Sasakian Finsler manifolds

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Abstract: In this study, almost contact Finsler structures on vector bundle are defined and the condition of normality in terms of the Nijenhuis torsion \(N_\phi\) of almost contact Finsler structure is obtained. It is shown that for a \(K\)-contact structure on Finsler manifold \(\nabla_X\xi = -\frac{1}{2}\phi X\) and the flag curvature for plane sections containing \(\xi\) are equal to \(\frac{1}{4}\). By using the Sasakian Finsler structure, the curvatures of a Finsler connection \(\nabla\) on \(V\) are obtained. We prove that a locally symmetric Finsler manifold with \(K\)-contact Finsler structure has a constant curvature \(\frac{1}{4}\). Also, the Ricci curvature on Finsler manifold with \(K\)-contact Finsler structure is given. As a result, Sasakian structures in Riemann geometry and Finsler condition are generalized.

As a conclusion we can state that Riemannian Sasakian structures are compared to Sasakian Finsler structures and it is proven that they are adaptable.

Key words: Finsler connection, vector bundle, almost contact manifold, Sasakian manifold, nonlinear connection, Ricci tensor

1. Introduction

Let \(V(M) = \{V, \pi, M\}\) be a vector bundle of total space \(V\) with a \((n+m)\)-dimensional \(C^\infty\) manifold and with a base space \(M\) that is an \(n\)-dimensional \(C^\infty\)-manifold. The projection map \(\pi: V \to M, u \in V \mapsto \pi(u) = x \in M\), where \(u = (x, y)\), and \(y \in R^m = \pi^{-1}(x)\) the fibre of \(V(M)\) over \(x\).

A non-linear connection \(N\) on the total space \(V\) of \(V(M)\) is a differentiable distribution \(N: V \to T_u(V), u \in V \mapsto N_u \subset T_u(V)\) such that

\[ T_u(V) = N_u \oplus V_u^v \quad \text{where} \quad V_u^v = \{X \in T_u(V): \pi_\ast (X) = 0\}. \tag{1.1} \]

\(N_u\) the horizontal distribution and \(V_u^v\) is the vertical distribution. Thus for all \(X \in T_u(V)\) can be separated by its components

\[ X = X^H + X^V \quad \text{where} \quad X^H \in N_u, \quad X^V \in V_u^v. \tag{1.2} \]

Let \(x^i, i=1,2,\ldots,n\) and \(y^a, a=1,2,\ldots,m\) be the coordinates of \(x\) and \(y\) such that \((x^i, y^a)\) are the coordinates of \(u \in V\). The local base of \(N_u\) is

\[ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_u^a(x, y) \frac{\partial}{\partial y^a} \tag{1.3} \]

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and that of $V_u^v$ is where $N_i^v(x, y)$ are the coefficients of $N$. Their dual bases are $(dx^i, dy^a)$ where

$$\delta y^a = dy^a + N_i^v(x, y) dx^i. \quad (1.4)$$

Let $X = X^i(x, y) \frac{\partial}{\partial x^i} + \bar{X}^v(x, y) \frac{\partial}{\partial y^a}, \forall X \in T_u(V)$. Then

$$X^H = X^i(x, y) \frac{\delta}{\delta x^i}, X^V = \bar{X}^v(x, y) \frac{\partial}{\partial y^a}, \bar{X}^a = X^a + N_i^v X^i. \quad (1.5)$$

Let $\omega$ be a 1-form $\omega = \tilde{\omega}_i(x, y)dx^i + \omega_a(x, y)dy^a$. Then

$$\omega^H = \tilde{\omega}_i dx^i, \tilde{\omega}_i = \omega_i - N_i^v(x, y) \omega_a; \omega^V = \omega_a \delta y^a \quad (1.6)$$

which gives

$$\omega^H(X^V) = 0, \omega^V(X^H) = 0 \text{ where } \omega = \omega^H + \omega^V. \quad (1.7)$$

The Finsler tensor field of type $\left(\begin{array}{ccc} p \n q \n r \n s \end{array}\right)$ on $V$ has the following local form [4]:

$$T = T^i_{j_1,...,j_p,a_1,...,a_r}(x, y) \frac{\delta}{\delta x^{i_1}} \otimes ... \otimes \frac{\delta}{\delta x^{i_p}} \otimes dx^{a_1} \otimes ... \otimes dx^{a_r} \otimes \frac{\partial}{\partial y^{j_1}} \otimes ... \otimes \frac{\partial}{\partial y^{j_p}} \otimes dy^{b_1} \otimes ... \otimes dy^{b_r}. \quad (1.8)$$

**Definition 1.1** A Finsler connection on $V$ is a linear connection $\nabla = \Gamma$ on $V$ with the property that the horizontal linear space $N_u$, $u \in V$ of the distribution $N$ is parallel with respect to $\nabla$ and the vertical spaces $V_u^v$, $u \in V$ are also parallel relative to $\nabla$ [3].

A linear connection $\nabla$ on $V$ is a Finsler connection on $V$ if and only if

$$(\nabla_X Y^H)^V = 0, (\nabla_X Y^V)^H = 0, \forall X, Y \in T_u(V). \quad (1.9)$$

A linear connection $\nabla$ on $V$ is a Finsler connection on $V$ if and only if [4]

$$\nabla_X Y = (\nabla_X Y^H)^V + (\nabla_X Y^V)^H, \forall X, Y \in T_u(V), \quad (1.10a)$$

$$\nabla_X \omega = (\nabla_X \omega^H)^H + (\nabla_X \omega^V)^V, \forall \omega \in T^*_u(V) \text{ and } X \in T_u(V). \quad (1.10b)$$

**Remark 1.1** Let $\nabla$ on $V$ is a Finsler connection on $V$. We get immediately that [6]

$$Y \in V_u^v \Rightarrow \forall X \in T_u(V); \nabla_X Y \in V_u^v, Y \in N_u \Rightarrow \forall X \in T_u(V); \nabla_X Y \in N_u. \quad (1.11)$$

For a Finsler connection $\nabla$ on $V$, there is an associated pair of operators; $h$- and $v$-covariant derivation in the algebra of Finsler tensor fields. For each $X \in T_u(V)$, set

$$\nabla^H_X Y = \nabla_X u Y, \nabla^H_X f = X^H (f), \forall Y \in T_u(V), \forall f \in F(V). \quad (1.12)$$

If $\omega \in T^*_u(V)$, we define

$$(\nabla^H_X \omega)(Y) = X^H (\omega(Y)) - \omega \left( \nabla^H_X Y \right), \forall Y \in T_u(V). \quad (1.13)$$
So, we may extend the action of the operator $\nabla^H_X$ to any Finsler tensor field by asking these questions: does $\nabla^H_X$ preserve the type of Finsler tensor fields, is it $\mathbb{R}$-linear, does it satisfy the Leibniz rule with respect to tensor product and does it commute with all contractions? We keep the notation $\nabla^H_X$ for this operator on the algebra of Finsler tensor fields. We call it the \textit{operator of $h$-covariant derivation}.

In a similar way, for every vector field $X \in T_u(V)$ set
\begin{equation}
\nabla_X^V Y = \nabla_{X^V} Y, \nabla_X^V f = X^V (f), \forall Y \in T_u(V), \forall f \in F(V).
\end{equation}

If $\omega \in T_u^q(V)$, we define
\begin{equation}
(\nabla^V_X \omega)(Y) = X^V (\omega (Y)) - \omega (\nabla_X^V Y), \forall Y \in T_u(V).
\end{equation}

We extend the action of $\nabla^V_X$ to any Finsler tensor field in a similar way, as for $\nabla^H_X$. We obtain an operator on the algebra of Finsler tensor fields on $V$; this will be denoted also by $\nabla^V_X$ and will be called the \textit{operator of $v$-covariant derivation} [1].

**Definition 1.2** Let $\omega \in T_u^q(V)$ be a differential $q$-form on $V$, $\nabla$ is a linear connection on $V$ and $T$ is the torsion tensor of $\nabla$. Then its exterior differential $d\omega$ is also defined as [4]:
\begin{equation}
d\omega (X_1, ..., X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} (\nabla_X^\omega)(X_1, ..., \hat{X}_i, ..., X_{q+1}), \forall X_i \in T_u(V)
- \sum_{1 \leq i \leq j \leq q+1} (-1)^{i+j} \omega \left( T(X_i, X_j), X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{q+1} \right).
\end{equation}

**Proposition 1.1** If $\omega \in T_u^1(V)$ is a 1-form and $\nabla$ is a Finsler connection on $V$, then its exterior differential is given by [3]
\begin{equation}
\begin{align*}
(d\omega)(X^H, Y^H) &= (\nabla^H_X \omega)(Y^H) - (\nabla^H_Y \omega)(X^H) + \omega \left( T(X^H, Y^H) \right), \\
(d\omega)(X^V, Y^H) &= (\nabla^V_X \omega)(Y^H) - (\nabla^V_Y \omega)(X^V) + \omega \left( T(X^V, Y^H) \right), \\
(d\omega)(X^V, Y^V) &= (\nabla^V_X \omega)(Y^V) - (\nabla^V_Y \omega)(X^V) + \omega \left( T(X^V, Y^V) \right), \forall X, Y \in T_u(V).
\end{align*}
\end{equation}

In the canonical coordinates $(x^i, y^a)$, there exists a well determined set of differentiable functions on $V$. $F^i_{jk}(x, y), F^a_{bb}(x, y); C^i_{ja}(x, y); C^a_{bc}(x, y)$ such that
\begin{align*}
\nabla^H_{\frac{\partial}{\partial x^i}} &= F^i_{jk}(x, y) \frac{\partial}{\partial y^j}, \nabla^H_{\frac{\partial}{\partial y^a}} = F^a_{bb}(x, y) \frac{\partial}{\partial y^b}, \\
\nabla^V_{\frac{\partial}{\partial x^i}} &= C^i_{ja}(x, y) \frac{\partial}{\partial x^j}, \nabla^V_{\frac{\partial}{\partial y^a}} = C^a_{bc}(x, y) \frac{\partial}{\partial y^b},
\end{align*}
where $F^i_{jk}(x, y), F^a_{bb}(x, y)$ are called coefficients of $h$-connections $\nabla^H$ and $C^i_{ja}(x, y), C^a_{bc}(x, y)$ are called coefficients of $v$-connections $\nabla^V$.

