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Derived and residual Sylvester-Hadamard designs and the Smith normal form

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Abstract: We computed the Smith normal form of Sylvester-Hadamard designs and its complement, their derived and residual Sylvester-Hadamard designs and their complementary derived and residual Sylvester-Hadamard designs.

Key words: Sylvester-Hadamard design, Smith normal form, derived, residual designs

1. Preliminaries

The p-ranks of Sylvester-Hadamard designs play an important role in shift registers and pseudo-noise matrices [1]. In this article by finding the Smith normal form we completely solve this problem, give formulas for their derived and residual Sylvester-Hadamard designs and their complementary derived and residual Sylvester-Hadamard designs.

By a balanced incomplete block design ($BIBD$) with parameters (v, b, r, k, λ) we mean an arrangement of v treatments into b subsets of these treatments called "blocks", such that

(i) each block consists of k distinct treatments;

(ii) each treatment occurs in r different blocks;

(iii) each pair of distinct treatments occur together in λ different blocks.

The following equations are satisfied by any $BIBD$:

$$vr = bk \quad \texttt{and} \quad \lambda(v-1) = r(k-1)$$

A $BIBD$ is said to be symmetric if $v = b$ and in consequence $r = k$. We call such a design a (v, k, λ) design.

Existence of a (v, k, λ) design implies the existence of its derived and residual design with parameters $(k, b-1, r-1, \lambda, \lambda-1)$ and $(v-k, b-1, r, k-\lambda, \lambda)$, respectively. They are obtained, respectively, by deleting a block of the (v, k, λ) design retaining all the treatments in $b-1$ blocks that appear (or do not appear) in the deleted block.

If $b_1, b_2, \ldots, b_v$ and $B_1, B_2 \ldots B_b$ denote the treatments and blocks of the $BIBD$ respectively then the incidence matrix $N = (n_{ij})$ of the design is defined by

$$n_{ij} = \begin{cases} 1, & \text{if } b_j \in B_i \ ; \\ 0, & \text{if } b_j \notin B_i . \end{cases}$$

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If N^T denotes the transpose of N then $N^T N = (r-\lambda)I + \lambda J$, where I is the identity matrix of order v and J is the square matrix of order v with all elements 1.

For any $BIBD$ with incidence matrix N there exists the complementary design with parameters $(v, b, b-r, v-k, b-2r+\lambda)$ with incidence matrix N^c, which is obtained by interchanging 0 and 1 in N.

Integral Equivalence: If A and B are matrices over the ring $\mathbb{Z}$ of integers, A and B are called *equivalent* $(A \sim B)$ if there are $\mathbb{Z}$-matrices P and Q, of determinant ± 1, such that

$$B = PAQ,$$

which means that one can be obtained from the other by a sequence of the following operations:

- Reorder the rows,
- Negate some row,
- Add an integer multiple of one row to another,

and the corresponding column operations.

Note: In the next section we use block $\mathbb{Z}$-equivalent row or column operations.

Smith Normal Form: If A is any n by n $\mathbb{Z}$-matrix, then there is a unique $\mathbb{Z}$-matrix

$$D = \mathrm{diag}(a_1, a_2, ..., a_n)$$

such that $A \sim D$ and

$$a_1 | a_2 | ... | a_r, a_{r+1} = \ldots = a_n = 0,$$

where the a_i are non-negative. The a_i are called *invariants factors* of A and D is the Smith normal form($SNF(A)$) of A.

A *Hadamard matrix* H of order m is an m by m matrix with elements ± 1 such that $HH^T = mI_m$. A *Sylvester-Hadamard matrix* of order 2^m is a Hadamard matrix that can be partitioned into

$$\begin{bmatrix} H & H \\ H & -H \end{bmatrix}, \qquad (1)$$

where $H_1 = \begin{bmatrix} 1 \end{bmatrix}$, $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and

$H_{2^m} = \begin{bmatrix} H_{2^{m-1}} & H_{2^{m-1}} \\ H_{2^{m-1}} & -H_{2^{m-1}} \end{bmatrix} = H_2 \otimes H_{2^{m-1}}$ for $2 \le m \in \mathbb{N}$ where $\otimes$ denotes the Kronecker product.

