A scheme over prime spectrum of modules

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Abstract: Let \( R \) be a commutative ring with nonzero identity and let \( M \) be an \( R \)-module with \( X = \text{Spec}(M) \). It is introduced a scheme \( O_X \) on the prime spectrum of \( M \) and some of its properties have been investigated.

Key words and phrases: Prime submodule, Zariski topology, primeful module, sheaf of rings, scheme

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unital. For a submodule \( N \) of an \( R \)-module \( M \), \( \langle N : M \rangle \) denotes the ideal \( \{ r \in R \mid rM \subseteq N \} \) and annihilator of \( M \), denoted by \( \operatorname{Ann}_R(M) \), is the ideal \( \langle 0 : M \rangle \). If there is no ambiguity we write \( \langle N : M \rangle \) (resp. \( \operatorname{Ann}(M) \)) instead of \( \langle N : R \rangle \) (resp. \( \operatorname{Ann}_R(M) \)). An \( R \)-module \( M \) is called faithful if \( \operatorname{Ann}(M) = \{0\} \).

A submodule \( N \) of an \( R \)-module \( M \) is said to be prime if \( N \neq M \) and whenever \( rm \in N \) (where \( r \in R \) and \( m \in M \) ) then \( r \in \langle N : M \rangle \) or \( m \in N \). If \( N \) is prime, then the ideal \( \mathfrak{p} = \langle N : M \rangle \) is a prime ideal of \( R \). In these circumstances, \( N \) is said to be \( \mathfrak{p} \)-prime (see [2]). The set of all prime submodules of an \( R \)-module \( M \) is called the prime spectrum of \( M \) and denoted by \( \text{Spec}(M) \). Similarly, the collection of all \( \mathfrak{p} \)-prime submodules of an \( R \)-module \( M \) for any \( \mathfrak{p} \in \text{Spec}(R) \) is designated by \( \text{Spec}_{\mathfrak{p}}(M) \). We remark that \( \text{Spec}(0) = \emptyset \) and that \( \text{Spec}(M) \) may be empty for some nonzero \( R \)-module module \( M \). For example, the \( \mathbb{Z}(p^\infty) \) as a \( \mathbb{Z} \)-module has no prime submodule for any prime integer \( p \) (see [3] and [7]). Such a module is said to be primeless. An \( R \)-module \( M \) is called primeful if either \( M = \{0\} \) or \( M \neq \{0\} \) and the natural map \( \psi : \text{Spec}(M) \to \text{Spec}(R/\operatorname{Ann}(M)) \) defined by \( \psi(P) = (P : M)/\operatorname{Ann}(M) \) for every \( P \in \text{Spec}(M) \), is surjective (see [6]). Let \( \mathfrak{p} \) be a prime ideal of \( R \), and \( N \subseteq M \). By the saturation of \( N \) with respect to \( \mathfrak{p} \), we mean the contraction of \( N_\mathfrak{p} \) in \( M \) and designate it by \( S_\mathfrak{p}(N) \) (see [5]).

Let \( M \) be an \( R \)-module. Throughout this paper \( X \) denotes the prime spectrum \( \text{Spec}(M) \) of \( M \). Let \( N \) be a submodule of \( M \). Then \( V(N) \) is defined as, \( V(N) = \{ P \in X \mid (P : M) \supseteq \langle N : M \rangle \} \) (see [4]). Set \( Z(M) = \{ V(N) : N \subseteq M \} \). Then the elements of the set \( Z(M) \) satisfy the axioms for closed sets in a topological space \( X \) (see [4]). The resulting topology is called the Zariski topology relative to \( M \).

We recall some preliminary results.

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Remark 1.2 (See [4, Theorem 6.1].) The following statements are equivalent:

1. $X$ is a $T_0$-space;
2. The natural map $\psi : \text{Spec}(M) \to \text{Spec}(R/\text{Ann}(M))$ is injective;
3. If $V(P) = V(Q)$, then $P = Q$ for any $P, Q \in \text{Spec}(M)$;
4. $|\text{Spec}_p(M)| \leq 1$ for every $p \in \text{Spec}(R)$.

