Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 500 – 510
© TÜBİTAK
doi:10.3906/mat-1102-16

Lie groupoids and generalized almost paracomplex manifolds

Fulya ŞAHİN,* **Mustafa Habil GÜRSOY, İlhan İÇEN**
İnönü University, Faculty of Science and Art, Department of Mathematics, 44280 Malatya, Turkey

Received: 10.02.2011 • **Accepted:** 02.03.2012 • **Published Online:** 26.04.2013 • **Printed:** 27.05.2013

Abstract: In this paper, we show that there is a close relationship between generalized paracomplex manifolds and Lie groupoids. We obtain equivalent assertions among the integrability conditions of generalized almost paracomplex manifolds, the condition of compatibility of source and target maps of symplectic groupoids with symplectic form and generalized paraholomorphic maps.

Key words: Lie groupoid, symplectic groupoid, generalized almost paracomplex manifold

1. Introduction

A groupoid is a small category in which all morphisms are invertible. More precisely, a groupoid (G, G_0) consists of two sets G and G_0, called arrows and objects, respectively, with maps $s, t : G \to G_0$ called source and target. It is equipped with a composition $m : G_2 \to G$ defined on the subset $G_2 = \{(g, h) \in G \times G \mid s(g) = t(h)\}$; an inclusion map of objects $e : G_0 \to G$ and an inverse map $i : G \to G$. For a groupoid, the following properties are satisfied: $s(gh) = s(h)$, $t(gh) = t(g)$, $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, $g(hf) = (gh)f$ whenever both sides are defined, $g^{-1}g = 1_{s(g)}$, $gg^{-1} = 1_{t(g)}$. Here we have used $gh, 1_x$ and g^{-1} instead of $m(g, h)$, $e(x)$ and $i(g)$, respectively. Generally, a groupoid (G, G_0) is denoted by the set of arrows G. A topological groupoid is a groupoid G whose set of arrows and set of objects are both topological spaces whose structure maps s, t, e, i, m are all continuous and s, t are open maps.

A Lie groupoid is a groupoid G whose set of arrows and set of objects are both manifolds whose structure maps s, t, e, i, m are all smooth maps and s, t are submersions. The latter condition ensures that s and t-fibres are manifolds.

One can see from the above definition that the space G_2 of composable arrows is a submanifold of $G \times G$. We note that the notion of Lie groupoids was introduced by Ehresmann [3].

On the other hand, Lie algebroids were first introduced by Pradines [8] as infinitesimal objects associated with the Lie groupoids. More precisely, a Lie algebroid structure on a real vector bundle A on a manifold M is defined by a vector bundle map $\rho_A : A \to TM$, the anchor of A, and an $\mathbb{R}$-Lie algebra bracket on $\Gamma(A)$, $[,]_A$ satisfying the Leibniz rule

$$[\alpha, f\beta]_A = f[\alpha, \beta]_A + L_{\rho_A(\alpha)}(f)\beta$$

for all $\alpha, \beta \in \Gamma(A), f \in C^\infty(M)$, where $L_{\rho_A(\alpha)}$ is the Lie derivative with respect to the vector field $\rho_A(\alpha)$, where $\Gamma(A)$ denotes the set of sections in A.

*Correspondence: fulya.sahin@inonu.edu.tr

2010 *AMS Mathematics Subject Classification:* 22A22, 53D17, 53D18.

On the other hand, Hitchin [5] introduced the notion of generalized complex manifolds by unifying and extending the usual notions of complex and symplectic manifolds. Later the notion of generalized Kähler manifold was introduced by Gualtieri [4] and submanifolds of such manifolds have been studied in many papers.

As an analogue of generalized complex structures on even dimensional manifolds, the concept of generalized almost paracomplex manifolds was introduced in [6] and such manifolds have been studied in [6] and [9].

Recently, Crainic [2] showed that there is a close relationship between the equations of a generalized complex manifold and a Lie groupoid. More precisely, he obtained that the complicated equations of such manifolds turn into simple structures for Lie groupoids.

In this paper, we investigate relationships between the complicated equations of generalized paracomplex structures and Lie groupoids. We showed that the equations of such manifolds are useful to obtain equivalent results on a symplectic groupoid.