The torsion tensor field $T$ of a Finsler-connection is characterised by five Finsler tensor fields:
\begin{equation}
[T(X^H, Y^H)]^H, [T(X^H, Y^H)]^V, [T(X^H, Y^V)]^H, [T(X^H, Y^V)]^V, [T(X^V, Y^V)]^V.
\end{equation}

**Proposition 1.2** If the Finsler connection on $V$ is without torsion then we have [3]
\begin{equation}
T(X^H, Y^H) = 0, T(X^H, Y^V) = 0, T(X^V, Y^V) = 0, \forall X, Y \in T_u(V).
\end{equation}
2. Almost contact Finsler structure on vector bundle

Let $\phi$ be an almost contact structure on $V$ given by the tensor field of type $\begin{pmatrix} 1 & 1 & 1 \\ \end{pmatrix}$ with the properties

1. $\phi \cdot \phi = -I_n + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V$
2. $\phi \xi^H = 0, \phi \xi^V = 0$
3. $\eta^H (\xi^H) + \eta^V (\xi^V) = 1$
4. $\eta^H (\phi X^H) = 0, \eta^V (\phi X^V) = 0, \eta^H (\phi X^V) = 0, \eta^V (\phi X^H) = 0$,

where $\eta$ is 1-form and $\xi$ is vector field [2].

Proposition 2.1 If $\phi$ is an almost contact Finsler structure on $V$, there exists a unique decomposition of $\phi$ in the Finsler tensor fields,

$$\phi = \phi^1 + \phi^2 + \phi^3 + \phi^4 = \begin{pmatrix} \phi^1 & \phi^2 \\ \phi^3 & \phi^4 \\ \end{pmatrix}$$

where

$$\phi^1 (\omega, X) = \phi (\omega^H, X^H), \phi^2 (\omega, X) = \phi (\omega^V, X^V),$$
$$\phi^3 (\omega, X) = \phi (\omega^H, X^H), \phi^4 (\omega, X) = \phi (\omega^V, X^V) \forall X \in T_u (V), \forall \omega \in T^*_u (V).$$

We can write

$$\phi (X^H) = \phi^1 (X^H) = \phi^3 (X^H), \phi (X^V) = \phi^2 (X^V) = \phi^4 (X^V).$$

Let $G$ be the Finsler metric structure on $V$ which is symmetric, positive definite and non-degenerate on $V$. The metric-structure $G$ on $V$ is decomposed as:

$$G = G^H + G^V$$

where $G^H$ is of type $\begin{pmatrix} 0 & 0 \\ 2 & 0 \\ \end{pmatrix}$, symmetric, positive definite and non-degenerate on $N_u$ and $G^V$ is of type

$$\begin{pmatrix} 0 & 0 \\ 0 & 2 \\ \end{pmatrix},$$

symmetric, positive definite and non-degenerate on $V^*_u i.e. for X, Y \in T_u (V)$

$$G (X, Y) = G^H (X, Y) + G^V (X, Y)$$

where $G^H (X, Y) = G (X^H, Y^H), G^V (X, Y) = G (X^V, Y^V).$

Now, if the Finsler metric structure $G$ on $V$ satisfies

$$G (\phi X, \phi Y) = G (X, Y) - \eta (X) \eta (Y),$$
$$G^H (\phi X, \phi Y) = G^H (X, Y) - \eta^H (X^H) \eta^H (Y^H),$$
$$G^V (\phi X, \phi Y) = G^V (X, Y) - \eta^V (X^V) \eta^V (Y^V),$$

which is equivalent to

$$G^H (X, \xi) = \eta^H (X), G^V (X, \xi) = \eta^V (X),$$
$$G^H (\phi X, \phi Y) = -G^H (\phi^2 X, Y), G^V (\phi X, \phi Y) = -G^V (\phi^2 X, Y),$$

322
then \((\phi, \eta, \xi, G)\) is called almost contact metrical Finsler structure on \(V\) [5]. Now, we define

\[
\Omega(X, Y) = G(X, \phi Y), \quad \Omega(X^H, Y^H) = G^H(X, \phi Y), \quad \Omega(X^V, Y^V) = G^V(X, \phi Y)
\]  

(2.9)

and call it the fundamental 2-form.

**Proposition 2.2** The fundamental 2-form, defined above, satisfies [5]

\[
\begin{align*}
\Omega(\phi X^H, \phi Y^H) &= \Omega(X^H, Y^H), \quad \Omega(\phi X^V, \phi Y^V) = \Omega(X^V, Y^V), \\
\Omega(X^H, Y^H) &= -\Omega(Y^H, X^H), \quad \Omega(X^V, Y^V) = -\Omega(Y^V, X^V) \quad \forall X, Y \in T_u(V). 
\end{align*}
\]

(2.10)

**Proposition 2.3** Let \(\nabla\) be a Finsler connection on \(V\) and \(\Omega\) be the fundamental 2-form which satisfies \(\Omega(X, Y) = d\eta(X, Y)\) i.e.

\[
\begin{align*}
\Omega(X^H, Y^H) &= (\nabla^H_X\eta)(Y^H) - (\nabla^H_Y\eta)(X^H) + \eta(T(X^H, Y^H)), \\
\Omega(X^V, Y^H) &= (\nabla^V_X\eta)(Y^H) - (\nabla^H_Y\eta)(X^V) + \eta(T(X^V, Y^H)), \\
\Omega(X^V, Y^V) &= (\nabla^V_X\eta)(Y^V) - (\nabla^V_Y\eta)(X^V) + \eta(T(X^V, Y^V)).
\end{align*}
\]

(2.11)

Then, the almost contact metrical Finsler structure is called almost Sasakian Finsler structure and the Finsler connection \(\nabla\) satisfying (2.11) is called almost Sasakian Finsler connection on \(V\) [5].

**Theorem 2.1** Let \(\Omega\) be the fundamental 2-form and almost Sasakian Finsler connection \(\nabla\) on \(V\) is torsion free. Then [5]

\[
\begin{align*}
\Omega(X^H, Y^H) &= (\nabla^H_X\eta)(Y^H) - (\nabla^H_Y\eta)(X^H), \\
\Omega(X^V, Y^H) &= (\nabla^V_X\eta)(Y^H) - (\nabla^H_Y\eta)(X^V), \\
\Omega(X^V, Y^V) &= (\nabla^V_X\eta)(Y^V) - (\nabla^V_Y\eta)(X^V), \quad \forall X, Y \in T_u(V).
\end{align*}
\]

(2.12)

**Proof** From Proposition 1.2 and equations in (2.11), we have (2.12). \(\square\)

**Definition 2.1** An almost Sasakian Finsler structure on \(V\) is said to be a Sasakian Finsler structure if the 1-form \(\eta\) is a killing vector field, i.e.

\[
\begin{align*}
(\nabla^H_X\eta)(Y^H) + (\nabla^H_Y\eta)(X^H) &= 0, \\
(\nabla^V_X\eta)(Y^V) + (\nabla^H_Y\eta)(X^V) &= 0, \quad (\nabla^V_Y\eta)(X^V) = 0 \forall X, Y \in T_u(V).
\end{align*}
\]

(2.13)

The Finsler connection \(\nabla\) on \(V\) is torsion free, which is called Sasakian Finsler connection [5].

**Theorem 2.2** Let \(\nabla\) be the torsion free Finsler connection together with a Sasakian Finsler structure on \(V\) and \(\Omega\) is to be the fundamental 2-form; then

\[
\begin{align*}
\Omega(X^H, Y^H) &= 2(\nabla^H_X\eta)(Y^H) - 2(\nabla^H_Y\eta)(X^H), \\
\Omega(X^H, Y^V) &= 2(\nabla^H_X\eta)(Y^V) - 2(\nabla^H_Y\eta)(X^H), \\
\Omega(X^V, Y^V) &= 2(\nabla^V_X\eta)(Y^V) - 2(\nabla^V_Y\eta)(X^V), \quad \forall X, Y \in T_u(V).
\end{align*}
\]

(2.14)
Example 2.1 Let $V(M) = \{V, \pi, M\}$ be a vector bundle with the total space $V = R^{10}$ is a 10-dimensional $C^\infty$-manifolds and the base space $M = R^5$ is a 5-dimensional $C^\infty$-manifold. Let $x^i, 1 \leq i \leq 5$ and $y^a, 1 \leq a \leq 5$ be the coordinates of $u = (x, y) \in V$, that is, $u = (x^1, x^2, x^3, x^4, x^5, y^1, y^2, y^3, y^4, y^5) \in V$.

The local base of $N_u$ is $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}, \frac{\partial}{\partial x^5})$ and the local base of $V^v_u$ is $(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \frac{\partial}{\partial y^3}, \frac{\partial}{\partial y^4}, \frac{\partial}{\partial y^5})$ such that $T_u(V) = N_u \oplus V^v_u$. Then

$$X^H = X^1 \frac{\partial}{\partial x^1} + X^2 \frac{\partial}{\partial x^2} + X^3 \frac{\partial}{\partial x^3} + X^4 \frac{\partial}{\partial x^4} + X^5 \frac{\partial}{\partial x^5},$$

$$X^V = \tilde{X}^1 \frac{\partial}{\partial y^1} + \tilde{X}^2 \frac{\partial}{\partial y^2} + \tilde{X}^3 \frac{\partial}{\partial y^3} + \tilde{X}^4 \frac{\partial}{\partial y^4} + \tilde{X}^5 \frac{\partial}{\partial y^5},$$

for $X^H \in N_u, X^V \in V^v_u$.

Let $\eta$ be a 1-form, $\eta = \eta_1 dx^1 + \eta_2 dx^2 + \eta_3 dx^3 + \eta_4 dx^4 + \eta_5 dx^5$ and $\eta^V = \tilde{\eta}_1 dy^1 + \tilde{\eta}_2 dy^2 + \tilde{\eta}_3 dy^3 + \tilde{\eta}_4 dy^4 + \tilde{\eta}_5 dy^5$ where $\eta = \eta^H + \eta^V$ and $\eta^H(X^H) = 0, \eta^V(X^H) = 0$.

We put $\eta^H = \frac{1}{3} (dx^5 - x^5 dx^4 - x^4 dx^3)$ and $\eta^V = \frac{1}{3} (\delta y^5 - y^5 \delta y^4 - y^4 \delta y^3)$.

The structure vector field $\xi$ is given by $\xi = 3 \left( \frac{\delta}{\delta x^5} + \frac{\partial}{\partial y^5} \right)$ and $\xi$ is decomposed as $\xi^H = 3 \frac{\delta}{\delta x^5}$ and $\xi^V = 3 \frac{\partial}{\partial y^5}$.

