Foliations and a class of metrics on tangent bundle

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Abstract: Let $M$ be a smooth manifold with Finsler metric $F$, and let $TM^0$ be the slit tangent bundle of $M$ with a generalized Riemannian metric $G$, which is induced by $F$. In this paper, we extract many natural foliations of $(TM^0, G)$ and study some of their geometric properties. Next we use this approach to obtain new characterizations of Finsler manifolds with positive constant curvature.

Key words: Finsler manifold, foliation, constant curvature, Riemannian metric

1. Introduction
Several monographs present methods of differential geometry used in the study of Finsler manifolds [1, 2, 3, 5, 6]. As the geometric objects that occur in Finsler geometry depend on both point and direction, the tangent bundle of a Finsler manifold plays a major role in this study. To emphasize this Bejancu and Farran in [4], by using Sasaki-Finsler metric $G_{S}$, initiate a study of interrelations between the geometry of foliations on the tangent bundle of a Finsler manifold and the geometry of the Finsler manifold itself. Then, Peyghan and Tayebi introduce new metric $G$ on slit bundle of Finsler manifold and they study geometric properties of this metric [10]. In this paper, we use this metric on $TM^0$ and we show that the vertical and horizontal Liouville vector fields $L$ and $L^*$ determine three totally geodesic foliations on $(TM^0, G)$. Finally, the main properties of the two foliations defined by $F$ on $(TM^0, G)$ are presented in Propositions 1 and 2. In the last section, for any $c > 0$ we consider the indicatrix-bundle $IM^0(c)$ and by using the horizontal Liouville foliation on $(IM^0(c), G)$ and the curvature-angular form we obtain three new characterizations of Finsler manifolds of positive constant curvature.

2. Preliminaries
Let $(M, F)$ be a Finsler manifold, where $M$ is a real $n$-dimensional smooth manifold and $F$ is the fundamental function of $(M, F)$ [2]. Consider $TM^0 = TM \setminus \{0\}$ and denote by $VTM^0$ the vertical vector bundle over $TM^0$, that is, $VTM^0 = \ker \pi_*$, where $\pi_*$ is the tangent mapping of the canonical projection $\pi : TM^0 \to M$.

We may think of the Finsler metric $(g = g_{ij}(x, y))$, where we set $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ as a Riemannian metric on $VTM^0$. The canonical nonlinear connection $HTM^0 = (N^j_i(x, y))$ of $(M, F)$ is given by $N^j_i = \frac{\partial G^j}{\partial y^i}$, where $G^j = \frac{1}{4} g^{ih}(\frac{\partial^2 F^2}{\partial y^i \partial y^j} y^h - \frac{\partial F^2}{\partial y^i} y^h)$. Then on any coordinate neighborhood $u \subset TM^0$ the vector fields

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Theorem 1 ([7]) A Finsler manifold \((M,F)\) is of constant curvature \(k\) if and only if the following holds
\[
R_{ij} = kF^2h_{ij}, \quad i,j = 1,\ldots,n.
\] (2.5)

Consider now the energy density \(2t(x,y) = F^2 = g_{ij}(x,y)y^iy^j\) defined by the Finsler metric \(F\) and also the smooth functions \(u,v:[0,\infty) \rightarrow \mathbb{R}\) such that \(u + 2tv > 0\) for every \(t\). The above conditions assure that the symmetric \((0,2)\)-type tensor field of \(TM^c\), \(G_{ij} = u(t)g_{ij} + v(t)y_iy_j\) is positive definite. The inverse of this matrix has the entries \(H^{kl} = \frac{1}{2}g^{kl} + \omega(t)y^ky^l\), where \((g^{kl})\) are the components of the inverse of the matrix \((g_{ij})\) and \(\omega(t) = -\frac{v}{u(a+2tv)}\). The components \(H^{kl}\) define symmetric \((0,2)\)-type tensor field of \(TM^c\). It is easy to see that if the matrix \((G_{ij})\) is positive definite, then matrix \(H^{kl}\) is positive definite, too. We use also the components \(H_{ij}\) of symmetric \((0,2)\)-type tensor field of \(TM^c\) obtained from the components \(H^{kl}\) by “lowering” the indices \(H_{ij} = g_{ik}H^{kl}g_{lj} = \frac{1}{2}g_{ij} + \omega y_iy_j\), where \(y_i = g_{ik}y^k\). The following Riemannian metric may be considered on \(TM^c\) (cf. [8]):
\[
G \left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) = G_{ij}, \quad G \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = H_{ij}, \quad G \left( \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right) = G \left( \frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j} \right) = 0.
\] (2.6)