2. Preliminaries

In this section we recall basic facts of Poisson geometry, Lie groupoids and Lie algebroids. More details can be found in [7] and [10]. A central idea in generalized geometry is that $TM \oplus T^*M$ should be thought of as a generalized tangent bundle to manifold M. If X and ξ denote a vector field and a dual vector field on M, respectively, then we write (X,ξ) (or $X+\xi$) as a typical element of $TM \oplus T^*M$. The Courant bracket of two sections $(X,\xi),(Y,\eta)$ of $TM \oplus T^*M = \mathcal{TM}$ is defined by

$$[(X,\xi),(Y,\eta)] = [X,Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi), \tag{2.1}$$

where d, L_X and i_X denote exterior derivative, Lie derivative and interior derivative with respect to X, respectively. The Courant bracket is antisymmetric but it does not satisfy the Jacobi identity. Here, we use the notations $\beta(\pi^\sharp\alpha) = \pi(\alpha,\beta)$ and $\omega_\sharp(X)(Y) = \omega(X,Y)$, which are defined as $\pi^\sharp : T^*M \to TM$, $\omega_\sharp : TM \to T^*M$ for any 1-forms α and β, 2-form ω and bivector field π, and vector fields X and Y. Also we denote by $[,]_\pi$ the bracket on the space of 1-forms on M defined by

$$[\alpha,\beta]_\pi = L_{\pi^\sharp\alpha}\beta - L_{\pi^\sharp\beta}\alpha - d\pi(\alpha,\beta).$$

On the other hand, a symplectic manifold is a smooth (even dimensional) manifold M with a nondegenerate closed 2-form $\omega \in \Omega^2(M)$. ω is called the symplectic form of M. Let G be a Lie groupoid on M and ω a form on Lie groupoid G; then ω is called multiplicative if

$$m^*\omega = pr_1^*\omega + pr_2^*\omega,$$

where $pr_i : G \times G \to G$, $i = 1,2$ are the canonical projections.

Similar to 2-forms, given a Lie groupoid G, a $(1,1)$-tensor $J : TG \to TG$ is called multiplicative [2] if for any $(g,h) \in G \times G$ and any $v_g \in T_gG$, $w_h \in T_hG$ such that (v_g, w_h) is tangent to $G \times G$ at (g,h), so is (Jv_g, Jw_h), and

$$(dm)_{g,h}(Jv_g, Jw_h) = J((dm)_{g,h}(v_g, w_h)).$$

A symplectic groupoid (G,ω) is a Lie groupoid G with a symplectic form ω on G such that ω is multiplicative. In this case, one says that G is a symplectic groupoid over G_0.

We now recall the notion of Poisson manifolds. A Poisson manifold is a smooth manifold M whose function space $C^{\infty}(M,\mathbb{R})$ is a Lie algebra with bracket $\{,\}$, such that the following properties are satisfied:

(i) $\{f,g\} = -\{g,f\}$

(ii) $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

(iii) $\{fg,h\} = f\{g,h\} + g\{f,h\}$.

If $(M,\{,\})$ is a Poisson manifold, then there is a unique bivector π called the Poisson bivector, defined by

$$\pi(df,dg) = \{f,g\},$$

and a unique homomorphism $\pi^{\sharp} : T^*M \to TM$ with the vector bundle map given by

$$\pi^{\sharp}(dh) = X_h = \{h,\cdot\},$$

where X_h is the Hamiltonian vector field determined by $h \in C^{\infty}(M,\mathbb{R})$.

It is also possible to define a Poisson manifold by using the bivector π. Indeed, a smooth manifold is a Poisson manifold if $[\pi,\pi] = 0$, where $[,]$ denotes the Schouten bracket on the space of multivector fields.

We now give a relation between Lie algebroid and Lie groupoid. Given a Lie groupoid G on M, the associated Lie algebroid $A = \mathrm{Lie}(G)$ has fibres $A_x = \mathrm{Ker}(ds)_x = T_x(G(-,x))$, for any $x \in M$. Any $\alpha \in \Gamma(A)$ extends to a unique right-invariant vector field on G, which will be denoted by same letter α. The usual Lie bracket on vector fields induces the bracket on $\Gamma(A)$, and the anchor is given by $\rho = dt : A \to TM$.

Given a Lie algebroid A, an integration of A is a Lie groupoid G together with an isomorphism $A \cong \mathrm{Lie}(G)$. If such a G exists, then it is said that A is integrable. In contrast with the case of Lie algebras, not every Lie algebroid admits an integration. However if a Lie algebroid is integrable, then there exists a canonical source-simply connected integration G, and any other source-simply connected integration is smoothly isomorphic to G. *From now on we assume that all Lie groupoids are source-simply connected.*

In this section, finally, we recall the notion of IM form (infinitesimal multiplicative form) on a Lie algebroid [1]. More precisely, an IM form on a Lie algebroid A is a bundle map

$$u : A \to T^*M$$

satisfying the properties

(i) $\langle u(\alpha), \rho(\beta)\rangle = -\langle u(\beta), \rho(\alpha)\rangle$,

(ii) $u([\alpha,\beta]) = L_{\alpha}(u(\beta)) - L_{\beta}(u(\alpha)) + d\langle u(\alpha), \rho(\beta)\rangle$

for $\alpha,\beta \in \Gamma(A)$, where $\rho = \rho_A$ and $\langle,\rangle$ denotes the usual pairing between a vector space and its dual.

