A note on unmixed ideals of Veronese bi-type

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Abstract: We classify the unmixed ideals of Veronese bi-type and in some cases we give a description of their associated prime ideals.

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1. Introduction

Let \( R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m] \) be a polynomial ring in two sets of variables over a field \( K \). In recent papers, monomial ideals of \( R \) are introduced and their connection to bipartite complete graphs is studied ([4], [6]). In this paper we study a class of monomial ideals of \( R \), so-called Veronese bi-type ideals. They are an extension of the ideals of Veronese type ([5]) in a polynomial ring in two sets of variables. More precisely, the ideals of Veronese bi-type are monomial ideals of \( R \) generated in the same degree: 
\[
L_{q,s} = \sum_{k+r=q} I_{k,s} J_{r,s},
\]
with \( k, r \geq 1 \), where \( I_{k,s} \) is the Veronese-type ideal generated on degree \( k \) by the set \( \{X_1^{a_{i1}} \cdots X_n^{a_{in}} \mid \sum_{j=1}^n a_{ij} = k, 0 \leq a_{ij} \leq s, s \in \{1, \ldots, k\}\} \) and \( J_{r,s} \) is the Veronese-type ideal generated on degree \( r \) by the set \( \{Y_1^{b_{i1}} \cdots Y_m^{b_{im}} \mid \sum_{j=1}^m b_{ij} = r, 0 \leq b_{ij} \leq s, s \in \{1, \ldots, r\}\} \) ([2], [3]). For \( s = 2 \) the Veronese bi-type ideals are the ideals associated to bipartite graphs with loops ([2]).

In this paper some properties of these class of monomial ideals are discussed. In particular, our aim is to classify the unmixed Veronese bi-type ideals.

Establishing whenever an ideal is unmixed in general is a difficult problem because it is necessary to know all its associated prime ideals. In [8] equidimensional and unmixed ideals of Veronese type are characterized. Now we are able to classify the unmixed Veronese bi-type ideals and in some cases we can give a description of the associated prime ideals.

This paper is organized as follows. In Section 1, unmixed ideals of Veronese bi-type are considered and the generalized ideals associated to the walks of special bipartite graphs, described by the Veronese by-type ideals \( L_{q,2} = \sum_{k+r=q} I_{k,2} J_{r,2} \), are considered. In Section 2, the toric ideal \( I(L_{q,s}) \) of the monomial subring \( K[L_{q,s}] \subset R \) is studied. Let \( L_{q,s} = (f_1, \ldots, f_p) \) and \( K[L_{q,s}] \) be the \( K \)-algebra spanned by \( f_1, \ldots, f_p \). There is a graded epimorphism of \( K \)-algebras: \( \varphi: S = K[T_1, \ldots, T_p] \to K[L_{q,s}] \) induced by \( \varphi(T_i) = f_i \), where \( S \) is a polynomial ring graded by \( \deg(T_i) = \deg(f_i) \). Let \( I(L_{q,s}) \) be the toric ideal of \( K[L_{q,s}] \), that is the kernel of \( \varphi \).
We are able to prove that \( I(L_{q,s}) \) has a quadratic Gröbner basis and as a consequence the \( K \)-algebra \( K[L_{q,s}] \) is Koszul. In order to formulate these results we have to recall the notion of sortability ([5]), and we apply it to the monomial ideals \( L_{q,s} \).

2. Unmixed ideals of Veronese bi-type

Let \( R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m] \) be the polynomial ring over a field \( K \) in two sets of variables with each \( \deg X_i = 1, \deg Y_j = 1 \), for all \( i = 1, \ldots, n \), \( j = 1, \ldots, m \).

We define the ideals of Veronese bi-type of degree \( q \) as the monomial ideals of \( R \)

\[
L_{q,s} = \sum_{r+k=q} I_{k,s}J_{r,s}, \quad r, k \geq 1,
\]

where \( I_{k,s} \) is the ideal of Veronese-type of degree \( k \) in the variables \( X_1, \ldots, X_n \) and \( J_{r,s} \) is the ideal of Veronese-type of degree \( r \) in the variables \( Y_1, \ldots, Y_m \).

\( L_{q,s} \) is not trivial for \( 2 \leq q \leq s(n + m) - 1, \ s \leq q \).