Remark 1.1 (See [4].) For any element $a$ of a ring $R$, the set $D_r = \text{Spec}(R) \setminus V(rR)$ is open in $\text{Spec}(R)$ and the family $F = \{D_r | r \in R\}$ forms a base for the Zariski topology on $\text{Spec}(R)$. Each $D_r$, in particular, $D_1 = \text{Spec}(R)$ is known to be quasi-compact. For each $r \in R$, we define $X_r = X - V(rM)$. Then every $X_r$ is an open set of $X$, $X_0 = \emptyset$, and $X_1 = X$. By [4, Corollary 4.2], for any $r, s \in R$, $X_{rs} = X_r \cap X_s$.

2. Main results

In this section we use the notion of prime spectrum of a module to define a sheaf of rings. Let $M$ be an $R$-module. For every open subset $U$ of $X$ we define $\text{Supp}(U) = \{(P : M) | P \in U\}$.

Definition 2.1 Let $M$ be an $R$-module. For every open subset $U$ of $X$ we define $\mathcal{O}_X(U)$ to be a subring of $\prod_{p \in \text{Supp}(U)} R_p$, the ring of functions $s : U \to \prod_{p \in \text{Supp}(U)} R_p$, where $s(P) \in R_p$, for each $P \in U$ and $p = (P : M)$ such that for each $P \in U$, there is a neighborhood $V$ of $P$, contained in $U$, and elements $a, f \in R$, such that for each $Q \in V$, $f \notin q := (Q : M)$, and $s(Q) = a/f$ in $R_q$.

It is clear that for an open set $U$ of $X$, $\mathcal{O}_X(U)$ is closed under sum and product. Thus $\mathcal{O}_X(U)$ is a commutative ring with identity (the identity element of $\mathcal{O}_X(U)$ is the function which sends all $P \in U$ to 1 in $R_{(P : M)}$). If $V \subseteq U$ are two open sets, the natural restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is a homomorphism of rings. It is then clear that $\mathcal{O}_X$ is a presheaf. Finally, it is clear from the local nature of the definition $\mathcal{O}_X$ is a sheaf. Hence

Lemma 2.2 Let $M$ be an $R$-module.

1. For each open subset $U$ of $X$, $\mathcal{O}_X(U)$ is a subring of $\prod_{p \in \text{Supp}(U)} R_p$.
2. $\mathcal{O}_X$ is a sheaf.

Next, we find the stalk of the sheaf.

Proposition 2.3 Let $M$ be an $R$-module. Then for each $P \in X$, the stalk $\mathcal{O}_P$ of the sheaf $\mathcal{O}_X$ is isomorphic to $R_P$, where $p := (P : M)$.

Proof Let $P$ be a $p$-prime submodule of $M$ and

$$m \in \mathcal{O}_P = \lim_{P \in U} \mathcal{O}_X(U).$$

Then there exists a neighborhood $U$ of $P$ and $s \in \mathcal{O}_X(U)$ such that $m$ is the germ of $s$ at the point $P$. We define a homomorphism $\varphi : \mathcal{O}_P \to R_p$ by $\varphi(m) = s(P)$. This is a well-defined homomorphism. Let $V$ be
another neighborhood of $P$ and $t \in \mathcal{O}_X(V)$ such that $m$ is the germ of $s$ at the point $P$. Then there exists an open subset $W \subseteq U \cap V$ such that $P \in W$ and $s|_W = t|_W$. Since $P \in W$, $s(P) = t(P)$. We claim that $\varphi$ is an isomorphism.

Let $x \in R_p$. Then $x = a/f$ where $a \in R$ and $f \in R \setminus p$. Since $f \not\in p$, $P \in X_f$. Now we define $s(Q) = a/f$ in $R_q$, where $q := (Q : M)$, for all $Q \in X_f$. Then $s \in \mathcal{O}(X_f)$. If $m$ is the equivalent class of $s$ in $\mathcal{O}_P$, then $\varphi(m) = x$. Hence $\varphi$ is surjective.