If A is a Lie algebroid of a Lie groupoid G, then a closed multiplicative 2-form ω on G induces an IM form u_{ω} of A by

$$\langle u_{\omega}(\alpha), X\rangle = \omega(\alpha, X).$$

For the relationship between IM form and closed 2-form we have the following theorem.

Theorem 1 *[1] If A is an integrable Lie algebroid and if G is its integration, then $\omega \mapsto u_{\omega}$ is a one to one correspondence between closed multiplicative 2-forms on G and IM forms of A.*

3. Lie groupoids and generalized paracomplex structures

In this section we first give a characterization for generalized paracomplex manifolds, then we obtain certain relationships between generalized paracomplex manifolds and symplectic groupoids. We recall that a generalized almost paracomplex structure $\mathcal{J}$ is an endomorphism on $\mathcal{TM}$ such that $\mathcal{J}^2 = Id$, where Id is the identity map on $\mathcal{TM}$. A generalized almost paracomplex structure can be represented by classical tensor fields as

$$\mathcal{J} = \begin{bmatrix} a & \pi^\sharp \\ \sigma_\sharp & -a^* \end{bmatrix}, \tag{3.1}$$

where π is a bivector on M, σ is a 2-form on M, $a : TM \to TM$ is a bundle map, and $a^* : T^*M \to T^*M$ is dual of a, for almost paracomplex structures; see [9] and [11].

A generalized almost paracomplex structure is called integrable (or just paracomplex structure) if $\mathcal{J}$ satisfies the condition

$$[\mathcal{J}\alpha, \mathcal{J}\beta] + [\alpha, \beta] - \mathcal{J}([\mathcal{J}\alpha, \beta] + [\alpha, \mathcal{J}\beta]) = 0, \tag{3.2}$$

for all sections $\alpha, \beta \in \mathcal{TM}$.

In the sequel, we give necessary and sufficient conditions for a generalized almost paracomplex structure to be integrable in terms of the above tensor fields. We note that the following result was stated in [9], but its proof was not given in there. In fact, the proof of the conditions of the following proposition is similar to the proposition given in [2] by Crainic for generalized complex structures. Although the conditions are similar to the generalized complex case, their proofs are slightly different from the complex case. Therefore we give one part of the proof of the following proposition.

Proposition 1 *A manifold with $\mathcal{J}$ given by (3.1) is a generalized paracomplex manifold if and only if*

(PC1) π satisfies the equation

$$\pi^\sharp([\xi, \eta]_\pi) = [\pi^\sharp(\xi), \pi^\sharp(\eta)];$$

(PC2) π and a are related by the two formulas

$$\begin{aligned} a\pi^\sharp &= \pi^\sharp a^*, \\ a^*([\xi, \eta]_\pi) &= L_{\pi^\sharp \xi}(a^*\eta) - L_{\pi^\sharp \eta}(a^*\xi) - d\pi(a^*\xi, \eta); \end{aligned} \tag{3.3}$$

(PC3) π, a and σ are related by the two formulas

$$a^2 + \pi^\sharp \sigma_\sharp = Id, \tag{3.4}$$

$$N_a(X, Y) = \pi^\sharp i_{X \wedge Y}(d\sigma), \tag{3.5}$$

where N_a is the torsion tensor of a; and

(PC4) σ and a are related by the two formulas

$$\begin{aligned} a^*\sigma_\sharp &= \sigma_\sharp a, \\ d\sigma_a(X, Y, Z) &= d\sigma(aX, Y, Z) + d\sigma(X, aY, Z) + d\sigma(X, Y, aZ) \end{aligned}$$

for all 1-forms ξ and η, and all vector fields X, Y and Z, where $\sigma_a(X, Y) = \sigma(aX, Y)$.