Remark 2.1 In general, \( I_{k,s} \subseteq I_k \), where \( I_k \) is the Veronese ideal of degree \( k \) generated by all the monomials in the variables \( X_1, \ldots, X_n \) of degree \( k \) ([6]).

One has \( I_{k,s} = I_k \) for any \( k \leq s \). If \( s = 1 \), \( I_{k,1} \) is the square-free Veronese ideal of degree \( k \) generated by all the square-free monomials in the variables \( X_1, \ldots, X_n \) of degree \( k \). Similar considerations hold for \( J_{r,s} \subseteq K[Y_1, \ldots, Y_m] \).

Example 2.2 Let \( R = K[X_1, X_2; Y_1, Y_2] \) be a polynomial ring.

1) \( L_{2,2} = I_{1,2}J_{1,2} = I_1J_1 = (X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2) \);
2) \( L_{4,2} = I_{3,2}J_{1,2} + I_{1,2}J_{3,2} + I_{2,2}J_2 = I_{3,2}J_1 + I_1J_{3,2} + I_2J_2 = (X_1^2Y_1, X_1^2Y_2, X_1Y_1^2Y_2, X_1X_2Y_1, X_1X_2Y_2, X_1Y_1^2Y_2, X_2Y_1^2Y_2, X_2Y_1Y_2^2, X_2Y_2^2, X_1Y_1^2Y_2, X_1Y_1Y_2^2, X_1Y_2^2, X_2^2Y_1Y_2, X_1X_2Y_1^2, X_1X_2Y_2^2, X_1X_2Y_2Y_2) \).

In this section we classify the unmixed Veronese bi-type ideals. First, we recall some preliminary notions.

Definition 2.3 Let \( G(L_{q,s}) \) be the unique minimal set of monic monomial generators of \( L_{q,s} \). A vertex cover of \( L_{q,s} \) is a subset \( W \) of \( \{X_1, \ldots, X_n; Y_1, \ldots, Y_m\} \) such that each \( u \in G(L_{q,s}) \) is divided by some variables of \( W \). Such a vertex cover \( W \) is called minimal if no proper subset of \( W \) is a vertex cover.

Denote by \( h(L_{q,s}) \) the minimal cardinality of the vertex covers of \( L_{q,s} \).

Definition 2.4 A monomial ideal is said to be unmixed if all its minimal vertex covers have the same cardinality.

Remark 2.5 We recall the one-to-one correspondence between the minimal vertex covers of an ideal and its minimal primes. Hence \( \varphi \) is a minimal prime ideal of \( L_{q,s} \) if and only if \( \varphi = (\mathcal{A}) \) for some minimal vertex cover \( \mathcal{A} \) of \( L_{q,s} \).

Now we are able to classify the unmixed Veronese bi-type ideals and in some cases we can give a description of the associated prime ideals.
Proposition 2.6 Let \( R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m] \). Let \( 2 \leq q < s(n + m) - i \) for \( i = 1, \ldots, s - 1 \). \( L_{q,s} \) is unmixed if and only if \( n = m \).

Proof By the structure of \( G(L_{q,s}) \) the minimal vertex covers of \( L_{q,s} \) are \( W_1 = \{X_1, \ldots, X_n\} \), \( W_2 = \{Y_1, \ldots, Y_m\} \). The minimal cardinality of the vertex covers of \( L_{q,s} \) is \( h(L_{q,s}) = \min\{n, m\} \). Hence all the minimal vertex covers have the same cardinality if and only if \( n = m \). \( \square \)

Example 2.7 \( R = K[X_1, X_2; Y_1, Y_2] \)
\[ L_{3,2} = (X_1^2Y_1, X_1^2Y_2, X_1X_2Y_1, X_1X_2Y_2, X_1^2Y_1^2, X_1Y_1Y_2, X_1Y_2^2, X_2Y_1^2, X_2Y_2^2). \]
The minimal vertex covers are: \( W_1 = \{X_1, X_2\} \); \( W_2 = \{Y_1, Y_2\} \).
\( h(L_{3,2}) = |W_1| = |W_2| = 2 \Rightarrow L_{3,2} \) is unmixed.

Proposition 2.8 Let \( R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m] \). If \( q = s(n + m) - i \) for \( i = 1, \ldots, s - 1 \), then \( L_{q,s} \) is unmixed.