If \(u = 1\) and \(v(t) = 0\), then the above metric gives us the Sasaki-Finsler metric \(G_S\) as follows [4]:
\[
G_S \left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) = G_S \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = g_{ij}, \quad G_S \left( \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right) = G_S \left( \frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j} \right) = 0.
\] (2.7)
**Lemma 1** The Levi-Civita connection of the Riemannian metric $G$ defined by (2.6) is as follows:

\[
\nabla_{\partial_i} \partial_j = \frac{1}{u^2} (-F^s_{ij} + G^s_{ij}) \frac{\delta}{\delta x^s} + (C^s_{ij} + \alpha_1 y_i y_j + \alpha_2 y_i \delta^s_j + \alpha_3 y_i y_j + \alpha_4 y_i y_j + \alpha_5 y_i y_j + \alpha_6 y_i y_j)
\]

\[
+ \alpha_7 y_i y_j y^s + \alpha_8 y_i y_j y^s + \alpha_9 y_i y_j y^s + \frac{1}{2u} R_{ij} H^{ks} \frac{\delta}{\delta x^s} + (F^s_{ij} - G^s_{ij}) \frac{\delta}{\delta y^s}, \tag{2.8}
\]

\[
\nabla_{\partial_i} \partial_j = (C^s_{ij} + \alpha_{10} y_i y_j y^s + \alpha_{11} y_i y_j y^s + \alpha_{12} y_i y_j y^s + \frac{1}{2u} R_{ij} H^{ks} \frac{\delta}{\delta x^s} + F^s_{ij} \frac{\delta}{\delta y^s}, \tag{2.9}
\]

where $\alpha_1 = \frac{u'}{2u}$, $\alpha_2 = \frac{u''}{2u}$, $\alpha_3 = \frac{u'' + uu'' + uu'' + uu''}{2u^2}$, $\alpha_4 = \frac{-u^2}{2u}$, $\alpha_5 = \frac{-u'' + uu'' + uu''}{2u^2}$, $\alpha_6 = \frac{u''}{2u}$, $\alpha_7 = \frac{uu'' + uu'' + uu''}{2u}$, $\alpha_8 = \frac{u(1 + 2uu)}{2u}$, $\alpha_9 = \frac{-u}{2u}$ and $C^s_{ij}$ is the $h$-covariant derivative of $C^s_{ij}$ with respect to Cartan connection.

3. **Foliations on** $(TM^\circ, G)$

In this section, we shall study various kinds of foliation which are naturally associated to $(TM^\circ, G)$. For this purpose, we consider two globally defined vector fields on $TM^\circ$ locally given by

\[
L = y^i \partial_i, \tag{3.12}
\]

\[
L^* = g^i \partial_i. \tag{3.13}
\]

$L$ and $L^*$ are called the *vertical* and *horizontal* Liouville vector fields, respectively. The line distribution $\mathcal{L} = \text{span}\{L\}$ and $\mathcal{L}^* = \text{span}\{L^*\}$ are called the *vertical* and *horizontal* Liouville distributions, respectively.

**Theorem 2** Let $(M, F)$ be a Finsler manifold. Then we have the following assertions:

(i) The vertical Liouville vector field determines a totally geodesic foliation on $(TM^\circ, G)$.

(ii) The horizontal Liouville vector field determines a totally geodesic foliation on $(TM^\circ, G)$ if and only if $u$ and $v$ satisfy in

\[
u v + \frac{1}{2} u' u + t(vu' + v'u) + 2tv^2 = 0. \tag{3.14}
\]

(iii) The distribution $\Gamma(\mathcal{L} \oplus \mathcal{L}^*)$ is integrable and its tangent foliation is totally geodesic on $(TM^\circ, G)$. 

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Proof By using Lemma 1, we get
\[
\tilde{\nabla}_L L = \left(1 + 2t(\alpha_1 - 2\alpha_2 + 2t\alpha_3)\right)L, \\
\tilde{\nabla}_L L^* = \left(2t(2\alpha_4 + \alpha_6 + 2t\alpha_5)\right)L, \\
\tilde{\nabla}_L L = \left(2t(\alpha_2 + \alpha_9 + \alpha_8 + 2t\alpha_7)\right)L^*, \\
\tilde{\nabla}_L L^* = \left(1 + 2t(\alpha_2 + \alpha_9 + \alpha_8 + 2t\alpha_7)\right)L^*.
\] (3.15) (3.16) (3.17) (3.18)