Proof We will obtain only the condition (PC3). The first equation of (PC3) is obtained from the equation $\mathcal{J}^2 = Id$. To prove the second part of (PC3), we take $\alpha = (X,0), \beta = (Y,0)$ in (3.2) such that X and Y are vector fields. Since $\mathcal{J}\alpha = (aX, \sigma_\sharp X)$ and $\mathcal{J}\beta = (aY, \sigma_\sharp Y)$, using (2.1) we get

$$[\mathcal{J}\alpha, \mathcal{J}\beta] = ([aX, aY], L_{aX}\sigma_\sharp Y - L_{aY}\sigma_\sharp X + \frac{1}{2}d(\sigma_\sharp X(aY) - \sigma_\sharp Y(aX)), \tag{3.6}$$

$$\begin{aligned} -\mathcal{J}[\mathcal{J}\alpha, \beta] &= (-a[aX, Y] - \pi^\sharp(-L_Y\sigma_\sharp X + \frac{1}{2}d(i_Y\sigma_\sharp X), -\sigma_\sharp[aX, Y] \\ &+ a^*(-L_Y\sigma_\sharp X + \frac{1}{2}d(i_Y\sigma_\sharp X)) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} -\mathcal{J}[\alpha, \mathcal{J}\beta] &= (-a[X, aY] - \pi^\sharp(-L_X\sigma_\sharp Y - \frac{1}{2}d(i_X\sigma_\sharp Y), -\sigma_\sharp[X, aY] \\ &+ a^*(L_X\sigma_\sharp Y - \frac{1}{2}d(i_X\sigma_\sharp Y)). \end{aligned} \tag{3.8}$$

Thus, from (3.6), (3.7) and (3.8) we obtain

$$\begin{aligned} [aX, aY] + [X, Y] - a[aX, Y] &- \pi^\sharp\big(- L_Y\sigma_\sharp X + \frac{1}{2}d(i_Y\sigma_\sharp X)\big) \\ -a[X, aY] - \pi^\sharp\big(- L_X\sigma_\sharp Y &- \frac{1}{2}d(i_X\sigma_\sharp Y)\big) = 0, \end{aligned}$$

where we have used $[\alpha, \beta] = ([X, Y], 0)$. Rearranging the above expression, we arrive at

$$\begin{aligned} [aX, aY] + [X, Y] - a[aX, Y] &- a[X, aY] - \pi^\sharp\big(- L_Y(i_X\sigma) + L_X(i_Y\sigma) \\ &+ \frac{1}{2}d(i_{X\wedge Y}\sigma) + \frac{1}{2}d(i_{X\wedge Y}\sigma)\big) = 0. \end{aligned} \tag{3.9}$$

On the other hand, making use of the formula

$$i_{X\wedge Y}(d\sigma) = L_X(i_Y\sigma) - L_Y(i_X\sigma) + d(i_{X\wedge Y}\sigma) - i_{[X,Y]}\sigma, \tag{3.10}$$

we get

$$\begin{aligned} -\pi^\sharp\big(L_X(i_Y\sigma) - L_Y(i_X\sigma) + d(i_{X\wedge Y}\sigma)\big) &= -\pi^\sharp\big(i_{X\wedge Y}(d\sigma) + i_{[X,Y]}\sigma\big) \\ &= -\pi^\sharp\big(i_{X\wedge Y}(d\sigma)\big) \\ &- \pi^\sharp\big(i_{[X,Y]}\sigma\big). \end{aligned} \tag{3.11}$$

Since

$$\pi^\sharp\big(i_{[X,Y]}\sigma\big) = \pi^\sharp\big(\sigma_\sharp[X, Y]\big),$$

from (3.4) we have

$$\pi^\sharp\big(i_{[X,Y]}\sigma\big) = -a^2[X, Y] + [X, Y]. \tag{3.12}$$

Using (3.12) in (3.11), we derive

$$\begin{aligned} -\pi^\sharp\big(L_X(i_Y\sigma) - L_Y(i_X\sigma) + d(i_{X\wedge Y}\sigma)\big) &= -\pi^\sharp\big(i_{X\wedge Y}(d\sigma) + i_{[X,Y]}\sigma \\ &= -\pi^\sharp\big(i_{X\wedge Y}(d\sigma)\big) + a^2[X,Y] \\ &- [X,Y]. \end{aligned} \tag{3.13}$$

Thus putting (3.13) in (3.9), we obtain (3.5). The other assertions can be obtained in a similar way. □

We note that if (3.1) is a generalized paracomplex structure, then

$$\overline{\mathcal{J}} = \begin{bmatrix} a & -\pi^\sharp \\ -\sigma_\sharp & -a^* \end{bmatrix}$$

is also a generalized paracomplex structure. $\overline{\mathcal{J}}$ is called the opposite of $\mathcal{J}$. In this paper we denote a generalized paracomplex manifold endowed with $\overline{\mathcal{J}}$ by $\overline{M}$.

As an analogue of a Hitchin pair on a generalized complex manifold, a Hitchin pair on a generalized almost paracomplex manifold M is a pair (ω, a) consisting of a symplectic form ω and a $(1,1)$-tensor a with the property that ω and a commute (i.e. $\omega(X, aY) = \omega(aX, Y)$) and $d\omega_a = 0$, where $\omega_a(X,Y) = \omega(aX, Y)$.