2. Sylvester-Hadamard Matrix

We have the following theorem from [3] that gives us the Smith normal of the matrix defined in (1).

Theorem 1. Let H denote the Sylvester-Hadamard matrix of order 2^m. Then the Smith normal form of H is

$$\mathrm{diag}[1, \underbrace{2, \ldots, 2}_{C(m,1)}, \underbrace{4, \ldots, 4}_{C(m,2)}, \underbrace{8, \ldots, 8}_{C(m,3)}, \ldots, \underbrace{2^{m-1}, \ldots, 2^{m-1}}_{C(m,m-1)}, 2^m]$$

where $C(m,k)$ denotes the binomial coefficients.

Without loss of generality, assume that H is of the form in (1). Then if we multiply row 1 with -1 and add row 1 to all other rows, and then subtract column 1 from all other columns, we see that H is integrally equivalent to the direct sum $[1] \oplus (-2A) = [1] \oplus (2A)$, where A is the incidence matrix of Sylvester-Hadamard $(2^m-1, 2^{m-1}, 2^{m-2})$-design. Since we have the direct sum we can state immediately the following theorem.

3. Sylvester-Hadamard Designs

Theorem 1 *Let A denote the incidence matrix of the Sylvester-Hadamard $(2^m-1, 2^{m-1}, 2^{m-2})$-design. Then the Smith normal form of A is*

$$diag[\underbrace{1,\ldots,1}_{C(m,1)}, \underbrace{2,\ldots,2}_{C(m,2)}, \underbrace{4,\ldots,4}_{C(m,3)}, \ldots, \underbrace{2^{m-2},\ldots,2^{m-2}}_{C(m,m-1)}, 2^{m-1}]$$

4. Complementary Sylvester-Hadamard Designs

Before computing the Smith normal form of the Complementary Sylvester-Hadamard $(2^m-1, 2^{m-1}-1, 2^{m-2}-1)$-design we need the following two results from [2].

Theorem 2 *Let A be the incidence matrix for a (v,k,λ) design with k and λ relatively prime and $n=k-\lambda$. Then $a_1=a_2=1$, $a_i=n/a_{v-i+2}$ for $3\leq i\leq v-i$, $a_v=nk$.*

Corollary 3 *Suppose that $n=p^t, p$ a prime. Let n_j be the number of invariant factors of A equal to p^j, $0\leq j\leq t$. Then*

$$n_0 = n_t + 2, \quad n_j = n_{t-j}, \quad 1 \leq j \leq t-1.$$

Now we state the theorem.

Theorem 4 *Let A denote the incidence matrix of the Complementary Sylvester-Hadamard $(2^m-1, 2^{m-1}-1, 2^{m-2}-1)$-design. Then the Smith normal form of A is*

$$diag[\underbrace{1,\ldots,1}_{C(m,1)+1}, \underbrace{2,\ldots,2}_{C(m,2)}, \underbrace{4,\ldots,4}_{C(m,3)}, \ldots, \underbrace{2^{m-2},\ldots,2^{m-2}}_{C(m,m-1)-1}, 2^{m-2}(2^{m-1}-1)]$$

Proof Since the parameters of our design satisfy the conditions of theorem 3 we get the last term $a_v = nk = (2^{m-1}-1-2^{m-2}+1)(2^{m-1}-1) = 2^{m-2}(2^{m-1}-1)$. By using the determinant of this design and corollary 1 we get the rest of the terms in the Smith normal form. □