Relation (3.15) tells us that \( \mathcal{L} \) is totally geodesic. Also, from (3.16) we derive that \( \mathcal{L}^* \) is totally geodesic if and only if (3.14) holds. Relations (3.17) and (3.18) give us \([L, L^*] = L^* \in \Gamma(\mathcal{L} \oplus \mathcal{L}^*)\). Now let \( X = XL + X^* L^* \) and \( Y = YL + Y^* L^* \) belong to \( \Gamma(\mathcal{L} \oplus \mathcal{L}^*) \), then by Dirac calculation we obtain
\[
\tilde{\nabla}_X Y = \left(XL(Y) + X^* L^*(Y) + XY(1 + 2t(\alpha_1 - 2\alpha_2 + 2t\alpha_3))
\right.
\]
\[
+ 2tX^* Y^*(2\alpha_4 + \alpha_6 + 2t\alpha_5)\big) + \left(XL(Y^*) + X^* L^*(Y^*)
\right.
\]
\[
+ XY^*(1 + 2t(\alpha_2 + \alpha_9 + \alpha_8 + 2t\alpha_7))
\]
\[
+ 2tX^* Y(\alpha_2 + \alpha_9 + \alpha_8 + 2t\alpha_7)\big) L^*.
\]
Hence we derive that \( \tilde{\nabla}_X Y \in \Gamma(\mathcal{L} \oplus \mathcal{L}^*) \) for any \( X, Y \in \Gamma(\mathcal{L} \oplus \mathcal{L}^*) \). Therefore the foliation determined by \( \Gamma(\mathcal{L} \oplus \mathcal{L}^*) \) is totally geodesic. \( \square \)

Remark 1 It is remarkable that the foliation in (ii) is also totally geodesic with respect to the Sasaki-Finsler metric (cf. [4]).

Also, by using (2.8)–(2.9) we can conclude the following:

Lemma 2 Let \((M, F)\) be a Finsler manifold. Then we have
\[
\tilde{\nabla}_X L = X^i \left((2t\alpha_1 + \alpha_9)g_i y^k + 2t\alpha_2 \delta_i^k\right) \delta_k
\]
\[
+ X^i \left((\alpha_1 - 2t\alpha_3)y_i y^k + (1 - 2t\alpha_2) \delta_i^k\right) \delta_k,
\] (3.19)
\[
\tilde{\nabla}_X L^* = X^i \left((1 + 2t\alpha_3)\delta_i^k + (2t\alpha_7 + \alpha_8 + \alpha_2) y_i y^k + \frac{1}{2u} y^j R_{ij} H^{jk}\right) \delta_k
\]
\[
+ X^i \left(\frac{1}{2} y^j R_{ij}^{k} + 2t\alpha_4 \delta_i^k + (\alpha_4 + 2t\alpha_5 + \alpha_6) y_i y^k\right) \delta_k,
\] (3.20)
where \( X = X^i \delta_i + X^i \delta_i \in \Gamma(TTM^\circ) \).

To introduce two more foliations on \((TM^\circ, G)\) we denote by \( \mathcal{L}' \) and \( \mathcal{L}^\perp \) the complementary orthogonal distributions to \( \mathcal{L} \) in \( VTM^\circ \) and \( TTM^\circ \), respectively. Now, let \( X, Y \in \Gamma(\mathcal{L}^\perp) \). Since \( G \) is parallel with
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respect to $\tilde{\nabla}$, then we get

$$G([X,Y], L) = G(\tilde{\nabla}X Y, L) - G(\tilde{\nabla}Y X, L) = G(X, \tilde{\nabla}Y L) - G(Y, \tilde{\nabla}X L). \quad (3.21)$$

By using (3.19), we derive

$$G(X, \tilde{\nabla}Y L) = \left[((\alpha_9 + \alpha_8 + 2t\alpha_7)(u + 2tv) + 2t\alpha_2)g_{ij} + 2tu\alpha_2 g_{ij}\right] \dot{X}^i \dot{Y}^j$$

$$+ \left[\alpha_1 - \alpha_2 + 2t\alpha_3\right]\left(\frac{t}{u} + 2tw\right) + w(1 - 2t\alpha_2)\left[g_{ij}\right]$$

$$+ \frac{1}{u}(1 - 2t\alpha_2)g_{ij} \dot{X}^i \dot{Y}^j. \quad (3.22)$$

Similarly we have

$$G(Y, \tilde{\nabla}X L) = \left[\left(\alpha_9 + \alpha_8 + 2t\alpha_7\right)(u + 2tv) + 2t\alpha_2 g_{ij}\right] Y^i X^j$$

$$+ \left[\alpha_1 - \alpha_2 + 2t\alpha_3\right]\left(\frac{t}{u} + 2tw\right) + w(1 - 2t\alpha_2)\left[g_{ij}\right]$$

$$+ \frac{1}{u}(1 - 2t\alpha_2)g_{ij} \dot{Y}^i \dot{X}^j. \quad (3.23)$$

Since $i, j, k$ in (3.22) and (3.23) are summation indices, then (3.22) is equal (3.23). Therefore, by according to (3.21) we infer

$$G([X,Y], L) = 0. \quad (3.24)$$

Hence $[X,Y] \in \Gamma(L^\perp)$, that is, $L^\perp$ is integrable. It is obvious that $L'$ is integrable, too. Therefore, we have the following theorem.