Lemma 1 *If π is a non-degenerate bivector on a generalized almost paracomplex manifold M, ω is the inverse 2-form (defined by $\omega_\sharp = (\pi^\sharp)^{-1}$) and π satisfies (3.4), then $\sigma = \omega - a^*\omega$.*

Proof For $X \in \chi(M)$, we apply $\omega_\sharp$ to (3.4) and using the dual paracomplex structure a^*, we have

$$(a^*)^2\omega_\sharp(X) + \sigma_\sharp(X) = \omega_\sharp(X).$$

Now for $Y \in \chi(M)$, since ω and a commute, we obtain

$$\omega(aX, aY) + \sigma(X,Y) = \omega(X,Y).$$

Thus we get

$$a^*\omega(X,Y) + \sigma(X,Y) = \omega(X,Y). \tag{3.14}$$

Since the equation (3.14) holds for all X and Y, we get

$$\sigma = \omega - a^*\omega.$$

□

We say that 2-form σ is the twist of Hitchin pair (ω, a).

A symplectic+paracomplex structure on M is a couple (ω, J) consisting of a symplectic form ω and a paracomplex structure J on M, which commute.

Lemma 2 *Let (M,ω) be a symplectic manifold. (ω, a) is a symplectic+paracomplex structure if and only if $d\omega_a = 0$, $\omega - a^*\omega = 0$.*

Proof We will only prove the sufficient condition. Since (M,ω) is a symplectic manifold, then $d\omega = 0$. Since $d\omega_a = 0$, $\omega = a^*\omega$, by using the equation (see [2])

$$i_{N_a(X,Y)}(\omega) = i_{aX\wedge Y + X\wedge aY}(d\omega_a) - i_{aX\wedge aY}(d\omega) - i_{X\wedge Y}(d(a^*\omega)) \tag{3.15}$$

we get $i_{N_a(X,Y)}(\omega) = -i_{X\wedge Y}(d(a^*\omega)) = -i_{X\wedge Y}(d\omega) = 0$. Hence, $\omega(N_a(X,Y),\bullet) = 0$. Since ω is non-degenerate, then $N_a = 0$. Thus a is a paracomplex structure. On the other hand, $\omega - a^*\omega = 0$ implies that ω and a commute.

The converse is clear. □

Next we relate (PC1) and the 2-form ω.

Lemma 3 *If π is a non-degenerate bivector on a generalized almost paracomplex manifold M, and ω is the inverse 2-form, then π satisfies (PC1) if and only if ω is closed.*

Proof Applying $\xi = i_X(\omega)$ and $\eta = i_Y(\omega)$ to (PC1), we get

$$\pi^\sharp(L_X(i_Y(\omega)) - L_Y(i_X(\omega)) - d(\omega_\sharp Y(\pi^\sharp\omega_\sharp X)) \quad = \quad [X,Y].$$

Since $\omega_\sharp = (\pi^\sharp)^{-1}$, we have

$$\pi^\sharp(L_X(i_Y(\omega)) - L_Y(i_X(\omega)) - d(\omega_\sharp Y)(X)) \quad = \quad [X,Y]. \tag{3.16}$$

Then applying $\omega_\sharp$ to (3.16), we derive

$$L_X(i_Y(\omega)) - L_Y(i_X(\omega)) - d(\omega_\sharp Y)(X) = \omega_\sharp([X,Y]).$$

Using (3.10), then we get

$$L_X(i_Y\omega) - L_Y(i_X\omega) + d(i_{X\wedge Y}\omega) - i_{[X,Y]}\omega = i_{X\wedge Y}d\omega.$$

Since the left hand side vanishes in the above equation, then $i_{X\wedge Y}d\omega = 0$. Thus $d\omega = 0$. The rest of the proof is completely analogous to the above calculations, and hence it can be omitted. □

Thus, we have the following result, which shows that there is close relationship between the condition (PC1) and a symplectic groupoid.

Theorem 2 *Let M be a generalized almost paracomplex manifold. There is a 1-1 correspondence between:*

(i) Integrable bivectors π on M satisfying (PC1),

(ii) Symplectic groupoids (Σ,ω) over M.

Since $\pi^\sharp$ and $[,]_\pi$ define a Lie algebroid structure on T^*M, one can obtain the above theorem by the following steps given in ([2],Theorem 3.2). We now give the conditions for (PC2) in terms of ω and ω_a.