Proof One has \( W_i = \{X_i\} \), \( i = 1, \ldots, n \), \( W_j = \{Y_j\} \), \( j = 1, \ldots, m \), are the minimal vertex covers of \( L_{q,s} \) by construction. \( \square \)

Example 2.9 \( R = K[X_1, X_2; Y_1, Y_2] \),
\[ L_{11,3} = (X_1^3X_2^2Y_1^3Y_2^3, X_1^3X_2^2Y_1^2Y_2^3, X_1^2X_2^2Y_1^3Y_2^3, X_1^3X_2^2Y_1^3Y_2^2). \]
The minimal vertex covers are:
\( W_1 = \{X_1\} \); \( W_2 = \{X_2\} \); \( W_3 = \{Y_1\} \); \( W_4 = \{Y_2\} \).
\( h(L_{11,3}) = |W_i| = 1 \) for all \( i = 1, 2, 3, 4 \Rightarrow L_{11,3} \) is unmixed.

Let \( A \subseteq \{1, 2, \ldots, n + m\} \), where \( n + m \) is the number of the variables of the polynomial ring \( R \). For a subset \( A \) we denote by \( P_A \) the prime ideal of \( R \) generated by the variables whose index is in \( A \).

Theorem 2.10 Let \( R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m] \), \( L_{q,s} \subset R \).
\[ P_A \in \text{Ass}_R(L_{q,s}) \Leftrightarrow |A| \leq i + 1, \]
for \( i = s(n + m) - q \), \( i = 1, \ldots, s - 1 \).

Proof In the following replace the set of variables \( \{X_1, \ldots, X_n\} \) with \( \{z_1, \ldots, z_n\} \) and \( \{Y_1, \ldots, Y_m\} \) with \( \{z_{n+1}, \ldots, z_{n+m}\} \).

Assume that \( P_A \in \text{Ass}_R(L_{q,s}) \). Then there exists a monomial \( f \notin L_{q,s} \) such that \( L_{q,s} : f = P_A \). Now we show we can choose such a monomial \( f \) of degree \( q - 1 \) such that \( L_{q,s} : f = P_A \). Suppose that \( f \notin L_{q,s} \), \( L_{q,s} : f = P_A \), \( \deg(f) \geq q \) and \( f = z_1^{a_1} \cdots z_{n+m}^{a_{n+m}} \). Then there exists \( j_0 \in \{1, 2, \ldots, n + m\} \) such that \( a_{j_0} > s \). Since \( L_{q,s} : f = P_A \), we have \( z_if \in L_{q,s} \) for all \( i \in A \) and \( z_if \notin L_{q,s} \) for all \( i \notin A \). Moreover, for all \( i \in A \) there exists a monomial \( u_i \in G(L_{q,s}) \) such that \( u_i(z_if) \). Being \( f \notin L_{q,s} \), this fact means that, for all \( i \in A \), the variable \( z_i \) appears in \( u_i \) with exponent \( a_i + 1 \). Therefore \( a_i < s \) for all \( i \in A \). It follows that \( j_0 \notin A \).

Now we claim that: I) \( \overline{f} = f/z_{j_0} \notin L_{q,s} \) and II) \( L_{q,s} : \overline{f} = P_A \).

The first fact follows from that \( f \notin L_{q,s} \) and \( a_{j_0} - 1 \geq s \). For the second assertion we proceed as follows.
\( L_{q,s} : \overline{f} \subseteq L_{q,s} : f \) because \( \overline{f} \) divides \( f \). Then \( L_{q,s} : \overline{f} \subseteq P_A \), being \( P_A = L_{q,s} : f \). Moreover, since \( a_{j_0} - 1 \geq s \)
then $u_i$ divides $z_if/zj_i$ for all $i \in A$, then $z_i \in L_{q,s} : (f/zj_i)$ for all $i \in A$. Hence $P_A \subseteq L_{q,s} : (f/xj_i)$. It follows the other inclusion $P_A \subseteq L_{q,s} : f$. Hence $P_A = L_{q,s} : f$. After a finite number of these reductions, we find $f \notin L_{q,s}$ of degree $q - 1$ such that $P_A = L_{q,s} : f$. From this fact follows that $fz_i \in L_{q,s}$ for all $i \in A$ and $fz_i \notin L_{q,s}$ for all $i \notin A$. In particular $a_i + 1 \leq s$ for all $i \in A$, and $a_i \leq s$ for all $i \notin A$. Then $a_i = s$ for all $i \notin A$. Therefore $f = \prod_{i \in A} z_i^{a_i} \prod_{i \notin A} z_i^s$ with $0 \leq a_i < s$ for all $i \in A$. We have $\text{deg}(\prod_{i \notin A} z_i^s) = (n + m - |A|) = t$. Then we obtain: $s(n + m) \geq (\sum_{i \in A} a_i + 1) + t = \sum_{i \in A} a_i + |A| + t = \sum_{i \in A} a_i + t + |A| = \text{deg}(f) + |A| = q - 1 + |A|.$