The tensor field $\phi^H$ of type $(1,1)$ and $\phi^V$ of type $(1,1)$ by a matrix form is given by

$$\phi^H = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 \\
-x^4 & x^3 & 0 & 0 & 0
\end{bmatrix}, \quad \phi^V = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 \\
y^4 & y^3 & 0 & 0 & 0
\end{bmatrix}.$$

We can see that $\eta^H(\xi^H) = 1, \phi^H(\xi^H) = 0$, $\eta^V(\xi^V) = 1, \phi^V(\xi^V) = 0$, $\eta^H(\xi^V) = 0, \eta^V(\xi^H) = 0$, $(\phi^H)^2 X^H = -X^H + \eta^H(X^H) \xi^H, (\phi^V)^2 X^V = -X^V + \eta^V(X^V) \xi^V$ and hence $(\phi, \xi, \eta)$ is almost contact Finsler structure on $R^{10}$.

3. Integrability tensor field of the almost contact Finsler structure

The integrability tensor field of the almost contact Finsler structure on $V$ is given by

$$\tilde{N}(X,Y) = [\phi X, \phi Y] - \phi [\phi X, Y] - \phi [X, \phi Y] + d\eta^H(X,Y) \xi^H + d\eta^V(X,Y) \xi^V, \forall X,Y \in T_u(V).$$

We define four tensors $N^{(1)}, N^{(2)}, N^{(3)}$ and $N^{(4)}$, respectively by

$$N^{(1)}(X^H,Y^H) = N_\phi(X^H,Y^H) + d\eta^H(X^H,Y^H) \xi^H,$$

$$N^{(2)}(X^H,Y^H) = (L^H_{\phi X} \eta^H)(Y^H) - (L^H_{\phi Y} \eta^H)(X^H),$$

$$N^{(3)}(X^H) = (L^H_{\xi \phi} \phi)(X^H), \quad N^{(4)}(X^H) = (L^H_{\xi \eta^H} \eta^H)(X^H),$$

324
If Lemma 3.1

It is clear that the almost contact Finsler structure \((\phi, \xi, \eta)\) is normal if and only if these four tensors vanish.

\[
N^{(1)}(X^V, Y^V) = N_\phi (X^V, Y^V) + d\eta^V (X^V, Y^V) \xi^V, \\
N^{(2)}(X^V, Y^V) = (L^V_{\phi X} \eta^V)(Y^V) - (L^V_{\phi Y} \eta^V)(X^V), \\
N^{(3)}(X^V) = (L^V_{\xi} \phi)(X^V), \quad N^{(4)}(X^V) = (L^V_{\xi} \eta^V)(X^V),
\]

\[
N^{(1)}(X^V, Y^H) = N_\phi (X^V, Y^H) + d\eta^V (X^V, Y^H) \xi^H + d\eta^H (X^V, Y^H) \xi^H,
\]

\[
N^{(2)}(X^V, Y^H) = (L^V_{\phi X} \eta^H)(Y^H) + (L^V_{\phi Y} \eta^H)(Y^H) - (L^H_{\phi Y} \eta^H)(X^V) - (L^H_{\phi Y} \eta^H)(Y^V),
\]

\[
N^{(3)}(X^V) = (L^V_{\xi} \phi)(X^V), \quad N^{(4)}(X^V) = (L^H_{\xi} \eta^V)(X^V),
\]

\[
N^{(3)}(Y^H) = (L^V_{\xi} \phi)(Y^H), \quad N^{(4)}(Y^H) = (L^H_{\xi} \eta^V)(Y^H).
\]

Lemma 3.1 If \(N^{(1)} = 0\), then \(N^{(2)} = N^{(3)} = N^{(4)} = 0\).

Proof If \(N^{(1)} = 0\), then for \(X^H, Y^H, \xi^H \in N_u\), from (3.1.a) we have

\[
\left[\xi^H, X^H\right] + \phi \left[\xi^H, \phi X^H\right] - \xi^H \left(\eta^H (X^H)\right) \xi^H = 0.
\]

(3.4)

Applying \(\eta^H\) to (3.4), we see that

\[
N^{(4)} (X^H) = (L^H_{\xi} \eta^H) (X^H) = \xi^H \left(\eta^H (X^H)\right) - \eta^H \left[\xi^H, X^H\right] = 0.
\]

From this equation, we also have

\[
\eta^H \left[\xi^H, \phi X^H\right] = 0.
\]

(3.5)

On the other hand, applying \(\phi\) to (3.4), we get

\[
N^{(3)} (X^H) = (L^V_{\xi} \phi) X^H = \phi \left[\xi^H, \phi X^H\right] - \left[\phi X^H, \xi^H\right] = 0.
\]

(3.6)

Finally, from \(N^{(1)} = 0\), by using (3.6), we derive

\[
0 = - \left[\phi X^H, Y^H\right] - \left[X^H, \phi Y^H\right] + \phi \left[X^H, Y^H\right] - \phi \left[\phi X^H, \phi Y^H\right] - \phi Y^H \left(\eta^H (X^H)\right) \xi^H + \phi \left(Y^H \left(\eta^H (X^H)\right)\right) \xi^H.
\]

(3.7)

Applying \(\eta^H\) to (3.7), we get \(N^{(2)}(X^H, Y^H) = 0\). Similarly, \(\forall X^V, Y^V, \xi^V \in V_u\), if \(N^{(1)} (X^V, Y^V) = 0\), then \(N^{(2)}(X^V, Y^V) = 0\), \(N^{(3)}(X^V) = 0\), \(N^{(4)}(X^V) = 0\).

If \(N^{(1)} (X^V, Y^H) = 0\), from (3.3.a) we obtain

\[
N^{(1)}(X^V, \xi^H) = \left[\xi^H, X^V\right] - \phi \left[\phi X^V, \xi^H\right] - \xi^H \left(\eta^V (X^V)\right) \xi^V = 0.
\]

(3.8)

Applying \(\eta^V\) and \(\eta^H\) to (3.8), we get (3.9):

\[
\eta^V \left[\xi^H, X^V\right] = \xi^H \left(\eta^V (X^V)\right), \quad \eta^H \left[\xi^H, X^V\right] = 0.
\]

(3.9)
Using (3.9) in (3.3.c), we obtain

\[ N^{(4)} (X^V) = (L^H_\xi \eta^V) (X^V) = \xi^H (\eta^V (X^V)) - \eta^V [\xi^H, X^V] = 0. \]

Applying \( \phi \) to (3.8), we get

\[ N^{(3)} (X^V) = (L^H_\phi \phi) X^V = [\xi^H, \phi X^V] + \phi [X^V, \xi^H] = 0. \]

On the other hand, replacing \( X \) by \( \xi \) in (3.3.a), we obtain

\[ [Y^H, \xi^V] - \phi [\xi^V, \phi Y^H] + \xi^V (\eta^H (Y^H)) \xi^H = 0. \] (3.10)

Applying \( \eta^H \) and \( \eta^V \) to (3.10), we get

\[ \eta^H [\xi^V, Y^H] = \xi^V (\eta^H (Y^H)), \eta^V [\xi^V, Y^H] = 0. \] (3.11)

Using (3.11) in (3.3.d), we obtain

\[ N^{(4)} (Y^H) = (L^V_\eta^H) (Y^H) = \xi^V (\eta^H (Y^H)) - \eta^H [\xi^V, Y^H] = 0. \]

Applying \( \phi \) to (3.10) and by using (3.11), we obtain

\[ N^{(3)} (Y^H) = (L^V_\phi) (Y^H) = [\xi^V, \phi Y^H] + \phi [Y^H, \xi^V] = 0. \]

By using (3.11), from (3.3.a), we calculate

\[
0 = N_\phi \left( \phi X^V, Y^H \right) + d\eta^V \left( \phi X^V, Y^H \right) \xi^V + d\eta^H \left( \phi X^V, Y^H \right) \xi^H \\
= [Y^H, \phi X^V] + [\phi Y^H, X^V] + \phi [X^V, Y^H] - \phi [\phi X^V, \phi Y^H] - \phi Y^H (\eta^V (X^V)) \xi^V + \phi X^V (\eta^H (Y^H)) \xi^H.
\] (3.12)

Applying \( \eta^V \) to (3.12), from (3.3.b), we obtain

\[
0 = N^{(2)} (X^V, Y^H) = \phi X^V \left( \eta^H (Y^H) \right) - \phi Y^H \left( \eta^V (X^V) \right) - \eta^V [\phi X^V, Y^H] + \eta^H [\phi Y^H, X^V] \\
+ \eta^V \left[ \phi Y^H, X^V \right] - \eta^H \left[ \phi X^V, Y^H \right].
\]

\[ \Box \]

**Proposition 3.1** The almost contact Finsler structure on \( V \) is normal if and only if

\[ N_\phi + d\eta^H \otimes \xi^H + d\eta^V \otimes \xi^V = 0. \] (3.13)

Let \( (\phi, \eta, \xi, G) \) be an almost metrical Finsler structure on \( V \) with contact metric. If the structure vector field \( \xi \) is a Killing vector field with respect to \( G \), the contact structure on \( V \) is called a K-contact Finsler structure and \( V \) is called a K-contact Finsler manifold.

**Lemma 3.2** Let \( (\phi, \eta, \xi, G) \) be a contact metrical Finsler structure on \( V \). Then \( N^{(2)} \) and \( N^{(4)} \) vanish. Moreover, \( N^{(3)} \) vanishes if and only if \( \xi \) is a Killing vector field with respect to \( G \).
Proof We have
\[ d\eta^H (\phi X^H, \phi Y^H) = \Omega \left( \phi X^H, \phi Y^H \right) = G \left( \phi X^H, \phi^2 Y^H \right) = -G \left( X^H, \phi Y^H \right) = G \left( X^H, \phi Y^H \right) = d\eta^H \left( X^H, Y^H \right) \]
from which \( d\eta^H (\phi X^H, Y^H) + d\eta^H (X^H, \phi Y^H) = 0 \).
This is equivalent to \( N^{(2)} \left( X^H, Y^H \right) = 0 \).

On the other hand, we have \( G \left( X^H, \phi Y^H \right) = d\eta^H (X^H, \xi^H) = X^H \eta^H (\xi^H) - \xi^H \eta^H (X^H) - \eta^H \left[ X^H, \xi^H \right] \).
Thus we obtain \( H^H \eta^H (X^H) = 0 \) and consequently, \( H^H \eta^H = 0 \) hence \( N^{(4)} (X^H) = 0 \).