Lemma 4 *Let M be a generalized almost paracomplex manifold and ω a symplectic form. Given a non-degenerate bivector π (i.e. $\pi^\sharp = (\omega_\sharp)^{-1}$) and a map $a : TM \to TM$, then π and a satisfy (PC2) if and only if ω and a commute and ω_a is closed.*

Proof For a 1-form ξ, we use $\xi = i_X\omega = \omega_\sharp X$ such that X is an arbitrary vector field. Since

$$a\pi^\sharp(i_X\omega) \quad = \quad \pi^\sharp a^*(i_X\omega), \tag{3.17}$$

applying $\omega_\sharp$ to (3.17) and using $\pi^\sharp = (\omega_\sharp)^{-1}$, we have

$$\omega(aX,Y) = a^*(i_X\omega)(Y),$$

for a vector field Y. Hence we get

$$\omega(aX, Y) \;=\; \omega(X, aY).$$

Let ξ and η be 1-forms such that $\xi = i_X\omega = \omega_\sharp(X)$ and $\eta = i_Y\omega = \omega_\sharp(Y)$ for arbitrary vector fields X and Y. Then from (3.3) we have

$$\begin{aligned} a^*(L_{\pi^\sharp\omega_\sharp(X)}(\omega_\sharp(Y)) - L_{\pi^\sharp\omega_\sharp(Y)}(\omega_\sharp(X)) - d\pi(i_X\omega, i_Y\omega)) &= L_{\pi^\sharp\omega_\sharp(X)}(a^*\omega_\sharp(Y)) \\ &- L_{\pi^\sharp\omega_\sharp(Y)}(a^*\omega_\sharp(X)) \\ &- d\pi(a^*i_X\omega, i_Y\omega). \end{aligned}$$

Since $\pi^\sharp = (\omega_\sharp)^{-1}$ and $a^*i_Y\omega = i_Y\omega_a$, we obtain

$$\begin{aligned} a^*(L_X(i_Y\omega) - L_Y(i_X\omega) - d(i_{Y\wedge X}\omega)) &= L_X(i_Y\omega_a) \\ &- L_Y(i_X\omega_a) \\ &- d(i_Y\omega(\pi^\sharp(\omega_\sharp(aX)))). \end{aligned}$$

Using (3.10) for the left hand side, we get

$$a^*(i_{X\wedge Y}(d\omega) + i_{[X,Y]}\omega) = L_X(i_Y\omega_a) - L_Y(i_X\omega_a) - d(\omega(Y, aX)).$$

Since ω and a commute, using this also for the right hand side, we have

$$a^*(i_{X\wedge Y}(d\omega)) + i_{[X,Y]}\omega_a = i_{X\wedge Y}(d\omega_a) + i_{[X,Y]}\omega_a. \tag{3.18}$$

Since ω is closed, (3.18) implies that $i_{X\wedge Y}(d\omega_a) = 0$, i.e. ω_a is closed. □

Let (M, ω) be a symplectic manifold. Then it is easy to see that there is a one to one correspondence between $(1,1)$-tensors a commuting with ω and 2-forms on M. On the other hand, it is easy to see that (PC2) is equivalent to the fact that a^* is an IM form on the Lie algebroid T^*M associated Poisson structure π. Thus, from the above discussion, Lemma 4 and Theorem 1, one can conclude with the following theorem.

Theorem 3 *Let M be a generalized almost paracomplex manifold. Let π be an integrable Poisson structure on M, and (Σ, ω) a symplectic groupoid over M. Then there is a natural 1-1 correspondence between*

(i) $(1,1)$-tensors a on M satisfying (PC2),

(ii) multiplicative $(1,1)$-tensors J on Σ with the property that (J, ω) is a Hitchin pair.

For the generalized complex manifold case, see [2].

We recall the notion of generalized paraholomorphic map between generalized paracomplex manifolds. This notion was given in [9] similar to the generalized holomorphic map given in [2].

Let $(M_i, \mathcal{J}_i)$, $i = 1, 2$, be two generalized paracomplex manifolds, and let a_i, π_i, σ_i be the components of $\mathcal{J}_i$ in the matrix representation (3.1). A map $f : M_1 \to M_2$ is called generalized paraholomorphic iff f maps π_1 into π_2, $f^*\sigma_2 = \sigma_1$ and $(df) \circ a_1 = a_2 \circ (df)$ [9].

We now state and prove the main result of this paper. This result gives equivalent assertions between the condition (PC3), twist σ of (ω, J) and paraholomorphic maps for a symplectic groupoid over M.