**Theorem 3** Let $(M, F)$ be a Finsler manifold. Then both distributions $L^\perp$ and $L'$ are integrable.

Also, similar to the proof of Proposition 2.1 in [4], we can prove the following:

**Proposition 1** (i) The fundamental foliation $F_F$ determined by the level hypersurfaces of the fundamental function $F$ of the Finsler manifold $(M, F)$ is just the foliation determined by the integrable distribution $L^\perp$.

(ii) The vertical Liouville vector field is orthogonal to foliation $F_F$.

(iii) The horizontal Liouville vector field is tangent to foliation $F_F$.

Next, we consider a fixed point $x_0 = (x_0^i)$ in $M$ and the hypersurfaces $I_{x_0}M(c) \subset T_{x_0}M^\circ = T_{x_0}M - \{0\}$ given by the equation

$$F(x_0, y) = c, \quad \forall y \in T_{x_0}M^\circ,$$

where $c$ is a positive constant. We call it the $c$-indicatrix of $(M, F)$ at $x_0$. Then the set of all $c$-indicatrices at $x_0$ determines a foliation of codimension one of the $m$-dimensional Riemannian manifold $(T_{x_0}M^\circ, g_{x_0})$, where $g_{x_0} = (g_{ij}(x_0, y))$ (see [4]). Now, let

$$l = \frac{1}{F}\sqrt{u + 2tvL} = \sqrt{u + 2tv^i \dot{\theta}_i}, \quad l^i = \frac{y^i}{F}.$$
then we have $G(l, l) = 1$. Also, we denote by the same symbol $g_{x_0}$ the induced Riemannian metric by $g_{x_0}$ on $I_{x_0}M(c)$. Now, we put
\[
\tilde{\nabla}_X Y = \nabla_X Y + b(X, Y),
\]
\[
\nabla'_X Y = \nabla''_X Y + B(X, Y)l,
\] (3.25) (3.26)
for any $X, Y \in \Gamma(TM)$, where $\nabla'$ and $\nabla''$ are the Levi-Civita connections on $(T_{x_0}M^o, g_{x_0})$ and $(I_{x_0}M(c), g_{x_0})$, respectively, while $h(\cdot, \cdot)$ and $B(\cdot, \cdot)l$ are the second fundamental forms of $T_{x_0}M^o$ and $I_{x_0}M(c)$ as submanifolds of $(TM^o, G)$ and $(T_{x_0}M^o, g_{x_0})$, respectively. Since $l$ is orthogonal to $I_{x_0}M(c)$, then we have $g_{x_0}(\nabla''_X Y, l) = 0$. Hence by using (3.26), we obtain
\[
g_{x_0}(\nabla'_X Y, l) = g_{x_0}(B(X, Y)l, l) = \frac{u + 2tv}{2t} B(X, Y) y^i y_j \dot{g}_{ij} = (u + 2tv) B(X, Y). \tag{3.27}
\]
Now, let $\tilde{\nabla}_X Y = (\tilde{\nabla}_X Y)\dot{i}_i + (\tilde{\nabla}_X Y)\dot{j}_j$. According to (2.6), we get
\[
G(\tilde{\nabla}_X Y, l) = G(((\tilde{\nabla}_X Y)\dot{i}_i + (\tilde{\nabla}_X Y)\dot{j}_j, \frac{1}{F}\sqrt{u + 2tv} \dot{y}_j)
= (\tilde{\nabla}_X Y)\dot{i}_i \sqrt{u + 2tv} y_j (\frac{1}{u} g_{ij} - \frac{v}{u(u + 2tv)} y_i y_j)
= \frac{1}{F\sqrt{u + 2tv}} (\tilde{\nabla}_X Y)\dot{i}_i y_j. \tag{3.28}
\]
Similarly, we obtain
\[
g_{x_0}(\tilde{\nabla}_X Y, l) = g_{x_0}(((\tilde{\nabla}_X Y)\dot{i}_i + (\tilde{\nabla}_X Y)\dot{j}_j, \frac{1}{F}\sqrt{u + 2tv} \dot{y}_j)
= (\tilde{\nabla}_X Y)\dot{i}_i \sqrt{u + 2tv} y_j \dot{g}_{ij} = (\tilde{\nabla}_X Y)\dot{i}_i \sqrt{u + 2tv} y_j. \tag{3.29}
\]
The relations (3.27), (3.28) and (3.29) give us
\[
B(X, Y) = \frac{1}{u + 2tv} g_{x_0}(\nabla'_X Y, l) = G(\tilde{\nabla}_X Y, l) = -G(Y, \tilde{\nabla}_X l)
= -G(Y, X(\sqrt{u + 2tv} \dot{Y}) + \sqrt{u + 2tv} \tilde{\nabla}_X Y)
= -\frac{\sqrt{u + 2tv}}{F} G(Y, \tilde{\nabla}_X Y). \tag{3.30}
\]
Let $X = \dot{X}^i \dot{i}_i \in \Gamma(TM)$. Since $L$ is orthogonal to $I_{x_0}M(c)$, then we have
\[
0 = G(X, L) = G(\dot{X}^i \dot{i}_i, y^j \dot{j}_j) = \dot{X}^i y^j (\frac{1}{u} g_{ij} - \frac{v}{u(u + 2tv)} y_i y_j) = \frac{1}{u + 2tv} \dot{X}^i y_i. \tag{3.31}
\]
Hence we infer that
\[
\dot{X}^i y_i = 0, \tag{3.31}
\]
because \( u + 2tv \neq 0 \). By using (3.19) and (3.31), we deduce