5. Derived and Residual Sylvester-Hadamard Designs

Theorem 5 *Let N_1 denote the incidence matrix of the derived design of the Sylvester-Hadamard $(2^{m+1}-1, 2^m, 2^{m-1})$-design. Then the Smith normal form of N_1 is of the form $SNF(N_1) = [D_1|0]$ where*

$$D_1 = diag\left[\underbrace{1,\ldots,1}_{C(m,1)+1}, \underbrace{2,\ldots,2}_{C(m,2)}, \underbrace{4,\ldots,4}_{C(m,3)}, \ldots, \underbrace{2^{m-2},\ldots,2^{m-2}}_{C(m,m-1)}, 2^{m-1}\right]$$

Theorem 6 *Let N_0 denote the the incidence matrix of residual design of the Sylvester-Hadamard $(2^{m+1}-1, 2^m, 2^{m-1})$-design. Then the Smith normal form of N_0 is of the form $SNF(N_0) = [D_0|0]$, where*

$$D_0 = diag\left[\underbrace{1,\ldots,1}_{C(m,1)},\underbrace{2,\ldots,2}_{C(m,2)},\underbrace{4,\ldots,4}_{C(m,3)},\ldots,\underbrace{2^{m-2},\ldots,2^{m-2}}_{C(m,m-1)},2^{m-1}\right].$$

Proof Since Sylvester-Hadamard design is transitive on points and on blocks, the incidence matrix of the Sylvester-Hadamard $(2^{m+1}-1, 2^m, 2^{m-1})$-design can be put in the form

$$\left[\begin{array}{c|c|c} A & \underline{0} & A \\ \hline \underline{0} & 1 & \underline{1} \\ \hline A & \underline{1} & A^c \end{array}\right],$$

where A is the incidence matrix of the Sylvester-Hadamard $(2^m-1, 2^{m-1}, 2^{m-2})$-design, A^c the complementary design of A, $\underline{1}, \underline{0}$ are the column or row vectors of appropriate size with all 1's and all 0's, respectively, and $\mathbf{0}$ is the appropriate size of the zero matrix. So the derived design takes the form

$$\left[\begin{array}{c|c} \underline{0} & \underline{1} \\ \hline A & A^c \end{array}\right].$$

and the residual design takes the form

$$\left[\begin{array}{c|c} A & A \end{array}\right].$$

We compute the SNF of the derived design by doing the following $\mathbb{Z}$-equivalent block operations:

1. Add the first block column to the second block column.
2. Then multiply the first row by -1 and add it to every other row.
3. Then swap the first block column and the second one.
4. Multiply the first column by -1 and add to every column in the first block.
5. Swap the second and the third column.

Namely, the operations we did are as follows:

$$\left[\begin{array}{c|c} \underline{0} & \underline{1} \\ \hline A & A^c \end{array}\right] \longrightarrow \left[\begin{array}{c|c} \underline{0} & \underline{1} \\ \hline A & A^c + A \end{array}\right] = \left[\begin{array}{c|c} \underline{0} & \underline{1} \\ \hline A & J \end{array}\right] \longrightarrow \left[\begin{array}{c|c} \underline{0} & \underline{1} \\ \hline A & \mathbf{0} \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{c|c} \underline{1} & \underline{0} \\ \hline \mathbf{0} & A \end{array}\right] \longrightarrow \left[\begin{array}{c|c|c} 1 & \mathbf{0} & \underline{0} \\ \hline \underline{0} & \mathbf{0} & A \end{array}\right] \longrightarrow \left[\begin{array}{c|c|c} 1 & \underline{0} & \mathbf{0} \\ \hline \underline{0} & A & \mathbf{0} \end{array}\right] \longrightarrow \left[\begin{array}{c|c|c} 1 & \underline{0} & \mathbf{0} \\ \hline \underline{0} & SNF(A) & \mathbf{0} \end{array}\right].$$