Theorem 4 *Let M be a generalized almost paracomplex manifold and (Σ, ω, J) an induced symplectic groupoid over M with the induced multiplicative $(1,1)$-tensor. Assume that (π, J) satisfy (PC1), (PC2) with integrable π. Then for a 2-form on M, the following assertions are equivalent.*

(i) (PC3) is satisfied,

(ii) $\omega - J^*\omega = t^*\sigma - s^*\sigma$,

(iii) $(t,s) : \Sigma \to M \times \overline{M}$ is a generalized paraholomorphic map; condition of generalized paraholomorphic map on M is $(dt) \circ a_1 = a_2 \circ (dt)$, this condition on $\overline{M}$ is $(ds) \circ a_1 = -a_2 \circ (ds)$).

Proof (i)$\Leftrightarrow$ (ii): Define $\phi = \widetilde{\sigma} - t^*\sigma + s^*\sigma$, such that $\widetilde{\sigma} = \omega - J^*\omega$ and $A = ker(ds)|_M$. We know from Theorem 1 that closed multiplicative 2-form θ on Σ vanishes if and only if IM form $u_\theta = 0$, i.e. $\theta(X, \alpha) = 0$, such that $X \in TM$, $\alpha \in A$. This case can be applied for forms with higher degree, i.e. 3-form θ vanishes if and only if $\theta(X, Y, \alpha) = 0$.

Since ω and ω_J are closed, from (3.15) we get $i_{X\wedge Y}(d(J^*\omega)) = -i_{N_J(X,Y)}\omega$. Putting $\widetilde{\sigma} = \omega - J^*\omega$, we obtain

$$i_{X\wedge Y}(d\widetilde{\sigma}) = i_{N_J(X,Y)}\omega. \tag{3.19}$$

Since $d\phi = 0 \Leftrightarrow d\phi(X, Y, \alpha) = 0$, we have

$$d\phi(X, Y, \alpha) = 0 \Leftrightarrow d\widetilde{\sigma}(X, Y, \alpha) - d(t^*\sigma)(X, Y, \alpha) + d(s^*\sigma)(X, Y, \alpha) = 0.$$

On the other hand, we obtain

$$d(t^*\sigma)(X, Y, \alpha) = d\sigma(dt(X), dt(Y), dt(\alpha)). \tag{3.20}$$

If we take $dt = \rho$ in (3.20) for A, we get

$$d(t^*\sigma)(X, Y, \alpha) = d\sigma(dt(X), dt(Y), \rho(\alpha)). \tag{3.21}$$

On the other hand, from [1] we know that

$$Id_\Sigma = m \circ (t, Id_\Sigma). \tag{3.22}$$

Differentiating (3.22), we obtain

$$X = dt(X). \tag{3.23}$$

Using (3.23) in (3.21), we get

$$d(t^*\sigma)(X, Y, \alpha) = d\sigma(X, Y, \rho(\alpha)).$$

In a similar way, we see that

$$d(s^*\sigma)(X, Y, \alpha) = d\sigma(ds(X), ds(Y), ds(\alpha)).$$

Since $\alpha \in \ker ds$, then $ds(\alpha) = 0$. Hence $d(s^*\sigma) = 0$.
Thus we obtain

$$d\widetilde{\sigma}(X, Y, \alpha) = d\sigma(X, Y, \rho(\alpha)). \tag{3.24}$$

Using (3.19) in (3.24), we derive

$$\omega(N_J(X, Y), \alpha) = d\sigma(X, Y, \rho(\alpha)). \tag{3.25}$$

On the other hand, it is clear that $\phi = 0 \Leftrightarrow \widetilde{\sigma} - t^*\sigma + s^*\sigma = 0$. Thus we obtain

$$\widetilde{\sigma}(X, \alpha) = \sigma(X, \rho(\alpha)).$$

Since $\widetilde{\sigma} = \omega - J^*\omega$, we get

$$\omega(\alpha, X) - \omega(JX, J\alpha) = \sigma(X, \rho(\alpha)). \tag{3.26}$$

Since Poisson bivector π is integrable, it defines a Lie algebroid whose anchor map is $\rho = \pi^\sharp$. Let us use $\pi^\sharp$ instead of ρ in (3.25) and (3.26); then we get

$$\omega(N_J(X, Y), \alpha) = d\sigma(X, Y, \pi^\sharp(\alpha)), \tag{3.27}$$

$$\omega(\alpha, X) - \omega(JX, J\alpha) = \sigma(X, \pi^\sharp(\alpha)).$$

Since $\omega(\alpha, X) = \alpha(X)$, $\omega_J(\alpha, X) = \alpha(JX)$, from (3.27) we have