\[ \tilde{\nabla}_X L = (1 - 2t\alpha_2)X. \quad (3.32) \]

The relation (3.32) in (3.30) implies

\[ B(X, Y) = (2t\alpha^2 - 1)\sqrt{u + 2tv}G(X, Y). \quad (3.33) \]

But by direct calculation we derive

\[ G(X, Y) = \frac{1}{u}g_{x_0}(X, Y). \]

Thus for any \( X, Y \in \Gamma(T_{x_0}M) \) we obtain

\[ B(X, Y) = (2t\alpha^2 - 1)\sqrt{u + 2tv}g_{x_0}(X, Y). \]

Therefore any \( c \)-indicatrix at \( x_0 \) is a totally umbilical manifold immersed in \( (T_{x_0}M^c, g_{x_0}) \). Finally, we deduce that the leaves of the integrable distribution \( L' \) are \( c \)-indicatrices, because \( L \) is the normal vector field to each \( c \)-indicatrix.

**Proposition 2** Let \((M, F)\) be a Finsler manifold. Then we have the following assertions:

(i) At any point \( x \in M \), the indicatrix foliation \( I_x M \) is a totally umbilical foliation of \((T_x M, g_x)\).

(ii) The foliation \( F_{L'} \) determined by the integrable distribution \( L' \) are \( c \)-indicatrices of \((M, F)\).

(iii) The foliation \( F_{L'} \) is a totally umbilical subfoliation of the vertical foliation \( F_V \).

### 4. Finsler manifolds of positive constant curvature

In this section, we give some necessary and sufficient conditions for \((M, F)\) to be of constant curvature.

Let \((M, F)\) be a Finsler manifold and consider the symmetric tensor fields \( R = (R_{ij}) \) and \( h = (h_{ij}) \), where \( R_{ij} \) and \( h_{ij} \) are given by (2.2) and (2.4). We define the symmetric Finsler tensor field \( \Lambda = (\Lambda_{ij}) \) by

\[ \Lambda_{ij} = R_{ij} - h_{ij}. \quad (4.34) \]

We consider \( \Lambda \) as a symmetric bilinear form on the \( \mathcal{F}(TM^c) \)-module \( \Gamma(HTM^c) \) and call it the curvature-angular form of \((M, F)\) (see [4]).

**Proposition 3** For any \( X \in \Gamma(HTM^c) \) we have

\[ \Lambda(L^*, X) = 0 = R(L^*, X). \quad (4.35) \]

**Proof** Let \( X = X^i\delta_i \in \Gamma(HTM^c) \). Using (ii) of (2.3) and (2.4), we have

\[ \Lambda(L^*, X) = g^i X^j\Lambda_{ij} = X^j g^i R_{ij} - X^j g^i g_{ij} + X^j g^i \frac{y_i y_j}{F^2} = -X^j y_i + X^j y_j = 0. \]

(4.36)

Also, part (ii) of (2.3) gives us

\[ R(L^*, X) = X^j g^i R_{ij} = 0. \]

(4.37)