Now, let $m \in \mathcal{O}_P$ and $\varphi(m) = 0$. Let $U$ be an open neighborhood of $P$ and $m$ be the germ of $s \in \mathcal{O}_X(U)$ at $P$. There is an open neighborhood $V \subseteq U$ of $P$ and elements $a, f \in R$ such that $s(Q) = a/f \in R_q$, where $q := (Q : M)$, for all $Q \in V$, $f \not\in q$. Thus $V \subseteq X_f$. Then $0 = \varphi(m) = s(P) = a/f$ in $R_p$. So, there is $h \in R \setminus p$ such that $ha = 0$. For $Q \in X_{fh} = X_f \cap X_h$ we have $s(Q) = a/f \in R_q$. Since $h \not\in q$, $s(Q) = \frac{a}{h} = \frac{h a}{q} = 0$. Thus $s|_{\mathcal{O}(X_{fh})} = 0$. Therefore, $s = 0 \in \mathcal{O}(X_{fh})$. Consequently $m = 0$. This completes the proof. □

As a direct consequence of Proposition 2.3, we have

**Corollary 2.4** If $M$ is an $R$-module, then $(\text{Spec}(M), \mathcal{O}_{\text{Spec}(M)})$ is a locally ringed space.

**Proposition 2.5** Let $M$ and $N$ be $R$-modules and $\phi : M \to N$ be an epimorphism. Then $\phi$ induces a morphism of locally ringed spaces

$$(f, f^\sharp) : (\text{Spec}(N), \mathcal{O}_{\text{Spec}(N)}) \to (\text{Spec}(M), \mathcal{O}_{\text{Spec}(M)}).$$

**Proof** By [4, Proposition 3.9], the map $f : \text{Spec}(N) \to \text{Spec}(M)$ which is defined by $P \mapsto \phi^{-1}(P)$, is continuous. Let $U$ be an open subset of $\text{Spec}(M)$ and $s \in \mathcal{O}_{\text{Spec}(M)}(U)$. Suppose $P \in f^{-1}(U)$. Then $f(P) = \phi^{-1}(P) \in U$. Assume that $W$ is an open neighborhood of $\phi^{-1}(P)$ with $W \subseteq U$ with $a, g \in R$, such that for each $Q \in W$, $g \not\in a := (Q : M)$, and $s(Q) = a/g$ in $R_q$. Since $\phi^{-1}(P) \in W$, $P \in f^{-1}(W)$. As we mentioned, $f$ is continuous, so $f^{-1}(W)$ is an open subset of $\text{Spec}(N)$. We claim that for each $Q' \in f^{-1}(W)$, $g \not\in (Q' : N)$. Suppose $g \in (Q' : N)$ for some $Q' \in f^{-1}(W)$. Then $\phi^{-1}(Q') = f(Q') \in W$. Since $\phi$ an epimorphism, $(Q' : N) = (\phi^{-1}(Q') : M)$. So, $g \in (\phi^{-1}(Q') : M)$. This is a contradiction. Therefore, we can define

$$f^\sharp(U) : \mathcal{O}_{\text{Spec}(M)}(U) \to \mathcal{O}_{\text{Spec}(N)}(f^{-1}(U))$$

by $f^\sharp(U)(s) = s \circ f$.

Assume that $V \subseteq U$ and $P \in f^{-1}(V)$. According to the commutativity of the diagram

$$\begin{CD}
   f^{-1}(U) @>{f}>> U @>{t}>> R_{(P, M)},
   @A{f^{-1}(V)}AA
   f^{-1}(V) @>{f}>> V
   @A{t|_V}AA
   @V{t|_V}VV
\end{CD}$$

we have

$$(t \circ f)|_{f^{-1}(V)}(P) = t|_V \circ f(P). \quad (2.1)$$

Consider the diagram

$$\text{204}$$
\[ \mathcal{O}_{\text{Spec}(M)}(U) \xrightarrow{\rho_U} \mathcal{O}_{\text{Spec}(N)}(f^{-1}(U)) \]
\[ \mathcal{O}_{\text{Spec}(M)}(V) \xrightarrow{\rho_V} \mathcal{O}_{\text{Spec}(N)}(f^{-1}(V)). \]