We mention that \( H^H \) is a Killing vector field if and only if \( N^{(3)} (Y^H) = 0 \).
Similarly, we consider that \( N^{(2)} (X^V, Y^V) = 0 \) and \( N^{(4)} (X^V) = 0 \).
Moreover, \( N^{(3)} (X^V) = 0 \) if and only if \( \xi^V \) is a Killing vector field with respect to \( G^V \).

Lemma 3.3 For an almost contact metric Finsler structure \((\phi, \eta, \xi, G)\) on \( V \), we have
\[
2G \left( \left( \nabla_X \phi \right) Y, Z \right) = d\Omega \left( X, \phi Y, \phi Z \right) - d\Omega \left( X, Y, Z \right) + G \left( N^{(1)} (Y, Z), \phi X \right) + N^{(2)} (Y, Z) \eta (X) + d\eta \left( \phi Y, X \right) \eta (Z) - d\eta \left( \phi Z, X \right) \eta (Y) . \tag{3.14}
\]

Proof The Finsler connection \( \nabla \) with respect to \( G \) is given by
\[
2G^H \left( \nabla_X^H Y^H, Z^H \right) = G^H \left( X^H, Z^H \right) + Y^H G^H \left( X^H, Z^H \right) - Z^H G \left( X^H, Y^H \right) + G^H \left( \left[ X^H, Y^H \right], Z^H \right) + G^H \left( Z^H, X^H \right) - Y^H G^H \left( X^H, X^H \right) \tag{3.15}
\]
\[
2G^V \left( \nabla_X Y^V, Z^V \right) = X^V G^V \left( Y^V, Z^V \right) + Y^V G^V \left( X^V, Z^V \right) - Z^V G \left( X^V, Y^V \right) + G^V \left( \left[ X^V, Y^V \right], Z^V \right) + G^V \left( Z^V, X^V \right) - Y^V G^V \left( X^V, X^V \right) \tag{3.16}
\]
\[
2G^H \left( \nabla_X Y^H, Z^H \right) = X^V G^H \left( Y^H, Z^H \right) + G^H \left( \left[ X^V, Y^V \right], Z^V \right) + G^H \left( Z^H, X^V \right) \tag{3.17}
\]
\[
2G^V \left( \nabla_X Y^V, Z^V \right) = X^V G^V \left( Y^V, Z^V \right) + G^V \left( \left[ X^V, Y^V \right], Z^V \right) + G^V \left( Z^V, X^V \right) \tag{3.18}
\]

Furthermore, we have
\[
d\Omega \left( X^H, Y^H, Z^H \right) = X^H \Omega \left( Y^H, Z^H \right) + Y^H \Omega \left( Z^H, X^H \right) + Z^H \Omega \left( X^H, Y^H \right) - \Omega \left( \left[ X^H, Y^H \right], Z^H \right) - \Omega \left( \left[ Z^H, X^H \right], Y^H \right) - \Omega \left( \left[ Y^H, Z^H \right], X^H \right) \tag{3.19}
\]
\[
d\Omega \left( X^V, Y^V, Z^V \right) = X^V \Omega \left( Y^V, Z^V \right) + Y^V \Omega \left( Z^V, X^V \right) + Z^V \Omega \left( X^V, Y^V \right) - \Omega \left( \left[ X^V, Y^V \right], Z^V \right) - \Omega \left( \left[ Z^V, X^V \right], Y^V \right) - \Omega \left( \left[ Y^V, Z^V \right], X^V \right) \tag{3.20}
\]
From (3.1.b), we have
\[ d\Omega (X^V, Y^H, Z^H) = X^V \Omega (Y^H, Z^H) - \Omega \left( [X^V, Y^H]^H, Z^H \right) - \Omega \left( [Z^H, X^V]^H, Y^H \right), \] (3.21)
\[ d\Omega (X^V, Y^V, Z^H) = Z^H \Omega (X^V, Y^V) - \Omega \left( [Z^H, X^V]^V, Y^V \right) - \Omega \left( [Y^V, Z^H]^V, X^V \right), \] (3.22)
\[ d\Omega (X^H, Y^V, Z^H) = Y^V \Omega (Z^H, X^H) - \Omega \left( [X^H, Y^V]^H, Z^H \right) - \Omega \left( [Y^V, Z^H]^H, X^H \right), \] (3.23)
\[ d\Omega (X^H, Y^V, Z^V) = X^H \Omega (Y^V, Z^V) - \Omega \left( [X^H, Y^V]^V, Z^V \right) - \Omega \left( [Z^V, X^H]^V, Y^V \right), \] (3.24)
\[ d\Omega (X^H, Y^H, Z^V) = Y^H \Omega (Z^V, X^H) - \Omega \left( [X^H, Y^H]^V, Z^V \right) - \Omega \left( [Y^H, Z^V]^V, X^V \right), \] (3.25)
\[ d\Omega (X^H, Y^H, Z^V) = Z^V \Omega (X^H, Y^H) - \Omega \left( [Z^V, X^H]^H, Y^H \right) - \Omega \left( [Y^H, Z^V]^H, X^H \right). \] (3.26)

By using (2.9), from (3.15) we get
\[ 2G^H ((\nabla^H_x \phi) Y^H, Z^H) = \phi Y^H G (X^H, Z^H) - Z^H \Omega (X^H, Y^H) + G^H ([X^H, \phi Y^H], Z^H) \]
\[ + \Omega ([Z^H, X^H], Y^H) - G^H ([\phi Y^H, Z^H], X^H) + Y^H \Omega (X^H, Z^H) - \phi Z^H G (X^H, Y^H) \]
\[ + \Omega ([X^H, Y^H], Z^H) G ([\phi Z^H, X^H], Y^H) - G^H ([Y^H, \phi Z^H], X^H). \]

Also from (3.19), we calculate
\[ d\Omega (X^H, \phi Y^H, \phi Z^H) = X^H \Omega (Y^H, Z^H) + \phi Y^H G (Z^H, X^H) - \phi Y^H (\eta^H (Z^H) \eta^H (X^H)) \]
\[ - \phi Z^H G (X^H, Y^H) + \phi Z^H (\eta^H (X^H) \eta^H (Y^H)) + G ([X^H, \phi Y^H], Z^H) \]
\[ - \eta^H [X^H, \phi Y^H] \eta^H (Z^H) + G ([\phi Z^H, X^H], Y^H) - \eta^H [\phi Z^H, X^H] \eta^H (Y^H) \]
\[ - \Omega ([\phi Y^H, \phi Z^H], X^H). \] (3.27)

Also from (3.1.a) by using (2.9), we obtain
\[ G (N^{(1)} (Y^H, Z^H), \phi X^H) = - \Omega ([Y^H, Z^H], X^H) + \Omega ([\phi Y^H, \phi Z^H], X^H) - G ([\phi Y^H, Z^H], X^H) \]
\[ + \eta^H [\phi Y^H, Z^H] \eta^H (X^H) - G ([Y^H, \phi Z^H], X^H) + \eta^H [Y^H, \phi Z^H] \eta^H (X^H). \] (3.28)

From (3.1.b), we have
\[ N^{(2)} (Y^H, Z^H) \eta^H (X^H) = \phi Y^H (\eta^H (Y^H)) \eta^H (X^H) - \phi Z^H (\eta^H (Y^H)) \eta^H (X^H) \]
\[ - \eta^H [\phi Y^H, Z^H] \eta^H (X^H) + \eta^H [Y^H, \phi Z^H] \eta^H (X^H). \] (3.29)

By using (3.27), (3.28) and (3.29), we have the equation.

Similarly by using (3.2.a), (3.2.b), (2.9), (3.16) and (3.20), we get
\[ 2G ((\nabla^V_x \phi) Y^V, Z^V) = d\Omega (X^V, \phi Y^V, \phi Z^V) - d\Omega (X^V, Y^V, Z^V) + G (N^{(1)} (Y^V, Z^V), \phi X^V) \]
\[ + N^{(2)} (Y^V, Z^V) \eta^V (X^V) + d\eta^V (\phi Y^V, X^V) \eta^V (Z^V) - d\eta^V (\phi Z^V, X^V) \eta^V (Y^V). \]

By using (2.9), (3.1.a), (3.1.b), (3.17) and (3.21), we calculate
\[ d\Omega (X^V, \phi Y^H, \phi Z^H) - \Omega (X^V, Y^H, Z^H) + d\eta^H (\phi Y^H, X^V) \eta^H (Z^H) - d\eta^H (\phi Z^H, X^V) \eta^H (Y^H) \]
\[ = 2G ((\nabla^V_x \phi) Y^H, Z^H). \]
By using (2.9) and (3.18), (3.24), (3.2.a) and (3.2.b), we obtain

\[ d\Omega(X^H,\phi Y^V,\phi Z^V) - d\Omega(X^H,Y^V,Z^V) + G^V(N^{(1)}(Y^V,Z^V),\phi X^H) \]

\[ + N^{(2)}(Y^V,Z^V)\eta^V(X^H) + d\eta^V(\phi Y^V,X^H)\eta^V(Z^V) + d\eta^V(\phi Z^V,X^H)\eta^V(Y^V) \]

\[ + d\eta^V(\phi Y^V,X^H)\eta^V(Z^V) - d\eta^V(\phi Z^V,X^H)\eta^V(Y^V) \]

\[ = G([X^H,\phi Y^V]^V,Z^V) - \eta^V(Z^V)\eta^V([X^H,\phi Y^V]^V) + G([\phi Z^V,X^H]^V,Y^V) \]

\[ - \eta^V(Y^V)\eta^V([\phi Z^V,X^H]^V) + \Omega([[X^H,Y^V]^V],Z^V) + \Omega([[Z^V,X^H]^V],Y^V) \]

\[ = G(\phi Y^V,X^H)^V\eta^V(Z^V) + \eta^V([\phi Z^V,X^H]^V)\eta^V(Y^V) \]

\[ = 2G^V((\nabla^H_X\phi)Y^V,Z^V). \]

**Lemma 3.4** For a contact metric Finsler structure \((\phi, \eta, \xi, G)\) of \(V\) with \(\Omega = d\eta\) and \(N^{(2)} = 0\), we get

\[ 2G((\nabla_X\phi)Y,Z) = G(N^{(1)}(Y,Z),\phi X) + d\eta(\phi Y,X)\eta(Z) - d\eta(\phi Z,X)\eta(Y). \]

Especially we have \(\nabla^H_X\phi = 0\).

**Proof** The first equation is trivial by the assumption. We prove that \(\nabla^H_X\phi = 0\).

From \(N^{(2)} = 0\) we have \(d\eta^H(X^H,\xi^H) = 0\) and \(d\eta^V(X^V,\xi^V) = 0\). Thus the first equation implies that \(\nabla^H_X\phi = 0\) and \(\nabla^V_X\phi = 0\).