Conversely, let $|A| \leq i + 1$, for $i = s(n + m) - q$, $i = 1, \ldots, s - 1$, that is $|A| \leq s(n + m) - q + 1$. Moreover, in these hypotheses one has $s(n + m - |A|) \leq q - 1$. In fact, $s(n + m - |A|) \leq s(n + m) - i - 1$; then $s|A| \geq i + 1$ that is true for $i = 1, \ldots, s - 1$. Being $q = s(n + m) - i$ for $i = 1, \ldots, s - 1$, then by the definition of $L_{q,s}$ it follows that for any monomial $u \in G(L_{q,s})$ there exists an integer $j \in A$ such that $z_j$ divides $u$. Therefore $L_{q,s} \in P_A$. The condition $s(n + m - |A|) \leq q - 1$ implies that $(s - 1)|A| + s(n + m - |A|) \geq q - 1$, which together with $s(n + m - |A|) \leq q - 1$ shows that there exists an integer $c_i < s$, for all $i \in A$ such that $c_i + (s(n + m - |A|)) = q - 1$. Then the monomial $f = \prod_{i \in A} z_i^{c_i} \prod_{i \notin A} z_i^s$ has degree $q - 1$. Hence $f \notin L_{q,s}$ and as a consequence $P_A \subseteq L_{q,s} : f$. Now we prove that $P_A = L_{q,s} : f$. Assume that $P_A$ is a proper subset of $L_{q,s} : f$. Then there exists a monomial $f'$, in the variables $z_i$, with $i \notin A$, of degree at least 1 such that $ff' \in L_{q,s}$. This means that there exists a monomial $u = z_i^{a_1} \cdots z_i^{a_{n+m}} \in G(L_{q,s})$ such that $u$ divides $ff'$. Therefore $a_i \leq c_i$ for any $i \in A$ because $f' \in K[z_i|i \notin A]$. It follows that $q = \text{deg}(u) = \sum_{i=1}^{n+m} a_i \leq \sum_{i \in A} c_i + s(n + m - |A|) = \text{deg}(f) = q - 1$, which is a contradiction. Hence $P_A$ is not a proper subset of $L_{q,s} : f$, but $P_A = L_{q,s} : f$. This equality means that $P_A \in \text{Ass}_R(L_{q,s})$.

Example 2.11 Let $R = K[X_1, X_2; Y_1, Y_2]$, $L_{15,4} = (X_1^4X_2^4Y_1Y_2, X_1^4X_2^4Y_1Y_2, X_1^4X_2^4Y_1Y_2, X_1^4X_2^4Y_1Y_2, X_1^3X_2^4 Y_1^4 Y_2)$. By Theorem 2.10 $\text{Ass}_R(L_{15,4}) = \{(X_1), (X_2), (Y_1), (Y_2), (X_1, X_2), (X_1, Y_1), (X_1, Y_2), (X_2, Y_1), (X_2, Y_2), (Y_1, Y_2)\}$.

As an application, we observe that the ideals of Veronese bi-type can be associated to graphs with loops. In fact for $s = 2$, the ideals $L_{q,s}$ are associated to the walks of length $q - 1$ of the strong quasi-bipartite graphs with loops ([2]).

Definition 2.12 A graph $G$ with loops is a strong quasi-bipartite if all vertices of $V_1$ are joined to all vertices of $V_2$ and for each vertex of $V$ there is a loop.