Since
\[ \rho_{f^{-1}(U)f^{-1}(V)}^\sharp(U)(P) = \rho_{f^{-1}(U)f^{-1}(V)}(t \circ f)(P) \]
\[ = (t \circ f)|_{f^{-1}(V)}(P) \]
\[ = t|_V \circ f(P) \quad \text{by equation 2.1} \]
\[ = \rho_{UV}(t) \circ f(P) \]
\[ = f^\sharp(V)\rho_{UV}(t)(P), \]
for each \( t \in \mathcal{O}_{\text{Spec}(M)}(U) \), the diagram (A) is commutative, and it follows that
\[ f^\sharp : \mathcal{O}_{\text{Spec}(M)} \longrightarrow f_*\mathcal{O}_{\text{Spec}(N)} \]
is a morphism of sheaves. By Proposition 2.3, the map on stalks
\[ f^\sharp_P : \mathcal{O}_{\text{Spec}(M), f(P)} \longrightarrow \mathcal{O}_{\text{Spec}(N), P} \]
is clearly the map of local rings
\[ R_{(f(P), M)} \longrightarrow R_{(P, N)}. \]
This implies that
\[ (\text{Spec}(N), \mathcal{O}_{\text{Spec}(N)}) \xrightarrow{(f, f^\sharp)} (\text{Spec}(M), \mathcal{O}_{\text{Spec}(M)}) \]
is a morphism of locally ringed spaces.

**Proposition 2.6** Let \( \Phi : R \to S \) be a ring homomorphism, \( N \) a \( S \)-module and \( M \) a primeful \( R \)-module such that \( \text{Spec}(M) \) is a \( T_0 \)-space and \( \text{Ann}_R(M) \subseteq \text{Ann}_R(N) \) (here, we consider \( N \) as an \( R \)-module by means of \( \Phi \)). Then \( \Phi \) induces a morphism of locally ringed spaces
\[ (\text{Spec}(N), \mathcal{O}_{\text{Spec}(N)}) \xrightarrow{(h, h^\sharp)} (\text{Spec}(M), \mathcal{O}_{\text{Spec}(M)}). \]

**Proof** Since \( \text{Ann}_R(M) \subseteq \text{Ann}_R(N) \), \( \Phi \) induces the map \( \Theta : R/\text{Ann}_R(M) \to S/\text{Ann}_S(N) \). It is well known that the maps \( f : \text{Spec}(S) \to \text{Spec}(R) \) by \( p \mapsto \Phi^{-1}(p) \) and \( d : \text{Spec}(S/\text{Ann}_S(N)) \to \text{Spec}(R/\text{Ann}_R(M)) \) by \( \bar{p} \mapsto \Theta^{-1}(\bar{p}) \) and \( \psi_N : \text{Spec}(N) \to \text{Spec}(S/\text{Ann}_S(N)) \) with \( \psi(N) = (P :_SN)/\text{Ann}_S(N) \) for each \( P \in \text{Spec}(N) \) are continuous maps. Also \( \psi_M : \text{Spec}(M) \to \text{Spec}(R/\text{Ann}_R(M)) \) is homeomorphism by [4, Theorem 6.5]. Therefore the map
\[ h : \text{Spec}(N) \longrightarrow \text{Spec}(M) \]
\[ P \mapsto \psi_M^{-1}d\psi_N(P) \]
is continuous. For each $P \in \text{Spec}(N)$, we get a local homomorphism