**Proposition 3.2** Let \((\phi, \eta, \xi, G)\) be a contact metrical Finsler structure on \(V\). Then \((\phi, \eta, \xi, G)\) is a K-contact Finsler structure if and only if \(N^{(3)}\) vanishes.

**Proposition 3.3** Let \((\phi, \eta, \xi, G)\) be contact metrical Finsler structure on \(V\). Then \((\phi, \eta, \xi, G)\) is a K-contact structure if and only if

\[ \nabla_X\xi^H = -\frac{1}{2}\phi X^H, \nabla_X\xi^V = -\frac{1}{2}\phi X^V. \tag{3.30} \]

**Proof** If the structure vector field \(\xi\) is a Killing vector field with respect to \(G\), then we have

\[ L^H_XG^V = 0, L^V_XG^V = 0. \tag{3.31} \]

\(\forall X^H, Y^H, \xi^H \in N_u\) and \(\forall X^V, Y^V, \xi^V \in V_w^e\) from (3.31), we can get

\[ G(\nabla^H_X\xi^H,Y^H) = -G(X^H,\nabla^H_X\xi^H), G(\nabla^V_X\xi^V,Y^V) = -G(X^V,\nabla^V_X\xi^V). \tag{3.32} \]

Replacing \(Y^H\) by \(\xi^H\) and \(Z^H\) by \(Y^H\) in (3.15), we have

\[ 2G(\nabla^H_X\xi^H,Y^H) = X^H\eta^H(Y^H) + \xi^H G(X^H,Y^H) - Y^H\eta^H(X^H) \]

\[ + G(\[X^H,\xi^H]\),Y^H) - \eta^H([X,Y]^H) - G([\xi^H,Y^H],X^H). \tag{3.33} \]

Replacing \(Y^H\) by \(\xi^H\), \(X^H\) by \(Y^H\) and \(Z^H\) by \(X^H\) in (3.15), we can get

\[ 2G(\nabla^H_X\xi^H,X^H) = Y^H\eta^H(X^H) + \xi^H G(X^H,Y^H) - X^H\eta^H(Y^H) \]

\[ + G(\[Y^H,\xi^H\],X^H) + \eta^H([X,Y]^H) - G([\xi^H,X^H],Y^H). \tag{3.34} \]
Using (3.33) and (3.34), we get $G(\nabla^H_\xi \xi, Y^H) - G(X^H, \nabla^H_\xi \xi) = d\eta^H(X^H, Y^H)$.

Since $\xi^H$ is a Killing vector field with respect to $G^H$, using (3.32), we obtain

$$d\eta^H(X^H, Y^H) = 2G(\nabla^H_\xi \xi, Y^H) = G(X^H, \phi Y^H) = -G(\phi X^H, Y^H)$$

and $\nabla^H_\xi \xi = -\frac{1}{2}\phi X^H$.

Similarly for $X^V, Y^V, \xi^V \in V^*_u$, from (3.16) and (3.32), we get $\nabla^V_\xi \xi^V = -\frac{1}{2}\phi X^V$.

Example 3.1 Let $V = \{V, \pi, M\}$ be a vector bundle with the total space $V = R^6$ is a 6-dimensional $C^\infty$-manifold and the base space $M = R^3$ is a 3-dimensional $C^\infty$-manifold. Let $x^i, 1 \leq i \leq 3$ and $y^a, 1 \leq a \leq 3$ be the coordinates of $u = (x, y) \in V$ that is $u = (x^1, x^2, x^3, y^1, y^2, y^3) \in V$.

The local base of $N_u$ is $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}\right)$ and that of $V_u$ is $\left(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial x^i}\right)$.

Let $X = X^i \frac{\partial}{\partial x^i} + \tilde{X}^a \frac{\partial}{\partial y^a} \forall X \in T_u(V)$, then $X^H = X^i \frac{\partial}{\partial x^i} + X^2 \frac{\partial}{\partial y^2} + X^3 \frac{\partial}{\partial y^3}$, $X^V = \tilde{X}^1 \frac{\partial}{\partial y^1} + \tilde{X}^2 \frac{\partial}{\partial y^2} + \tilde{X}^3 \frac{\partial}{\partial y^3}$ where $X^H \in N_u$ and $X^V \in V^*_u$. Similarly $Y$ can be written as

$$Y^H = Y^1 \frac{\partial}{\partial x^1} + Y^2 \frac{\partial}{\partial x^2} + Y^3 \frac{\partial}{\partial x^3}, Y^V = Y^1 \frac{\partial}{\partial y^1} + Y^2 \frac{\partial}{\partial y^2} + Y^3 \frac{\partial}{\partial y^3}$$

Let $\eta$ be a 1-form, $\eta = \eta_1 dx^1 + \eta_2 dx^2 + \eta_3 dx^3$ and $\eta^V = \tilde{\eta}_1 \delta y^1 + \tilde{\eta}_2 \delta y^2 + \tilde{\eta}_3 \delta y^3$ where $\eta = \eta^H + \eta^V$ and $\eta^H(X^V) = 0$ and $\eta^V(X^H) = 0$. We put $\eta^H = \frac{1}{2}(dx^3 - x^2 dx^1)$ and $\eta^V = \frac{1}{2}(\delta y^3 - y^2 \delta y^1)$. Then the structure vector field $\xi$ is given by $\xi = 2\left(\frac{\partial}{\partial x^3} + \frac{\partial}{\partial y^1}\right)$ and $\xi$ is decomposed as $\xi^H = 2\frac{\partial}{\partial x^3}$ and $\xi^V = 2\frac{\partial}{\partial y^1}$. The tensor field $\phi^H$ of type (1, 1) and $\phi^V$ of type (1, 1) by a matrix form is given by

$$\phi^H = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & x^2 & 0 \end{bmatrix}, \phi^V = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y^2 & 0 \end{bmatrix}$$

The Riemann metric tensor field $G = G^H + G^V$ is given by

$$G^H = \frac{1}{4}(dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + \eta^H \otimes \eta^H) = \frac{1}{4}\left((1 + (x^3)^2)(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - 2x^2(dx^1)(dx^3)\right)$$

$$G^V = \frac{1}{4}(\delta y^1 \otimes \delta y^1 + \delta y^2 \otimes \delta y^2 + \eta^V \otimes \eta^V) = \frac{1}{4}\left((1 + (y^3)^2)(\delta y^1)^2 + (\delta y^2)^2 + (\delta y^3)^2 - 2y^2(\delta y^1)(\delta y^3)\right)$$

Thus we give a metric tensor field $G$ by a matrix form

$$G^H = \frac{1}{4}\begin{bmatrix} 1 + (x^3)^2 & 0 & -x^2 \\ 0 & 1 & 0 \\ -x^2 & 0 & 1 \end{bmatrix}, G^V = \frac{1}{4}\begin{bmatrix} 1 + (y^3)^2 & 0 & -y^2 \\ 0 & 1 & 0 \\ -y^2 & 0 & 1 \end{bmatrix}.$$
On the other hand, we formalize that
\[
\eta^H (X^H) = G^H (X^H, \xi^H), \quad \eta^V (X^V) = G^V (X^V, \xi^V)
\]
\[
G^H (\phi X^H, \phi Y^H) = G^H (X^H, Y^H) - \eta^H (X^H) \eta^H (Y^H), \quad G^V (\phi X^V, \phi Y^V) = G^V (X^V, Y^V) - \eta^V (X^V) \eta^V (Y^V)
\]
\[
\eta^H (X^H) = \frac{1}{2} (dx^3 - x^2 dx^1) \left( X^1 \frac{\partial}{\partial x^1} + X^2 \frac{\partial}{\partial x^2} + X^3 \frac{\partial}{\partial x^3} \right) = \frac{1}{2} (X^3 - X^1 x^2) \quad (3.35)
\]
\[
G^H (X^H, \xi^H) = \frac{1}{4} \left[ \begin{array}{ccc} X^1 & X^2 & X^3 \end{array} \right] \left[ \begin{array}{ccc} 1 + (x^2)^2 & 0 & -x^2 \\ 0 & 1 & 0 \\ -x^2 & 0 & 1 \end{array} \right] = \frac{1}{4} \left[ \begin{array}{ccc} X^1 & X^2 & X^3 \end{array} \right] \left[ \begin{array}{c} -2x^2 \\ 0 \\ 2 \end{array} \right] = \frac{1}{4} (-2X^1 x^2 + 2X^3) = \frac{1}{4} (X^3 - X^1 x^2).
\]

From (3.35) and (3.36) we get \( \eta^H (X^H) = G^H (X^H, \xi^H) \). Similarly, we have
\[
\eta^V (X^V) = \frac{1}{2} (dy^3 - y^2 dy^1) \left( \tilde{X}^1 \frac{\partial}{\partial y^1} + \tilde{X}^2 \frac{\partial}{\partial y^2} + \tilde{X}^3 \frac{\partial}{\partial y^3} \right) = \frac{1}{2} \left( \tilde{X}^3 - \tilde{X}^1 y^2 \right) \eta^V = G^V (X^V, \xi^V),
\]
\[
\phi^H (X^H) = \left( X^2, -X^1, X_2^x \right), \quad \phi^V (Y^H) = \left( Y^2, -Y^1, Y_2^x \right),
\]
\[
\phi^V (X^V) = \left( \tilde{X}^2, -\tilde{X}^1, \tilde{X}_2^y \right), \quad \phi^V (Y^V) = \left( \tilde{Y}^2, -\tilde{Y}^1, \tilde{Y}_2^y \right),
\]
\[
G^H (\phi X^H, \phi Y^H) = \frac{1}{4} \left( \tilde{X}^1 Y^1 + \tilde{X}^2 Y^2 \right), \quad G^V (\phi X^V, \phi Y^V) = \frac{1}{4} \left( \tilde{X}^1 Y^1 + \tilde{X}^2 Y^2 \right).
\]
\[
G^H (X^H, Y^H) = \frac{1}{4} \left\{ \left( Y^1 \left( 1 + (x^2)^2 \right) - Y^3 x^2 \right) X^1 + X^2 Y^2 + X^3 (Y^3 - Y^1 x^2) \right\},
\]
\[
G^V (X^V, Y^V) = \frac{1}{4} \left\{ \left( \tilde{Y}^1 \left( 1 + (y^2)^2 \right) - \tilde{Y}^3 y^2 \right) \tilde{X}^1 + \tilde{X}^2 \tilde{Y}^2 + \tilde{X}^3 \left( \tilde{Y}^3 \tilde{Y}^1 y^2 \right) \right\},
\]
\[
\eta^H (X^H) \eta^H (Y^H) = \frac{1}{4} \left[ X^3 Y^3 + X^1 Y^1 \left( x^2 \right)^2 - X^1 Y^3 x^2 - X^3 Y^1 x^2 \right],
\]
\[
\eta^V (X^V) \eta^V (Y^V) = \frac{1}{4} \left[ \tilde{X}^3 \tilde{Y}^3 + \tilde{X}^1 \tilde{Y}^1 \left( y^2 \right)^2 - \tilde{X}^1 \tilde{Y}^3 y^2 - \tilde{X}^3 \tilde{Y}^1 y^2 \right].
\]