The relations (4.36) and (4.37) imply (4.35). \( \square \)
Next, we consider a leaf $IM(c)$ of the fundamental foliation $\mathcal{F}_F$ on $(TM^\circ, G)$. As we can write

$$IM(c) = \bigcup_{x \in M} I_x M(c),$$

we call $IM(c)$ the $c$-indicatrix bundle over $M$. Also, we consider the horizontal Liouville foliation $\mathcal{F}_{L^*}$ determined by the integral curves of $L^*$. According to Theorem 2, $\mathcal{F}_{L^*}$ is a totally geodesic foliation on $(TM^\circ, G)$ if and only if

$$uv + \frac{1}{2} u'u + t(vu' + v'u) + 2t^2 v + 2tu^2 = 0.$$

Therefore we infer that $\mathcal{F}_{L^*}$ is totally geodesic on any $c$-indicatrix bundle $(IM(c), G)$ if and only if

$$uv + \frac{1}{2} u'u + \frac{1}{2} (vu' + v'u) + \frac{1}{2} v'u + v^2 = 0.$$

Here and in the sequel, we denote by the same symbol $G$ the Riemannian metric on $IM(c)$ which is induced by the metric $G$ on $TM^\circ$.

**Theorem 4** Let $(M, F)$ be a Finsler manifold and $IM(c)$ be a $c$-indicatrix over $M$. Then the Riemannian metric $G$ on $IM(c)$ is bundle-like for horizontal Liouville foliation $\mathcal{F}_{L^*}$ on $IM(c)$ if and only if $\Lambda = (1 - \frac{1}{2t})R$ on $IM(c)$.

**Proof** First, we note that all the vector bundles in this proof are considered to be over $IM(c)$. Let $\mathcal{L}'$ be the complementary orthogonal distribution to the horizontal Liouville distribution $\mathcal{L}^*$ in $HTM^\circ$. Then $\mathcal{L}^\perp = \mathcal{L}' \oplus \mathcal{L}'' \oplus \mathcal{L}^*$ is the tangent bundle of $IM(c)$. It is known that the Riemannian metric $G$ is bundle-like for $\mathcal{F}_{L^*}$ on $IM(c)$ if and only if

$$G(\nabla_X Y, L^*) + G(\nabla_Y X, L^*) = 0,$$

(4.38)

where $X, Y \in \Gamma(L' \oplus L'')$ and $\nabla$ is the Levi-Civita connection on $(IM(c), G)$. Since $\nabla$ is parallel with respect to $G$ and $G(X, L^*) = G(Y, L^*) = 0$, then we have $G(\nabla_X Y, L^*) = G(Y, \nabla_X L^*)$ and $G(\nabla_Y X, L^*) = G(X, \nabla_Y L^*)$. Therefore (4.38) is equivalent to

$$G(\nabla_X L^*, Y) + G(\nabla_Y L^*, X) = 0, \ \forall X, Y \in \Gamma(L' \oplus L''),$$

(4.39)

where $\nabla$ is the Levi-Civita connection on $(IM(c), G)$. Since $L$ is the normal bundle to $IM(c)$, then (4.39) is equivalent to

$$G(\nabla_X L^*, Y) + G(\nabla_Y L^*, X) = 0, \ \forall X, Y \in \Gamma(L' \oplus L''),$$

(4.40)

where $\nabla$ is the Levi-Civita connection on $(TM^\circ, G)$.

Now, we consider three cases to analyze (4.40). In the first case, let $X$ and $Y$ belong to $\Gamma(L')$. Then by using (3.20), we conclude that $\nabla_X L^*$ and $\nabla_Y L^*$ belong to $\Gamma(HTM^\circ)$. Thus we have $G(\nabla_X L^*, Y) = G(\nabla_Y L^*, X) = 0$, because $L'$ and $HTM^\circ$ are orthogonal vector bundles with respect to $G$. Consequently, in this case (4.40) is identically satisfied. In the second case, we let $X$ and $Y$ belong to $\Gamma(L'')$. Then by using (3.20), we conclude that $\nabla_X L^*$ and $\nabla_Y L^*$ belong to $\Gamma(VTM^\circ)$. Similar to the previous case, we can deduce
that (4.40) is again identically satisfied. In the third case, we let $X = X^i \partial_i \in \Gamma(\mathcal{L}')$ and $Y = Y^i \delta_i \in \Gamma(\mathcal{L}'')$. Since $\mathcal{L}''$ is the complementary orthogonal distribution to $\mathcal{L}^*$ in $HTM^\circ$, then we have

$$0 = G(Y, L^*) = Y^i y^j G(\delta_i, \delta_j) = Y^i y^j (ug_{ij} + vy_{ij}) = (u + 2tv)Y^i y_i.$$  \hfill (4.41)