$$\Phi_{(P:S^i N)} : R_f(P:S^i N) \longrightarrow S_{(P:S^i N)}.$$ 

Let $U$ be an open subset of $\text{Spec}(M)$ and let $t \in O_{\text{Spec}(M)}(U)$. Suppose that $P \in h^{-1}(U)$. Then $h(P) \in U$ and there exists a neighborhood $W$ of $h(P)$ with $W \subseteq U$ and elements $r, g \in R$ such that for each $Q \in W$, $g \notin (Q :_R M)$, and $t(Q) = \frac{r}{g} \in R_{\langle Q :_R M \rangle}$. Hence $g \notin (h(P) :_R M)$. By definition of $h$, $(h(P) :_R M) = \Phi^{-1}(P :_{SN} N)$. So, $\Phi(g) \notin (P :_{SN} N)$. Thus $\Phi_{(P:S^i N)}(\frac{t}{g})$ define a section on $O_{\text{Spec}(N)}(h^{-1}(W))$.

Since $R \xrightarrow{g} S_{\Phi(g)} \xrightarrow{\Phi(P:S^i N)} S_{(P:S^i N)}$ is commutative, we can define

$$h^2(U) : O_{\text{Spec}(M)}(U) \longrightarrow h_* O_{\text{Spec}(N)}(U) = O_{\text{Spec}(N)}(h^{-1}(U))$$

by $h^2(U)(t)(P) = \Phi_{(P:S^i N)}(t(h(P)))$ for each $t \in O_{\text{Spec}(M)}(U)$ and $P \in h^{-1}(U)$. Assume that $V \subseteq U$ and $P \in h^{-1}(V)$.

According to the commutative diagram

we have

$$\Phi_{(P:S^i N)}(t|_V \circ h)(P) = (\Phi_{(P:S^i N)}t \circ h)|_{h^{-1}(V)}(P). \quad (2.2)$$

Considering the diagram

we have

$$(B)$$

206
it is easy to see that
\[
\begin{align*}
\rho'_{h^{-1}(V)h^{-1}(U)}h^t(U)(t)(P) &= \rho'_{h^{-1}(U)h^{-1}(V)}\Phi(\mathcal{P}, \mathcal{N})t \circ h(P) \\
&= (\Phi(\mathcal{P}, \mathcal{N})t \circ h)|_{h^{-1}(V)}(P) \\
&= \Phi(\mathcal{P}, \mathcal{N})t|_V \circ h(P) \quad \text{by equation 2.2} \\
&= h^t(V)|_V \circ h(P) \\
&= h^t(V)\rho_{UV}(t)(P).
\end{align*}
\]

So, the diagram (B) is commutative, and it follows that
\[
h^t : \mathcal{O}_{\text{Spec}(M)} \longrightarrow h_*\mathcal{O}_{\text{Spec}(N)}
\]
is a morphism of sheaves. By Proposition 2.3, the map on stalks
\[
h^t_P : \mathcal{O}_{\text{Spec}(M), h(P)} \longrightarrow \mathcal{O}_{\text{Spec}(N), P}
\]
is clearly
\[
R_f(\mathcal{P}, \mathcal{N}) \longrightarrow S(\mathcal{P}, \mathcal{N}).
\]
This implies that
\[
(\text{Spec}(N), \mathcal{O}_{\text{Spec}(N)}) \xrightarrow{(h^t)} (\text{Spec}(M), \mathcal{O}_{\text{Spec}(M)})
\]
is a morphism of locally ringed spaces. \(\square\)

**Example 2.7** Let \(\Omega\) be the set of all prime integers \(p\), \(M = \prod_p \mathbb{Z}/p\mathbb{Z}\) and \(N = \bigoplus_p \mathbb{Z}/p\mathbb{Z}\) where \(p\) runs through \(\Omega\). By [6, p.136, Example 1], \(N\) is a faithful \(\mathbb{Z}\)-module and \(M\) is a faithful primeful \(\mathbb{Z}\)-module. It is also shown that
\[
\text{Spec}(M) = \{ S(\mathfrak{p}) \} \cup \{ pM | p \in \Omega \}.
\]
Therefore by Remark 1.1, \(\text{Spec}(M)\) is a \(T_0\)-space. Hence by Proposition 2.6, there exists a morphism of locally ringed spaces
\[
(\text{Spec}(\bigoplus_p \mathbb{Z}/p\mathbb{Z}), \mathcal{O}_{\text{Spec}(\bigoplus_p \mathbb{Z}/p\mathbb{Z})}) \rightarrow (\text{Spec}(\prod_p \mathbb{Z}/p\mathbb{Z}), \mathcal{O}_{\text{Spec}(\prod_p \mathbb{Z}/p\mathbb{Z})}).
\]

