^i_{R^{op}}(Y, C) \cong \operatorname{Ext}^{i+n-1}_{R^{op}}(X, C)$. Therefore we get the desired exact sequence. $\square$

Now we show that the converse of Proposition 4.2 is also true.

Proposition 4.5 *Let* $M \in \operatorname{mod} R^{op}$ *and* $A \in \operatorname{mod} S$. *If there exists an exact sequence* $0 \to M \to \operatorname{Tr}^\varepsilon_C A \to G \to 0$ *in* $\operatorname{mod} R^{op}$ *with* G G_C*-projective and* $\operatorname{Tr}^\varepsilon_C A$ *a* C*-transpose of* A, *then* M *is a* C*-Gorenstein transpose of* A.

Proof Let $P_1 \xrightarrow{f} P_0 \to A \to 0$ be a projective presentation of A in $\operatorname{mod} S$ with $\operatorname{Tr}^\varepsilon_C A = \operatorname{Coker} f^\dagger$. Then we have the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A^\dagger & = & A^\dagger & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & P_0^\dagger & = & P_0^\dagger & & & & \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f^\dagger} & & & & \\
0 & \to & K & \to & P_1^\dagger & \to & G & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M & \to & \operatorname{Tr}^\varepsilon_C A & \to & G & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Since both G and $P_1^\dagger$ are G_C-projective, K is G_C-projective by [12, Theorem 2.8]. Again since G is G_C-projective, by applying the functor $(\ )^\dagger$ to the above commutative diagram, we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^\dagger & \longrightarrow & (\operatorname{Tr}_C^\varepsilon A)^\dagger & \longrightarrow & M^\dagger & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G^\dagger & \longrightarrow & P_1^{\dagger\dagger} & \longrightarrow & K^\dagger & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle f^{\dagger\dagger}} & & \downarrow{\scriptstyle g^\dagger} & & \\
 & & & & P_0^{\dagger\dagger} & = & P_0^{\dagger\dagger} & & \\
 & & & & \downarrow & & & & \\
 & & & & A & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0. & & & &
\end{array}$$

By the Snake Lemma, we have $\operatorname{Im} g^\dagger \cong \operatorname{Im} f^{\dagger\dagger}$. Thus we get $\operatorname{Coker} g^\dagger = P_0^{\dagger\dagger}/\operatorname{Im} g^\dagger \cong P_0^{\dagger\dagger}/\operatorname{Im} f^{\dagger\dagger} \cong A$, and therefore we get a G_C-projective presentation of A in $\operatorname{mod} S$:

$$K^\dagger \xrightarrow{g^\dagger} P_0^{\dagger\dagger} \to A \to 0.$$

Since both K and $P_0^\dagger$ are C-reflexive, we get an exact sequence $0 \to A^\dagger \to P_0^{\dagger\dagger\dagger} \xrightarrow{g^{\dagger\dagger}} K^{\dagger\dagger} \to M \to 0$ in $\operatorname{mod} R^{op}$ and M is a C-Gorenstein transpose of A. □

Combining Propositions 4.2 and 4.5, we get the following theorem.

Theorem 4.6 *Let $M \in \operatorname{mod} R^{op}$ and $A \in \operatorname{mod} S$. Then M is a C-Gorenstein transpose of A if and only if M can be embedded into a C-transpose $\operatorname{Tr}_C^\varepsilon A$ of A with the cokernel G_C-projective, that is, there exists an exact sequence $0 \to M \to \operatorname{Tr}_C^\varepsilon A \to G \to 0$ in $\operatorname{mod} R^{op}$ with G G_C-projective.*

Corollary 4.7 *Let $A \in \operatorname{mod} S$. Then for any G_C-projective module $G \in \operatorname{mod} R^{op}$ and any C-transpose $\operatorname{Tr}_C^\varepsilon A$ of A, $G \oplus \operatorname{Tr}_C^\varepsilon A$ is a C-Gorenstein transpose of A.*

Proof Assume that $G \in \operatorname{mod} R^{op}$ is G_C-projective. Then there exists an exact sequence $0 \to G \to C_1 \to G' \to 0$ in $\operatorname{mod} R^{op}$ with $C_1 \in \operatorname{add}_{R^{op}} C$ and G' G_C-projective, which induces an exact sequence $0 \to G \oplus \operatorname{Tr}_C^\varepsilon A \to C_1 \oplus \operatorname{Tr}_C^\varepsilon A \to G' \to 0$. Since $C_1 \oplus \operatorname{Tr}_C^\varepsilon A$ is again a C-transpose of A, $G \oplus \operatorname{Tr}_C^\varepsilon A$ is a C-Gorenstein transpose of A by Theorem 4.6. □