Also, (3.31) gives us

$$X^i y_i = 0.$$  \hfill (4.42)

According to (3.20), we get

$$G(\tilde{\nabla}_X L^*, Y) = (u + tv)X^k Y^r g_{kr} - \frac{1}{2u} X^i Y^r R_{ir}.$$  \hfill (4.43)

Similarly, we obtain

$$G(\tilde{\nabla}_Y L^*, X) = -tv X^r Y^i g_{ir} - \frac{1}{2u} Y^i R_{ri} X^r.$$  \hfill (4.44)

Using (4.43) and (4.44), we obtain the following expression of (4.40):

$$\left(ug_{ij} - \frac{1}{u} R_{ij}\right) X^i Y^j = 0.$$  \hfill (4.45)

On the other hand, (4.42) implies

$$h_{ij} X^i Y^j = g_{ij} X^i Y^j.$$  \hfill (4.46)

By using (4.43), (4.45) and (4.46) we obtain

$$\Lambda_{ij} X^i Y^j = R_{ij} X^i Y^j - h_{ij} X^i Y^j = R_{ij} X^i Y^j - g_{ij} X^i Y^j$$

$$= R_{ij} X^i Y^j - \frac{1}{u^2} R_{ij} X^i Y^j = \left(1 - \frac{1}{u^2}\right) R_{ij} X^i Y^j.$$  \hfill (4.47)

Now, we consider the isomorphism of vector bundles $\Phi : \mathcal{L}' \to \mathcal{L}''$ defined by $\Phi(X^i \partial_i) = X^i \delta_i = X^*$. Then (4.47) is equivalent to

$$\Lambda(X^*, Y) = (1 - \frac{1}{u^2}) R(X^*, Y), \quad \forall X^*, Y \in \Gamma(\mathcal{L}'').$$  \hfill (4.48)

Finally, from (4.35) and (4.48) we deduce that (4.40) is equivalent to $\Lambda = (1 - \frac{1}{u^2}) R$ on $IM(c)$. \hfill $\square$

Taking into account that $\mathcal{L}^\perp$ is orthogonal to the vertical Liouville distribution $\mathcal{L}$ we deduce that $L^*$ is a Killing vector field on $IM(c)$ if and only if (see [11])

$$G(\tilde{\nabla}_X L^*, Y) + G(\tilde{\nabla}_Y L^*, X) = 0, \quad \forall X, Y \in \Gamma(\mathcal{L}^\perp).$$  \hfill (4.49)

Now, we can prove the following theorem.

**Theorem 5** Let $(M, F)$ be a Finsler manifold and $IM(c)$ be a $c$-indicatrix bundle over $M$. Then the horizontal Liouville vector field $L^*$ is a Killing vector field on $IM(c)$ if and only if $\Lambda = (1 - \frac{1}{u^2}) R$ on $IM(c)$.

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Proof If \( L^* \) is a Killing vector field on \( IM(c) \), then according to (4.49), the relation (4.40) is held and consequently from Theorem 4 we infer that \( \Lambda = (1 - \frac{1}{u^2})R \) on \( IM(c) \). Conversely let \( \Lambda = (1 - \frac{1}{u^2})R \) on \( IM(c) \). Then (4.40) gives us (4.49), only for any \( X, Y \) belong to \( \Gamma(\mathcal{L}' \oplus \mathcal{L}'') \). Also, if \( X = Y = L^* \) then (3.16) implies (4.49). Hence in order to complete the proof we need to show that (4.49) is held for \( X = L^* \) and \( Y \in \Gamma(\mathcal{L}' \oplus \mathcal{L}'') \). According to (3.16), since \( \tilde{\nabla}_{L^*}L^* = 2t(2\alpha_4 + \alpha_6 + 2t\alpha_5)L \) then we deduce that we should prove that

\[
G(\tilde{\nabla}_Y L^*, L^*) = 0, \quad \forall Y \in \Gamma(\mathcal{L}' \oplus \mathcal{L}'').
\]

We consider two cases to analyze (4.50).

Case 1. \( Y \in \Gamma(\mathcal{L}'') \). Then from (3.20) we infer that \( \tilde{\nabla}_Y L^* \in \Gamma(VTM^s) \), and consequently (4.50) is held in this case.