**Proposition 2.8** Let \(M\) be a faithful and primeful \(R\)-module. For any element \(f \in R\), the ring \(\mathcal{O}_X(X_f)\) is isomorphic to the localized ring \(R_f\).

**Proof** We define the map \(\Theta : R_f \rightarrow \mathcal{O}_X(X_f)\) by
\[
\frac{a}{f^m} \mapsto (s : Q \mapsto \frac{a}{f^m} \in R(Q, M)).
\]
Indeed \(\Theta\) sends \(\frac{a}{f^m}\) to the section \(s \in \mathcal{O}_X(X_f)\) which assigns to each \(Q\) the image of \(\frac{a}{f^m} \in R(Q, M)\). It is easy to see \(\Theta\) is a well-defined homomorphism. We are going to show that \(\Theta\) is an isomorphism.
We first show that $\Theta$ is injective. If $\Theta(\frac{a}{b}) = \Theta(\frac{c}{d})$, then for every $P \in X_f$, $\frac{a}{b}$ and $\frac{c}{d}$ have the same image in $R_p$, where $p = (P : M)$. Thus there exists $h \in R \setminus p$ such that $h(f^m a - f^n b) = 0$ in $R$. Let $I = \{0 : R \ f^m a - f^n b\}$. Then $h \in I$ and $h \notin p$, so $I \notin p$. This happens for any $P \in X_f$, so we conclude that 

$$V(I) \cap \text{Supp}(X_f) = \emptyset$$

hence 

$$\text{Supp}(X_f) \subseteq D(I) := \text{Spec}(R) \setminus V(I).$$

Since $M$ is faithful primeful, 

$$D_f = \text{Supp}(X_f) \subseteq D(I).$$

Therefore $f \in \sqrt{I}$ and so $f^l \in I$ for some positive integer $l$. Now we have 

$$f^l(f^m a - f^n b) = 0$$

which shows that $\frac{a}{b} = \frac{a}{b}$ in $R_p$. Hence $\Theta$ is injective.

Let $s \in \mathcal{O}_X(X_f)$. Then we can cover $X_f$ with open subset $V_i$, on which $s$ is represented by $\frac{a_i}{b_i}$, with $g_i \notin (P : M)$ for all $P \in V_i$, in other words $V_i \subseteq X_{g_i}$. By [4, Proposition 4.3], the open sets of the form $X_{g_i}$ form a base for the topology. So, we may assume that $V_i = X_{h_i}$ for some $h_i \in R$. Since $X_{h_i} \subseteq X_{g_i}$, by [4, Proposition 4.1], $h_i \in \sqrt{(g_i)}$. Thus $h_i^n \in (g_i)$ for some $n \in \mathbb{N}$. So, $h_i^n = cg_i$ and

$$\frac{a_i}{g_i} = \frac{ca_i}{cg_i} = \frac{ca_i}{h_i^n}.$$ 

We see that $s$ is represented by $\frac{b_i}{k_i}$, $(b_i = ca_i, k_i = h_i^n)$ on $X_{h_i}$ and (since $X_{h_i} = X_{h_i}$) the $X_{h_i}$ cover $X_f$. The open cover $X_f = \bigcup X_{h_i}$ has a finite subcover by [4, Proposition 4.4]. Suppose, $X_f \subseteq X_{k_1} \cup \cdots \cup X_{k_n}$. For $1 \leq i, j \leq n$, $\frac{b_i}{k_i}$ and $\frac{b_j}{k_j}$ both represent $s$ on $X_{k_i} \cap X_{k_j}$. By Remark 1.2, $X_{k_i} \cap X_{k_j} = X_{k_i,k_j}$ and by injectivity of $\Theta$, we get $\frac{b_i}{k_i} = \frac{b_j}{k_j}$ in $R_{k_i,k_j}$. Hence for some $n_{ij}$,