Now the result for the derived design follows by theorem 2. Similarly, if we multiply the first column block by -1 and add it to the second block column we get

$$\left[\begin{array}{c|c} A & A \end{array}\right] \longrightarrow \left[\begin{array}{c|c} A & \mathbf{0} \end{array}\right] \longrightarrow \left[\begin{array}{c|c} SNF(A) & \mathbf{0} \end{array}\right],$$

and the result for the residual design follows by theorem 2. □

6. Derived and Residual Complementary Sylvester-Hadamard Designs

Theorem 7 *Let N_1^c denote the incidence matrix of the derived design of the complementary Sylvester-Hadamard $(2^{m+1}-1, 2^m-1, 2^{m-1}-1)$-design. Then the Smith normal form of N_1^c is of the form $SNF(N_1^c) = [D_1^c|0]$, where*

$$D_1^c = diag\left[\underbrace{1,\ldots,1}_{C(m,1)+1}, \underbrace{2,\ldots,2}_{C(m,2)}, \underbrace{4,\ldots,4}_{C(m,3)}, \ldots, \underbrace{2^{m-2},\ldots,2^{m-2}}_{C(m,m-1)-1}, 2^{m-2}(2^{m-1}-1)\right].$$

Theorem 8 *Let N_0^c denote the incidence matrix of the residual design of the complementary Sylvester-Hadamard $(2^{m+1}-1, 2^m-1, 2^{m-1}-1)$-design. Then the Smith normal form of N_0^c is of the form $SNF(N_0^c = [D_0^c|0]$, where*

$$D_0^c = diag\left[\underbrace{1,\ldots,1}_{C(m,1)+1}, \underbrace{2,\ldots,2}_{C(m,2)}, \underbrace{4,\ldots,4}_{C(m,3)}, \ldots, \underbrace{2^{m-2},\ldots,2^{m-2}}_{C(m,m-1)}, 2^{m-1}\right].$$

Proof The incidence matrix of the complementary Sylvester-Hadamard $(2^{m+1}-1, 2^m-1, 2^{m-1}-1)$-design can be put in the form

$$\left[\begin{array}{c|c|c} B & \underline{1} & B \\ \hline \underline{1} & 0 & \underline{0} \\ \hline B & \underline{0} & A \end{array}\right],$$

where $B = A^c$ is the incidence matrix of the complementary Sylvester-Hadamard $(2^m-1, 2^{m-1}-1, 2^{m-2}-1)$-design. So the derived design takes the form

$$\left[\begin{array}{c|c} B & B \end{array}\right].$$

and the residual design takes the form

$$\left[\begin{array}{c|c} \underline{1} & \underline{0} \\ \hline B & A \end{array}\right]$$

We compute the SNF of the derived design by doing similar $\mathbb{Z}$-equivalent block operations described in the proof of theorem 6. Namely,

$$\left[\begin{array}{c|c} B & B \end{array}\right] \longrightarrow \left[\begin{array}{c|c} B & 0 \end{array}\right] \longrightarrow \left[\begin{array}{c|c} SNF(B) & 0 \end{array}\right].$$

Now the result for the derived design follows from theorem 4. Similarly,

$$\left[\begin{array}{c|c} \underline{1} & \underline{0} \\ \hline B & A \end{array}\right] \longrightarrow \left[\begin{array}{c|c} \underline{1} & \underline{0} \\ \hline B+A & A \end{array}\right] \longrightarrow \left[\begin{array}{c|c} \underline{1} & \underline{0} \\ \hline J & A \end{array}\right] \longrightarrow \left[\begin{array}{c|c} \underline{1} & \underline{0} \\ \hline \mathbf{0} & A \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{c|c|c} 1 & \mathbf{0} & \underline{0} \\ \hline \underline{0} & \mathbf{0} & A \end{array}\right] \longrightarrow \left[\begin{array}{c|c|c} 1 & \underline{0} & \mathbf{0} \\ \hline \underline{0} & A & \mathbf{0} \end{array}\right] \longrightarrow \left[\begin{array}{c|c|c} 1 & \underline{0} & \mathbf{0} \\ \hline \underline{0} & SNF(A) & \mathbf{0} \end{array}\right]$$

and the result follows from theorem 2. □

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