Thus, we get \( G^H (\phi X^H, \phi Y^H) = G^H (X^H, Y^H) - \eta^H (X^H) \eta^H (Y^H), \quad G^V (\phi X^V, \phi Y^V) = G^V (X^V, Y^V) - \eta^V (X^V) \eta^V (Y^V) \) and hence \( (\phi, \xi, \eta, G) \) is an almost contact Finsler metric structure.

\[
G^H (X^H, \phi Y^H) = \frac{1}{4} \left[ \begin{array}{ccc} X^1 & X^2 & X^3 \end{array} \right] \left[ \begin{array}{ccc} 1 + (x^2)^2 & 0 & -x^2 \\ 0 & 1 & 0 \\ -x^2 & 0 & 1 \end{array} \right] \left[ \begin{array}{c} Y^2 \\ -Y^1 \\ Y^2 x^2 \end{array} \right] = \frac{1}{4} (X^1 Y^2 - X^2 Y^1).
\]

Also, we know that \( dx^H = \frac{1}{2} \left( dx^1 \wedge dx^2 \right) \). By using this equality, we obtain
\[
d\eta^H (X^H, Y^H) = G^H (X^H, \phi Y^H). \quad \text{Similarly we get}
\]
\[
G^V (X^V, \phi Y^V) = \frac{1}{4} \left[ \begin{array}{ccc} \tilde{X}^1 & \tilde{X}^2 & \tilde{X}^3 \end{array} \right] \left[ \begin{array}{ccc} 1 + (y^2)^2 & 0 & -y^2 \\ 0 & 1 & 0 \\ -y^2 & 0 & 1 \end{array} \right] \left[ \begin{array}{c} \tilde{Y}^2 \\ -\tilde{Y}^1 \\ \tilde{Y}^2 y^2 \end{array} \right] = \frac{1}{4} \left( \tilde{X}^1 \tilde{Y}^2 - \tilde{X}^2 \tilde{Y}^1 \right).
By using \(d\eta^V = \frac{1}{2} (\delta y^1 \wedge \delta y^2)\), we derived \(d\eta^V (X^V, Y^V) = G^V (X^V, \phi Y^V)\). As a result we come up with the following equation:

\[
d\eta^H (X^H, Y^H) = G^H (X^H, \phi Y^H) = \Omega (X^H, Y^H), \quad d\eta^V (X^V, Y^V) = G^V (X^V, \phi Y^V) = \Omega (X^V, Y^V). \tag{3.37}
\]

Then the almost contact metrical Finsler structure \((\phi, \xi, \eta, G)\) is called almost Sasakian Finsler structure.

Because of \(\eta \wedge (d\eta) \neq 0\), \((\phi, \xi, \eta, G)\) is a contact metrical Finsler structure. The vector fields

\[
X_1 = 2 \left( \frac{\delta}{\delta x^2} + \frac{\partial}{\partial y^2} \right), \quad X_2 = 2 \left( \frac{\delta}{\delta x^2} + x^2 \frac{\delta}{\delta x^1} + \frac{\partial}{\partial y^1} + y^2 \frac{\partial}{\partial y^1} \right), \quad \xi = 2 \left( \frac{\delta}{\delta x^1} + \frac{\partial}{\partial y^1} \right)
\]

form a \(\phi\)-basis for the contact metrical Finsler structure, where these are decomposed as

\[
\xi^H = 2 \left( \frac{\delta}{\delta x^1} \right), \quad \xi^V = 2 \left( \frac{\partial}{\partial y^1} \right).
\]

On the other hand, we can see that \(N_\phi + d\eta \otimes \xi = 0\), that is \(N_\phi^H + d\eta^H \otimes \xi^H = 0\) and \(N_\phi^V + d\eta^V \otimes \xi^V = 0\). Hence the contact metrical Finsler structure is normal.

4. The curvature of a Finsler connection

The curvature of a Finsler connection \(\nabla\) is given by:

\[
R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - [X, Y] Z, \forall X, Y, Z \in T_u (V). \tag{4.1}
\]

As \(\nabla\) preserves by parallelism the horizontal and the vertical distributions, from (4.1) we have that the operator \(R(X, Y)\) carries horizontal vector fields into horizontal vector fields and vertical vector fields into verticals. Consequently,

\[
R(X, Y) Z = R^H (X, Y) Z^H + R^V (X, Y) Z^V, \forall X, Y, Z \in T_u (V). \tag{4.2}
\]

Noting that the operator \(R(X, Y)\) is skew-symmetric with respect to \(X\) and \(Y\), a theorem follows [1]:

**Theorem 4.1** The curvature of a Finsler connection \(\nabla\) on the tangent space \(T_u (V)\) is completely determined by the following six Finsler tensor fields:

\[
R(X^H, Y^H) Z^H = \nabla_X^H \nabla_Y^H Z^H - \nabla_Y^H \nabla_X^H Z^H - \nabla_{[X^H, Y^H]} Z^H, \\
R(X^H, Y^V) Z^H = \nabla_X^H \nabla_Y^V Z^H - \nabla_Y^H \nabla_X^V Z^H - \nabla_{[X^H, Y^V]} Z^H, \\
R(X^V, Y^H) Z^H = \nabla_X^V \nabla_Y^H Z^H - \nabla_Y^V \nabla_X^H Z^H - \nabla_{[X^V, Y^H]} Z^H, \\
R(X^V, Y^V) Z^H = \nabla_X^V \nabla_Y^V Z^H - \nabla_Y^V \nabla_X^V Z^H - \nabla_{[X^V, Y^V]} Z^H, \\
R(X^V, Y^V) Z^V = \nabla_X^V \nabla_Y^V Z^V - \nabla_Y^V \nabla_X^V Z^V - \nabla_{[X^V, Y^V]} Z^V, \\
R(X^V, Y^V) Z^V = \nabla_X^V \nabla_Y^V Z^V - \nabla_Y^V \nabla_X^V Z^V - \nabla_{[X^V, Y^V]} Z^V. \tag{4.3}
\]

Then the curvature tensor of a Finsler connection \(\nabla\) has only three different components with respect to the Berwald basis. These are given by:

\[
R \left( \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right) \frac{\delta}{\delta x^k} = R^i_{hj} \frac{\delta}{\delta x^i}, R \left( \frac{\delta}{\delta y^k}, \frac{\delta}{\delta x^j} \right) \frac{\delta}{\delta x^h} = P^i_{hj} \frac{\delta}{\delta x^i}, R \left( \frac{\partial}{\partial y^k}, \frac{\partial}{\partial x^j} \right) \frac{\delta}{\delta x^h} = S^i_{hj} \frac{\delta}{\delta x^i}. \tag{4.4}
\]
These three components are the first, third and fifth Finsler tensors from (4.3). The other three Finsler tensors from (4.3) have the same local components $R_{hjk}^i$, $P_{hjk}^i$, and $S_{hjk}^i$.

\[
\left( \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^k} \right) \frac{\partial}{\partial y^i} = R_{hjk}^i \frac{\partial}{\partial y^j}, \quad R \left( \frac{\partial}{\partial y^k}, \frac{\partial}{\partial x^p} \right) \frac{\partial}{\partial y^q} = P_{hjk}^i \frac{\partial}{\partial y^p}, \quad R \left( \frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^p} \right) \frac{\partial}{\partial y^q} = S_{hjk}^i \frac{\partial}{\partial y^p}. \quad (4.5)
\]

So, a Finsler connection $\nabla \Gamma = (N^i_j, F^i_{jk}, C^i_{jk})$ has only three local components $R_{hjk}^i$, $P_{hjk}^i$, $S_{hjk}^i$ [1].

For a Finsler connection $\nabla$, consider the torsion $T$, defined as usual

\[
T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \forall X, Y \in T_u(V). \quad (4.6)
\]

Breaking $T$ down into horizontal and vertical parts gives the torsion of a Finsler connection, $\nabla$ on $T_u(V)$ is completely determined by the following Finsler tensor fields [1]:

\[
T^H (X^H, Y^H) = \nabla^H_X Y^H - \nabla^H_Y X^H - [X^H, Y^H]^H, \quad T^V (X^H, Y^H) = - [X^H, Y^H]^V, \\
T^H (X^V, Y^V) = - \nabla^V_X Y^V - [X^H, Y^V]^H, \quad T^V (X^H, Y^V) = \nabla^H_X Y^V - [X^H, Y^V]^V, \\
T^V (X^V, Y^V) = \nabla^V_X Y^V - \nabla^V_Y X^V - [X^V, Y^V]^V. \quad (4.7)
\]

Let $\nabla$ be the torsion free Finsler connection, then we get

\[
[X^H, Y^H]^H = \nabla^H_X Y^H - \nabla^H_Y X^H, \quad [X^H, Y^H]^V = 0, \quad [X^H, Y^V]^H = - \nabla^V_X X^H, \\
[X^H, Y^V]^V = \nabla^V_X Y^V, \quad [X^V, Y^V]^V = \nabla^V_X Y^V - \nabla^V_Y X^V. \quad (4.8)
\]

**Theorem 4.2** In order for a $(n+m)$-dimensional Finsler manifold $V$ to be $K$-contact, it is necessary and sufficient that the following two conditions are satisfied:

1. $V$ admits a unit Killing vector field $\xi$;
2. The flag curvature for plane sections containing $\xi$ are equal to $\frac{1}{4}$ at every point of $V$.