Corollary 4.7 provides a method to construct a C-Gorenstein transpose of a module from a C-transpose of the same module. It is interesting to know whether any C-Gorenstein transpose is obtained in this way. If the answer to this question is positive, then we can conclude that the C-Gorenstein transpose of a module is unique up to G_C-projective equivalence.

Let $A \in \operatorname{mod} S$. It is clear that the C-Gorenstein transpose of A depends on the choice of the G_C-projective presentation of A. In the following, as applications of Theorem 4.6, we will investigate the relation between two C-Gorenstein transposes of A.

For a positive integer n, by Proposition 4.2, we have that $A \in \operatorname{mod} S$ is n-C-torsionfree if and only if $\operatorname{Ext}^i_{R^{op}}(\operatorname{Tr}^{\pi}_{G_C} A, C) = 0$ for any (or some) C-Gorenstein transpose $\operatorname{Tr}^{\pi}_{G_C} A$ of A and $1 \le i \le n$.

The following result shows that some homological properties of any two C-Gorenstein transposes of a given module are identical.

Proposition 4.8 *Let $A \in \operatorname{mod} S$. Then for any two C-Gorenstein transposes $\operatorname{Tr}^{\pi_1}_{G_C} A$ and $\operatorname{Tr}^{\pi_2}_{G_C} A$ and any C-transpose $\operatorname{Tr}^{\varepsilon}_{C} A$ of A, we have*

(1) $\operatorname{Ext}^i_{R^{op}}(\operatorname{Tr}^{\pi_1}_{G_C} A,\ C) \cong \operatorname{Ext}^i_{R^{op}}(\operatorname{Tr}^{\pi_2}_{G_C} A,\ C) \cong \operatorname{Ext}^i_{R^{op}}(\operatorname{Tr}^{\varepsilon}_{C} A,\ C)$ for any $i \ge 1$.

(2) For any $n \ge 1$, $\operatorname{Tr}^{\pi_1}_{G_C} A$ is n-C-torsionfree if and only if so is $\operatorname{Tr}^{\pi_2}_{G_C} A$, and if and only if so is $\operatorname{Tr}^{\varepsilon}_{C} A$.

(3) Some C-Gorenstein transpose of A is zero if and only if A is G_C-projective, if and only if any C-Gorenstein transpose of A is G_C-projective.

(4) G_C-$pd_{R^{op}}(\operatorname{Tr}^{\pi_1}_{G_C} A) = G_C$-$pd_{R^{op}}(\operatorname{Tr}^{\pi_2}_{G_C} A) = G_C$-$pd_{R^{op}}(\operatorname{Tr}^{\varepsilon}_{C} A)$

Proof (1) It is an immediate consequence of Remark 3.2(3) and Proposition 4.2.

(2) Let $\operatorname{Tr}^{\pi}_{G_C} A$ be any C-Gorenstein transpose of A. By Theorem 4.6, without loss of generality we may assume that there is an exact sequence $0 \to \operatorname{Tr}^{\pi}_{G_C} A \to \operatorname{Tr}^{\varepsilon}_{C} A \to G \to 0$ in $\operatorname{mod} R^{op}$ with G G_C-projective.