$$\begin{aligned} -\alpha(N_J(X, Y)) &= d\sigma(X, Y, \pi^\sharp(\alpha)) \\ &= i_{X\wedge Y} d\sigma(\pi^\sharp(\alpha)) \\ &= \pi(\alpha, i_{X\wedge Y} d\sigma) \\ &= -\alpha(\pi^\sharp(i_{X\wedge Y} d\sigma)), \end{aligned}$$

i.e. $\alpha(N_J(X, Y)) = \alpha(\pi^\sharp(i_{X\wedge Y} d\sigma))$.
Since the above equation holds for all non-degenerate α, we get

$$N_J(X, Y) = \pi^\sharp(i_{X\wedge Y} d\sigma). \tag{3.28}$$

On the other hand, from (3.26) we obtain

$$\begin{aligned} -\alpha(X) + \alpha(a^2 X) &= i_X \sigma(\pi^\sharp(\alpha)) \\ &= \pi(\alpha, i_X \sigma) \\ &= -\alpha(\pi^\sharp \sigma_\sharp X). \end{aligned}$$

Thus we get

$$a^2 + \pi^\sharp \sigma_\sharp = I. \tag{3.29}$$

Then (i)$\Leftrightarrow$(ii) follows from (3.28) and (3.29).

(ii)$\Leftrightarrow$(iii): $\omega - J^*\omega = t^*\sigma - s^*\sigma$ says that (t, s) is compatible with 2-forms. Also it is clear that (t, s) and bivectors are compatible due to Σ being a symplectic groupoid. We will check the compatibility of (t, s) and $(1,1)$-tensors. From the compatibility condition of t and s, we will get $dt \circ J = a \circ dt$ and $ds \circ J = -a \circ ds$. For all $\alpha \in A$ and $V \in \chi(\Sigma)$, we have

$$\omega(\alpha, V) = \omega(\alpha, dtV)$$

which is equivalent to

$$\alpha(V) = \langle u_\omega(\alpha), dtV \rangle.$$

Since $u_\omega = Id$ and $u_{\omega_J} = a^*$, we get

$$\begin{aligned}
\langle \alpha, a(dt(V)) \rangle &= \alpha(a(dt(V))) \\
&= a^*\alpha(dt(V)) \\
&= \langle u_{\omega_J}\alpha, V \rangle \\
&= \omega(\alpha, JV) \\
&= \omega(\alpha, dt(JV)) \\
&= \langle \alpha, dt(JV) \rangle
\end{aligned}$$

since this equation holds for all $\alpha \in A$, $a(dt) = dt(J)$. Using $s = t \circ i$,

$$\begin{aligned}
a(ds(V)) &= ad(t \circ i)V \\
&= a(dt(di(V))) \\
&= -a(dt(V)) \\
&= -ds(JV),
\end{aligned}$$

which shows that $a(ds) = -ds(J)$. Thus the proof is completed. □

Acknowledgement

The authors are grateful to the referee for his/her valuable comments and suggestions.

References

[1] Bursztyn, H., Crainic, M., Weinstein, A., Zhu, C.: Integration of twisted Dirac brackets, Duke Math. J., 123, 549–607, (2004).

[2] Crainic, M.: Generalized complex structures and Lie brackets, Bull. Braz. Math. Soc., New Series 42(4), 559–578, (2011).

[3] Ehresmann, C.: Catègories topologiques et catègories differentiables, Coll. Geom. Diff. Globales, Bruxelles, 137–150, (1959).

[4] Gualtieri, M.: Generalized complex geometry, Ph.D. thesis, Univ. Oxford, arXiv:math.DG/0401221, (2003).

[5] Hitchin, N.: Generalized Calabi-Yau manifolds, Q. J. Math., 54, 281–308, (2003).

[6] Iglesias-Ponte, D., Wade, A.: Contact manifolds and generalized complex structures, J. Geom. Phys., 53, 249–258, (2005).

[7] Mackenzie K.: Lie groupoids and Lie algebroids in differential geometry, London Mathematical Society Lecture Note Series, vol. 124, Cambridge University Press, Cambridge, (1987).

[8] Pradines, J.: Théorie de Lie pour les groupoïdes différentiables, Calcul différentiel dans la catégorie des groupoïdes infinitésimaux, Comptes rendus Acad. Sci. Paris 264 A, 245–248, (1967).

[9] Vaisman, I.: Reduction and submanifolds of generalized complex manifolds, Dif. Geom. and its Appl., 25, 147–166, (2007).

[10] Vaisman, I.: Lectures on the geometry of Poisson manifolds, Progress in Math., vol. 118, Birkhäuser Verlag, Boston, (1994).

[11] Wade, A.: Dirac structures and paracomplex manifolds, C. R. Acad. Sci. Paris, Ser. I, 338, 889–894, (2004).