**Proof** Let $V$ be a $K$-contact manifold. From (4.3) and (3.30), we have

\[
G^H \left( R \left( X^H, \xi^H \right), X^H \right) = G^H \left( \nabla^H_X \xi^H - \nabla^H_X \xi^H - \nabla^H_{[X^H, \xi^H]} X^H \right) = \frac{1}{4} G^H \left( X^H, X^H \right) = \frac{1}{4} G^V \left( R \left( X^V, \xi^V \right), X^V \right) = \frac{1}{4} G^V \left( X^V, X^V \right) = \frac{1}{4}.
\]

where $X^H$ is a unit vector field orthogonal to $\xi^H$ and $X^V$ is a unit vector field orthogonal to $\xi^V$. Hence

\[
G \left( R \left( X, \xi \right), X \right) = G^H \left( R^H \left( X, \xi \right), X^H \right) + G^V \left( R^V \left( X, \xi \right), X^V \right) = \frac{1}{4} \left( G^H \left( X^H, X^H \right) + G^V \left( X^V, X^V \right) \right) = \frac{1}{4} G \left( X, X \right) = \frac{1}{4}.
\]

Thus we obtain $K \left( X, \xi \right) = \frac{G \left( R \left( X, \xi \right), X \right)}{G \left( X, X \right)} = \frac{1}{4}$. Conversely, we suppose that $M$ satisfies the conditions (1.1) and (1.2). Since $\xi$ is a Killing vector field, we have

\[
d\eta \left( X^H, Y^H \right) = \left( G^H \left( \nabla^H_X \xi^H, X^H \right) - G^H \left( \nabla^H_Y \xi^H, X^H \right) \right) = -2G \left( \nabla^H_X \xi^H, X^H \right) = G \left( X^H, \phi Y^H \right),
\]

333
\[ d\eta(X^V, Y^V) = G(X^V, \phi Y^V). \]

Consequently, \((\phi, \eta, \xi, G)\) is a \(K\)-contact Finsler structure on \(V\).

Let \((\phi, \eta, \xi, G)\) be a contact metrical Finsler structure on \(V\). If the metric structure of \(V\) is normal, then \(V\) is mentioned to have a Sasakian Finsler structure and \(V\) is called a Sasakian Finsler manifold. \(\square\)

**Theorem 4.3** An almost contact metrical Finsler structure \((\phi, \eta, \xi, G)\) on \(V\) is a Sasakian Finsler structure if and only if

\[
(\nabla_X^H)Y^H = \frac{1}{2} \left[ G^H(X^H, Y^H) \xi^H - \eta^H(Y^H)X^H \right], \tag{4.9}
\]

\[
(\nabla_X^V)Y^V = \frac{1}{2} \left[ G^V(X^V, Y^V) \xi^V - \eta^V(Y^V)X^V \right]. \tag{4.10}
\]

**Proof** If the structure is normal, we have \(\Omega = d\eta\) and \(N^{(1)} = N^{(2)} = 0\). Thus, by using (3.14), (3.18) and (3.19), we get

\[ 2G^H((\nabla_X^H)Y^H, \xi^H) - \eta^H(\xi^H)X^H = -\partial_3 (X^H, Y^H, \xi^H) + d\eta(\phi Y^H, X^H) = G^H(\xi^H, X^H) - \eta^H(X^H)G^H. \]

Thus we have

\[ (\nabla_X^H)Y^H = \frac{1}{2} \left[ G^H(X^H, Y^H) \xi^H - \eta^H(Y^H)X^H \right]. \]

Similarly, from Lemma 3.3, we have

\[ 2G^V((\nabla_X^V)Y^V, \xi^V) = G^V(Y^V, X^V) - \eta^V(X^V)Y^V. \]

Thus we get

\[ (\nabla_X^V)Y^V = \frac{1}{2} \left[ G^V(X^V, Y^V) \xi^V - \eta^V(Y^V)X^V \right]. \]

Conversely, we suppose that the structure satisfies (4.9) and (4.10). Putting \(Y^H = \xi^H\) in (4.9) we have

\[ -\phi \nabla_X^H \xi^H = \frac{1}{2} (\eta^H(X^H, \xi^H) - X^H, \xi^H) \]

and putting \(Y^V = \xi^V\) in (4.10), we can get

\[ -\phi \nabla_X^V \xi^V = \frac{1}{2} (\eta^V(X^V, \xi^V) - X^V, \xi^V) \]

and hence, applying \(\phi\) to this, we obtain

\[ \nabla_X^V \xi^V = \frac{1}{2} \phi Y^V. \]

Since \(\xi\) is skew-symmetric, we prove that \(\xi^H\) and \(\xi^V\) is a Killing vector field. Moreover, we obtain

\[ d\eta(X^H, Y^H) = \frac{1}{2} \left[ (\nabla_X^H \eta)Y^H - (\nabla_Y^H \eta)X^H \right] = G(X^H, \phi Y^H) = \Omega(X^H, Y^H), \]

\[ d\eta(X^V, Y^V) = \frac{1}{2} \left[ (\nabla_X^V \eta)Y^V - (\nabla_Y^V \eta)X^V \right] = G(X^V, \phi Y^V) = \Omega(X^V, Y^V). \]

Thus the structure is a contact metric Sasakian structure.

If \((\phi, \eta, \xi, G)\) is a Sasakian Finsler structure on \(V\), from (4.9) and (4.10) we obtain

\[ R(X^H, Y^H) \xi^H = \frac{1}{4} (\eta^H(Y^H)X^H - \eta^H(X^H)Y^H), \tag{4.11} \]

\[ R(X^V, Y^V) \xi^V = \frac{1}{4} (\eta^V(Y^V)X^V - \eta^V(X^V)Y^V). \tag{4.12} \]

That is, we have

\[ R(X, Y) \xi = R^H(X, Y) \xi^H + R^V(X, Y) \xi^V = R(X^H, Y^H) \xi^H + R(X^V, Y^V) \xi^V = \frac{1}{4} (\eta^H(Y^H)X^H + \eta^V(Y^V)X^V - \eta^H(X^H)Y^H - \eta^V(X^V)Y^V). \tag{4.13} \]

\(\square\)
Theorem 4.4 Let $V$ be a $(n+m)$-dimensional Finsler manifold admitting a unit Killing vector field $\xi$. Then $V$ is a Sasakian Finsler manifold if and only if

$$R(X, \xi)Y = \frac{1}{4}[-G^H(X, Y)\xi^H - G^V(X, Y)\xi^V + \eta^H(Y^H)X^H + \eta^V(Y^V)X^V].$$ \hspace{1cm} (4.14)

Proof

From (4.15), (4.16), (4.17), (4.18), (4.19), we also have the following equations:

\begin{align*}
R^H(X, \xi)Y^H &= \nabla^H\nabla^H\xi^H - \nabla^H\nabla^H\xi^V - \nabla^H_{[X, \xi]}Y^H = -\frac{1}{4}(\nabla^H_{\xi}\phi)Y^H \\
R^V(X, \xi)Y^V &= -\frac{1}{2} (\nabla^V_X\phi)Y^V = -\frac{1}{4}[G(X^V, Y^V)\xi^V - \eta^V(Y^V)X^V].
\end{align*}

From these equations mentioned above, we have the equation.

Let $(\phi, \eta, \xi, G)$ be a Sasakian Finsler structure on $V$. From (4.9) and (4.10), we realize that

\begin{align*}
R(X^H, Y^H)\phi Z^H &= \phi R(X^H, Y^H)Z^H + \frac{1}{4}\{G(\phi X^H, Z^H)Y^H - G(Y^H, Z^H)\phi X^H \\
&+ G(X^H, Z^H)\phi Y^H - G(\phi Y^H, Z^H)X^H\}, \\
R(X^V, Y^V)\phi Z^V &= \phi R(X^V, Y^V)Z^V + \frac{1}{4}\{G(\phi X^V, Z^V)Y^V - G(Y^V, Z^V)\phi X^V \\
&+ G(X^V, Z^V)\phi Y^V - G(\phi Y^V, Z^V)X^V\}, \\
R(X^H, Y^H)\phi Z^H &= \phi R(X^H, Y^H)Z^H, \\
R(X^V, Y^V)\phi Z^H &= \phi R(X^V, Y^V)Z^H + \frac{1}{4}\{G(\phi X^H, Z^H)Y^V - G(X^H, Z^H)\phi Y^V \}
\end{align*}

From (4.15), (4.16), (4.17), (4.18), (4.19), we also have the following equations:

\begin{align*}
R(X^H, Y^H)Z^H &= -\phi R(X^H, Y^H)\phi Z^H + \frac{1}{4}\{G(Y^H, Z^H)X^H - G(X^H, Z^H)Y^H \\
&- G(\phi Y^H, Z^H)\phi X^H + G(\phi X^H, Z^H)\phi Y^H\}, \\
R(X^V, Y^V)Z^V &= -\phi R(X^V, Y^V)\phi Z^V + \frac{1}{4}\{G(Y^V, Z^V)X^V - G(X^V, Z^V)Y^V \\
&- G(\phi Y^V, Z^V)\phi X^V + G(\phi X^V, Z^V)\phi Y^V\}, \\
R(X^H, Y^H)Z^V &= -\phi R(X^H, Y^H)\phi Z^V, \\
R(X^H, Y^V)Z^V &= -\phi R(X^H, Y^V)\phi Z^V + \frac{1}{4}\{G(Y^V, Z^V)X^H - G(\phi Y^V, Z^V)\phi X^H\},
\end{align*}
Proof

For $X$ vector $\xi$ is a Sasakian Finsler manifold with constant curvature

$\phi, \eta, \xi, G$

Proposition 4.1

Let $\phi$ be a horizontal section if there exists a unit vector $X$ in $N_u$ orthogonal to $\xi$ such that $\{X, \phi X\}$ and a plane section in $V_u$ is called a vertical section if there exists a unit vector $X$ in $V_u$ orthogonal to $\xi$ such that $\{X, \phi X\}$. Then the horizontal flag curvature

$K(X, \phi X) = G(R(X, \phi X) \phi X, X)$

is called a horizontal sectional curvature, which will be denoted by $K(X)$. Vertical flag curvature

$K(X, \phi X) = G(R(X, \phi X) \phi X, X)$

is called a vertical sectional curvature, which will be denoted by $K(X)$. On a Sasakian Finsler manifold the sectional curvature is $K(X) = K(X) + K(X)$.

Proposition 4.1

Let $(\phi, \eta, \xi, G)$ be a $K$-contact Finsler structure on $V$. If $V$ is locally symmetric, then $V$ is a Sasakian Finsler manifold with constant curvature $\frac{1}{4}$.