$$(k_i,k_j)^{n_{ij}}(k_jb_i - k_ib_j) = 0.$$ 

Let $m = \max\{n_{ij}|1 \leq i, j \leq n\}$. Then

$$k_j^{m+1}(k_i^m b_i) - k_i^{m+1}(k_j^m b_j) = 0.$$ 

By replacing each $k_i$ by $k_i^{m+1}$, and $b_i$ by $k_i^m b_i$, we still see that $s$ is represented on $X_{k_i}$ by $\frac{b_i}{k_i}$, and furthermore, we have $k_j b_i = k_i b_j$ for all $i, j$. Since $X_f \subseteq X_{k_1} \cup \cdots \cup X_{k_n}$, by [4, Proposition 4.1], we have 

$$D_f = \psi(X_f) \subseteq \bigcup_{i=1}^{n} \psi(X_{k_i}) = \bigcup_{i=1}^{n} D_{k_i},$$

where $\psi$ is the natural map $\psi : \text{Spec}(M) \to \text{Spec}(R)$. So, there are $c_1, \cdots, c_n$ in $R$ and $t \in \mathbb{N}$, such that $f^t = \sum c_i k_i$. Let $a = \sum c_i b_i$. Then for each $j$ we have

$$k_j a = \sum c_i b_i k_j = \sum c_i k_i b_j = b_j f^t.$$ 

This implies that $\frac{a}{b} = \frac{b_j}{k_j}$ on $X_{k_j}$. So $\Theta(\frac{a}{b}) = s$ everywhere, which shows that $\Theta$ is surjective. \hfill $\square$
Corollary 2.9 Let $M$ be a faithful and primeful $R$-module. Then $\mathcal{O}(\text{Spec}(M))$ is isomorphic to $R$.

We recall that a scheme $X$ is locally Noetherian if it can be covered by open affine subsets $\text{Spec}(A_i)$, where each $A_i$ is a Noetherian ring. $X$ is Noetherian if it is locally Noetherian and quasi-compact [1].

Theorem 2.10 Let $M$ be a faithful and primeful $R$-module such that $X$ is a $T_0$-space. Then $(X, \mathcal{O}_X)$ is a scheme. Moreover, if $R$ is Noetherian, then $(X, \mathcal{O}_X)$ is a Noetherian scheme.

Proof Let $g \in R$. Since the natural map $\psi : \text{Spec}(M) \to \text{Spec}(R)$ is continuous by [4, Proposition 3.1], the map $\psi|_{X_g} : X_g \to \psi(X_g)$ is also continuous. By assumption and Remark 1.1, $\psi|_{X_g}$ is a bijection. Let $E$ be a closed subset of $X_g$. Then $E = X_g \cap V(N)$ for some submodule $N$ of $M$. Hence $\psi(E) = \psi(X_g \cap V(N)) = \psi(X_g) \cap V(N : M)$ is a closed subset of $\psi(X_g)$. Therefore, $\psi|_{X_g}$ is a homeomorphism.

Suppose $X = \bigcup_{i \in I} X_{g_i}$. Since $M$ is faithful primeful and $X$ is a $T_0$-space, for each $i \in I$

$$X_{g_i} \cong \psi(X_{g_i}) = \text{Supp}(X_{g_i}) = D_{g_i} \cong \text{Spec}(R_{g_i}).$$

Thus by Proposition 2.8, $X_{g_i}$ is an affine scheme and this implies that $(X, \mathcal{O}_X)$ is a scheme. For the last statement, we note that since $R$ is Noetherian, so is $R_{g_i}$ for each $i \in I$. Hence $(X, \mathcal{O}_X)$ is a locally Noetherian scheme. By [4, Theorem 4.4], $X$ is quasi-compact. Therefore, $(X, \mathcal{O}_X)$ is a Noetherian scheme.

References