Case 2. \( Y \in \Gamma(\mathcal{L}') \). In this case we have \( Y = Y^i \partial_i \), where \( Y^i \) satisfy (4.42). Then by using (3.20), we obtain

\[
\tilde{\nabla}_Y L^* = Y^i \left( (1 + 2t\alpha_0)\delta^k_i + \frac{1}{2u^2} y^j R_{i+j+k} \right) \delta_k.
\]

Hence we get

\[
G(\tilde{\nabla}_Y L^*, L^*) = Y^i \left( (1 + 2t\alpha_0)(u + 2tv)y_i + \frac{1}{2u^2} y^j y^r R_{i+j+r} \right).
\]

But by using (ii) of (2.2), (ii) of (2.3) and (4.42) we have \( Y^k y_k = 0 \) and \( R_{i+j+r} y^j = 0 \). Hence \( G(\tilde{\nabla}_Y L^*, L^*) = 0 \), where \( Y \in \Gamma(\mathcal{L}') \).

By using the above cases, we deduce that (4.49) is identically satisfied, and therefore \( L^* \) is a Killing vector field on \( IM(c) \).

Theorem 6 A Finsler manifold \( (M, F) \) is of positive constant curvature \( k \) if and only if \( \Lambda = (1 - \frac{1}{u^2})R \) on the indicatrix bundle \( IM(c) \) where \( c = \frac{1}{\sqrt{k}} \).

Proof Let \( (M, F) \) be a Finsler manifold of constant curvature \( k \). Then by Theorem 1, we have

\[
R_{ij} = kF^2 h_{ij}.
\]

But on \( IM(c) \) we have \( F(x, y) = c = \frac{u}{\sqrt{k}} \). Hence we obtain \( F^2 = \frac{u^2}{k} \) or equivalently

\[
kF^2 = u^2.
\]

Substituting the above equation into (4.52), we obtain

\[
h_{ij} = \frac{1}{u^2} R_{ij}.
\]

Substituting (4.54) into (4.34), we get

\[
\Lambda_{ij} = R_{ij} - \frac{1}{u^2} R_{ij} = (1 - \frac{1}{u^2})R_{ij}.
\]
Conversely, let $\Lambda = (1 - \frac{1}{u^2})R$ on $IM(c)$. Then it follows from (4.53) and (4.34) that

$$R_{ij}(x, y) = u^2 h_{ij}(x, y) = kF^2(x, y) h_{ij}(x, y), \quad \forall(x, y) \in IM(c).$$

(4.56)

Now, we take a point $(x, y) \in TM^c \setminus IM(c)$. Since $TM^c$ admits the fundamental foliation $\mathcal{F}_F$, there exist $c^* > 0$ such that $(x, y) \in IM(c^*)$, that is, $F(x, y) = c^*$. Since $F$ is positively homogeneous of degree one, we have $F(x, \frac{c}{c^*}y) = \frac{c}{c^*}F(x, y) = c$, i.e., $(x, \frac{c}{c^*}y) \in IM(c)$. Hence by (4.56), we obtain

$$R_{ij}(x, \frac{c}{c^*}y) - h_{ij}(x, \frac{c}{c^*}y) = (1 - \frac{1}{u^2})R_{ij}(x, \frac{c}{c^*}y),$$

or equivalently

$$R_{ij}(x, \frac{c}{c^*}y) = u^2 h_{ij}(x, \frac{c}{c^*}y).$$

(4.57)

(4.58)

Since $h_{ij}$ and $R_{ij}$ are positively homogeneous of degree zero and two, respectively, equation (4.58) implies

$$R_{ij}(x, y) = u^2 c^2 h_{ij}(x, y).$$

(4.59)

Since $c = \frac{u}{\sqrt{k}}$ and $F(x, y) = c^*$, it follows from (4.59) that

$$R_{ij}(x, y) = kF^2(x, y) h_{ij}(x, y), \quad \forall(x, y) \in TM^c \setminus IM(c).$$

(4.60)

Thus it follows from (4.56), (4.60) and Theorem 1 that $(M, F)$ is a Finsler manifold of positive constant curvature $k$. 

\[ \square \]

**Theorem 7** Let $(M, F)$ be a Finsler manifold, and $k, c$ two positive numbers such that $c = \frac{u}{\sqrt{k}}$. Then the following assertions are equivalent:

(i) $(M, F)$ is a Finsler manifold of constant curvature $k$.

(ii) The Sasaki-Finsler metric $G$ on the indicatrix bundle $IM(c)$ is bundle-like for the horizontal Liouville foliation $IM(c)$.

(iii) The horizontal Liouville vector field is a Killing vector field on $(IM(c), G)$.

(iv) The curvature-angular form $\Lambda$ of $(M, F)$ satisfy $\Lambda = (1 - \frac{1}{u^2})R$ on $IM(c)$.

References