Proof

For $X, Y, Z, \xi, H \in N_u$ from (4.9), (4.10), (4.11) and (4.12), we get

$(\nabla^H R)(X, Y, Z) = \frac{1}{4}\left\{G(Z, X) Y - G(Z, Y) X - R(X, Y) Z\right\}.$

Since $V$ is locally symmetric, that is, $\nabla^H R = 0$, from (4.36) we obtain

$R(X, Y) Z = \frac{1}{4}\left\{G(Z, X) Y - G(Z, Y) X\right\}.$

Thus for any orthonormal pair $\{X, Y\}$, we get

$K(X, Y) = G(R(X, Y) Y, X) = \frac{1}{4}.$
Similarly for $X^V, Y^V, Z^V, \xi^V \in V_u^v$, we get
\[
R(X^V, Y^V) Z^V = \frac{1}{4} \{ G(Z^V, Y^V) X^V - G(Z^V, X^V) Y^V \},
\]
and for any orthonormal pair $\{X, Y\}$, we obtain $K(X, Y^V) = G(R(X, Y^V) Y^V, X^V) = \frac{1}{4}$.

For any orthonormal pair $\{X, Y\}$, we get $K(X, Y) = \frac{G_H(r(x^H, y^H) y^H, y^H) + G_V(r(x^V, y^V) y^V, y^V)}{G_H(x^H, x^H) G_V(x^V, x^V)} = \frac{1}{4}$ which shows us that the sectional curvature of $V$ is $\frac{1}{4}$. The horizontal Ricci tensor $S^H$ of a $(4n+2)$-dimensional Sasakian Finsler manifold $V$ is given by
\[
S^H(X^H, Y^H) = \sum_{i=1}^{2n} G(R(X^H, E_i^H) E_i^H, Y^H) + G(R(X^H, \xi^H) \xi^H, Y^H)
\]
where $\{E_1^H, E_2^H, ..., E_{2n}^H, \xi^H\}$ is a local orthonormal frame of $N_u$.

The vertical Ricci tensor of a $(4n+2)$-dimensional Sasakian Finsler manifold $V$ is given by
\[
S^V(X^V, Y^V) = \sum_{i=1}^{2n} G(R(X^V, E_i^V) E_i^V, Y^V) + G(R(X^V, \xi^V) \xi^V, Y^V)
\]
where $\{E_1^V, E_2^V, ..., E_{2n}^V, \xi^V\}$ is a local orthonormal frame of $V_u^v$. Thus the Ricci tensor $S$ of a $(4n+2)$-dimensional Sasakian Finsler manifold $V$ is given by
\[
S(X, Y) = S^H(X, Y) + S^V(X, Y) = S(X^H, Y^H) + S(X^V, Y^V) = \sum_{i=1}^{2n} G(R(X^H, E_i^H) E_i^H, Y^H) + G(R(X^H, \xi^H) \xi^H, Y^H) + \sum_{i=1}^{2n} G(R(X^V, E_i^V) E_i^V, Y^V) + G(R(X^V, \xi^V) \xi^V, Y^V).
\]

Proposition 4.2 A contact metric structure $(\phi, \eta, \xi, G)$ on a Finsler manifold of dimension $(4n+2)$ is $K$-contact if and only if $S(\xi^H, \xi^H) = \frac{n}{2}S(\xi^V, \xi^V) = \frac{n}{2}$.

Proof From (4.37) and (4.14), we have
\[
S(\xi^H, \xi^H) = \sum_{i=1}^{2n} G(R(E_i^H, \xi^H) \xi^H, E_i^H) = \frac{1}{4} \sum_{i=1}^{2n} G(E_i^H, E_i^H) - \frac{1}{4} \sum_{i=1}^{2n} \eta^H(E_i^H) \eta^H(E_i^H).
\]
Since $E_i^H$ and $\xi^H$ orthogonal, we can take $\eta^H(E_i^H) = 0$, thus we have $S(\xi^H, \xi^H) = \frac{n}{2}$.

\[
S(\xi^V, \xi^V) = \sum_{i=1}^{2n} G(R(E_i^V, \xi^V) \xi^V, E_i^V) = \frac{1}{4} \sum_{i=1}^{2n} G(E_i^V, E_i^V) - \frac{1}{4} \sum_{i=1}^{2n} \eta^V(E_i^V) \eta^V(E_i^V).
\]
Since $E_i^V$ and $\xi^V$ orthogonal, we can take $\eta^V(E_i^V) = 0$, thus we have $S(\xi^V, \xi^V) = \frac{n}{2}$.

\[337\]
Lemma 4.1 The Ricci tensor $S$ of a $(4n+2)$-dimensional Sasakian Finsler manifold satisfies the following equations:

\[ S(X, \xi) = S(X^H, \xi^H) + S(X^V, \xi^V) = \frac{n}{2} \eta^H (X^H) + \frac{n}{2} \eta^V (X^V) = \frac{n}{2} (\eta^H (X^H) + \eta^V (X^V)) = \frac{n}{2} \eta(X), \]

\[ S(\phi X, \phi Y) = S(\phi X^H, \phi Y^H) + S(\phi X^V, \phi Y^V) - \frac{n}{2} \eta^H (X^H) \eta^H (Y^H) - \frac{n}{2} \eta^V (X^V) \eta^V (Y^V). \]

5. Conclusion

For the Sasakian Finsler structure $(\phi, \eta, \xi, G)$ on $V$, the following relations hold:

\[ \phi \cdot \phi = -I_n + \eta^H \otimes \xi^H + \eta^V \otimes \xi^V, \quad \phi \xi^H = 0, \quad \phi \xi^V = 0, \quad \eta^H (\xi^H) + \eta^V (\xi^V) = 1, \]

\[ \eta^H (\phi X^H) = 0, \quad \eta^V (\phi X^V) = 0, \quad \eta^H (\phi X^V) = 0, \quad \eta^V (\phi X^V) = 0, \]

\[ G^H (\phi X, \phi Y) = G^H (X, Y) - \eta^H (X^H) \eta^H (Y^H), \quad G^V (\phi X, \phi Y) = G^V (X, Y) - \eta^V (X^V) \eta^V (Y^V), \]

\[ G^H (X, \xi) = \eta^H (X^H), \quad G^V (X, \xi) = \eta^V (X^V), \quad N_\phi + d\eta^H \otimes \xi^H + d\eta^V \otimes \xi^V = 0, \]

\[ \Omega (X^H, Y^H) = G^H (X, \phi Y) = d\eta (X^H, Y^H), \quad \Omega (X^V, Y^V) = G^V (X, \phi Y) = d\eta (X^V, Y^V), \]

\[ \nabla^H_H \phi = 0, \quad \nabla^V_H \phi = 0, \quad \nabla^H_H \xi^H = -\frac{1}{2} \phi X^H, \quad \nabla^V_H \xi^V = -\frac{1}{2} \phi X^V, \]

\[ (\nabla^H_H \phi) X^H =\frac{1}{4} [G^H (X^H, Y^H) \xi^H - \eta^H (Y^H) X^H] (\nabla^V_H \phi) Y^V \]

\[ = \frac{1}{4} [G^V (X^V, Y^V) \xi^V - \eta^V (Y^V) X^V] R (X^H, Y^H) Z^H \]

\[ = \frac{1}{4} \{ G (Z^H, Y^H) X^H - G (Z^H, X^H) Y^H \} (V \text{ is locally symmetric}), \quad \eta^H (X^H, Y^H) \xi^H \]

\[ = \frac{1}{4} (\eta^H (Y^H) X^H - \eta^H (X^H) Y^H), \quad R^V (X^V, Y^V) \xi^V \]

\[ = \frac{1}{4} (\eta^V (Y^V) X^V - \eta^V (X^V) Y^V), \quad R^H (X^H, \xi^H) Y^H \]

\[ = \frac{1}{4} [-G (X^H, Y^H) \xi^H + \eta^H (Y^H) X^H], \quad S (\xi^H, \xi^H) = \frac{n}{2}, \]

\[ R^V (X^V, \xi^V) Y^V = \frac{1}{4} [\eta (X^V, Y^V) \xi^V + \eta^V (Y^V) X^V], \quad S (\xi^V, \xi^V) = \frac{n}{2}, \]

\[ K (X^V, Y^V) = G (R (X^V, Y^V) Y^V, X^V) = \frac{1}{4} K (X^H, Y^H) = G (R (X^H, Y^H) Y^H, X^H) = \frac{1}{4}, \]

\[ S(X, \xi) = S(X^H, \xi^H) + S(X^V, \xi^V) = \frac{n}{2} (\eta^H (X^H) + \eta^V (X^V)) = \frac{n}{2} \eta(X). \]

$\forall X^V, Y^V, \xi^H \in V_n$ and $\forall X^H, Y^H, \xi^H \in N_n$, where a linear connection $\nabla$ on $V$ denotes Finsler connection, $\phi$ is the tensor field of type $\left( \frac{1}{2} \frac{1}{1} \right)$ on $V$, $\eta$ is a 1-form, $S$ is the Ricci tensor, $R$ is the Riemann curvature tensor, $G$ is the Finsler metric structure on $V$, $K$ is the flag curvature on $V$. Next, let us set the equation below.
$2d\tilde{\eta}(X,Y) = X(\tilde{\eta}(Y)) - Y(\tilde{\eta}(X)) - \tilde{\eta}[X,Y], \forall X = X^H \in N_u, Y = Y^H \in N_u, \forall \tilde{\eta} = \eta^H \in N_u^*$. If we get $\tilde{\phi} = \phi^H, \tilde{\eta} = \eta^H, \tilde{\xi} = \xi^H, \tilde{g} = g^H$, the standard Sasakian structure of the base space $M^{2n+1}$ is $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$.

Then we have the following equations:

\begin{align*}
N_{\tilde{\phi}} + 2d\tilde{\eta} \otimes \tilde{\xi} &= 0, \\
\nabla_X \tilde{\phi} Y &= \tilde{g}(X,Y) \tilde{\xi} - \tilde{\eta}(Y) X, \\
\nabla_{\tilde{\xi}} \tilde{\phi} &= 0, \\
\tilde{g}\left(R\left(X,\tilde{\xi}\right)\tilde{\xi},X\right) &= 1, S\left(\tilde{\xi},\tilde{\xi}\right) = 2n, R(X,Y)\tilde{\xi} = \tilde{\eta}(Y) X - \tilde{\eta}(X) Y,
\end{align*}

\[K(X,Y) = \tilde{g}(R(X,Y)Y,X) = 1 \text{ (sectional curvature for orthonormal pair } \{X,Y\}).\]

The structure $(\phi^H, \eta^H, \xi^H, G^H)$ on $N_u$ is Sasakian Finsler if and only if the base manifold $M^{2n+1}$ with the structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ has positive constant curvature 1 in which case $M^{2n+1}$ is Sasakian manifold and $N_u$ is Sasakian Finsler manifold.

References