If $\operatorname{Ext}^1_S(\operatorname{Tr}^{\varepsilon'}_{C}(\operatorname{Tr}^{\varepsilon}_{C} A),\ C) = 0$, then $\operatorname{Tr}^{\varepsilon}_{C} A$ is C-torsionless. So $\operatorname{Tr}^{\pi}_{G_C} A$ is also C-torsionless and $\operatorname{Ext}^1_S(\operatorname{Tr}^{\varepsilon_2}_{C}(\operatorname{Tr}^{\pi}_{G_C} A),\ C) = 0$. Since G is G_C-projective, we get an exact sequence $0 \to \operatorname{Tr}^{\varepsilon_1}_{C} G \to \operatorname{Tr}^{\varepsilon'}_{C}(\operatorname{Tr}^{\varepsilon}_{C} A) \to \operatorname{Tr}^{\varepsilon_2}_{C}(\operatorname{Tr}^{\pi}_{G_C} A) \to 0$ in $\operatorname{mod} S$ with $\operatorname{Tr}^{\varepsilon_1}_{C} G$ G_C-projective. So we have that $\operatorname{Ext}^i_S(\operatorname{Tr}^{\varepsilon_2}_{C}(\operatorname{Tr}^{\pi}_{G_C} A),\ C) = \operatorname{Ext}^i_S(\operatorname{Tr}^{\varepsilon'}_{C}(\operatorname{Tr}^{\varepsilon}_{C} A),\ C)$ for any $i \ge 2$, and $\operatorname{Ext}^1_S(\operatorname{Tr}^{\varepsilon_2}_{C}(\operatorname{Tr}^{\pi}_{G_C} A),\ C) \to \operatorname{Ext}^1_S(\operatorname{Tr}^{\varepsilon'}_{C}(\operatorname{Tr}^{\varepsilon}_{C} A),\ C) \to 0$ is exact. Thus we have that, for any $i \ge 1$, $\operatorname{Ext}^i_S(\operatorname{Tr}^{\varepsilon_2}_{C}(\operatorname{Tr}^{\pi}_{G_C} A),\ C) = 0$ if and only if $\operatorname{Ext}^i_S(\operatorname{Tr}^{\varepsilon'}_{C}(\operatorname{Tr}^{\varepsilon}_{C} A)\ C) = 0$. And we conclude that for any $n \ge 1$, $\operatorname{Tr}^{\pi}_{G_C} A$ is n-C-torsionfree if and only if so is $\operatorname{Tr}^{\varepsilon}_{C} A$. The assertion follows from (1) and the fact that A is a C-Gorenstein transpose of any C-Gorenstein transpose of A.

(3) Note that A is a C-Gorenstein transpose of any C-Gorenstein transpose of A, applying Theorem 3.6, the assertion follows from (1) and (2).

(4) Let $\operatorname{Tr}^{\pi}_{G_C} A$ be any C-Gorenstein transpose of A. If $\operatorname{Tr}^{\pi}_{G_C} A = 0$, then the assertion follows from (3). Now suppose that $\operatorname{Tr}^{\pi}_{G_C} A \neq 0$. By Theorem 4.6, there exists a C-transpose $\operatorname{Tr}^{\varepsilon}_{C} A$ of A satisfying the exact sequence $0 \to \operatorname{Tr}^{\pi}_{G_C} A \to \operatorname{Tr}^{\varepsilon}_{C} A \to G \to 0$ in $\operatorname{mod} R^{op}$ with G G_C-projective. Then we have that G_C-$pd_{R^{op}}(\operatorname{Tr}^{\pi}_{G_C} A) = G_C$-$pd_{R^{op}}(\operatorname{Tr}^{\varepsilon}_{C} A)$ by Lemma 2.6 and Remark 3.2 (1). □

As the end of this paper we show that any double C-Gorenstein transpose of A shares some homological properties of A.

Corollary 4.9 *Let $A \in \operatorname{mod} S$. Then for any C-Gorenstein transpose $\operatorname{Tr}^{\pi}_{G_C} A$ of A and any C-Gorenstein transpose $\operatorname{Tr}^{\pi_1}_{G_C}(\operatorname{Tr}^{\pi}_{G_C} A)$ of $\operatorname{Tr}^{\pi}_{G_C} A$, we have*

(1) $\operatorname{Ext}^i_S(\operatorname{Tr}^{\pi_1}_{G_C}(\operatorname{Tr}^{\pi}_{G_C} A), C) \cong \operatorname{Ext}^i_S(A, C)$ for any $i \ge 1$.

(2) For any $n \geq 1$, $\mathrm{Tr}^{\pi_1}_{G_C}(\mathrm{Tr}^{\pi}_{G_C} A)$ *is* n*-*C*-torsionfree if and only if so is* A.

(3) G_C*-*$pd_S(\mathrm{Tr}^{\pi_1}_{G_C}(\mathrm{Tr}^{\pi}_{G_C} A)) = G_C$*-*$pd_S A$.

Proof Note that A is a C-Gorenstein transpose of any C-Gorenstein transpose $\mathrm{Tr}^{\pi}_{G_C} A$ of A. So all of the assertions follow from Proposition 4.8. □

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