Definition 2.13 Let $G$ be a strong quasi-bipartite graph on the vertex set $V = \{v_1, \ldots, v_n\}$. A walk of length $q$ in $G$ is an alternating sequence $w = \{v_{i_0}, l_{i_1}, v_{i_1}, l_{i_2}, \ldots, v_{i_{q-1}}, l_{i_q}, v_{i_q}\}$, where $v_{i_j}$ is a vertex of $G$ and $l_{i_j} = \{v_{i_{j-1}}, v_{i_j}\}$ is the edge joining $v_{i_{j-1}}$ and $v_{i_j}$ or a loop if $v_{i_{j-1}} = v_{i_j}$, $1 \leq i_1 \leq i_2 \leq \ldots \leq i_q \leq n$.

Example 2.14 Let $G$ be a strong quasi-bipartite graph on vertices $\{x_1, x_2; y_1, y_2\}$. A walk of length 2 is

$$w = \{x_1, l_1, x_2, y_1\},$$

where $l_1 = \{x_1, x_1\}$ is the loop on $x_1$ and $l_2 = \{x_1, y_1\}$ is the edge joining $x_1$ and $y_1$. 4
Let $G$ be a strong quasi-bipartite graph on vertex set $\{x_{1}, \ldots, x_{n}; y_{1}, \ldots, y_{m}\}$. The generalized ideal $I_q(G)$ associated to $G$ is the ideal of the polynomial ring $R = K[x_{1}, \ldots, x_{n}; y_{1}, \ldots, y_{m}]$ generated by the monomials of degree $q$ corresponding to the walks of length $q - 1$. Hence the generalized ideal $I_q(G)$ is generated by all monomials of degree $q \geq 3$ corresponding to the walks of length $q - 1$ and the variables in each generator of $I_q(G)$ have at most degree 2. Therefore:

$$I_q(G) = L_{q,2} = \sum_{k+r=q} I_{k,2}J_{r,2}, \text{ for } q \geq 3 \ ([2]).$$

**Example 2.15** Let $R = K[x_1, x_2; y_1, y_2]$ be a polynomial ring over a field $K$ and $G$ be the strong quasi-bipartite graph on vertices $x_1, x_2, y_1, y_2$:

![Diagram of a graph with vertices $x_1, x_2, y_1, y_2$ and edges connecting $x_1$ and $x_2$ to $y_1$ and $y_2$.]

$I_3(G) = I_1J_1 + I_2J_2 = (X_1Y_2, X_2Y_1, X_1Y_1^2, X_2Y_2^2, X_1Y_1, X_1X_2Y_2, X_1X_2Y_1, X_1X_2Y_1^2, X_1X_2Y_2^2, X_1X_2, X_1Y_1X_2Y_2, X_1Y_1X_2Y_1, X_1Y_1X_2Y_1^2, X_1Y_1X_2Y_2^2, X_1Y_1, X_1X_2Y_1X_2Y_2, X_1X_2Y_1X_2Y_1^2, X_1X_2Y_1X_2Y_2^2, X_1X_2, X_1Y_1X_2Y_1X_2Y_2, X_1Y_1X_2Y_1X_2Y_1^2, X_1Y_1X_2Y_1X_2Y_2^2).$

$I_4(G) = I_3J_1 + I_1J_3 + I_2J_2 = (X_1^2X_2Y_1, X_2^2X_1Y_2, X_1X_2^2Y_1, X_1X_2^2Y_2, X_1X_2Y_1^2, X_1X_2Y_1^2, X_1X_2Y_2^2, X_1X_2Y_2^2, X_1X_2^2, X_1X_2^2, X_1X_2^2, X_1X_2^2, X_1X_2^2, X_1X_2^2, X_1X_2^2, X_1X_2^2, X_1X_2^2, X_1X_2^2, X_1X_2^2, X_1X_2^2, X_1X_2^2).$

The following result classifies the ideals $I_q(G)$ that are unmixed.

**Proposition 2.16** Let $R = K[x_1, \ldots, x_n; y_1, \ldots, y_m]$.  

1) If $2 \leq q < 2(n + m) - 1$ then $L_{q,2}$ is unmixed if and only if $n = m$.

2) If $q = 2(n + m) - 1$, then $L_{q,2}$ is unmixed.

**Proof**  By Propositions 2.6 and 2.8.

We consider the case $q \geq 3$. In fact, for $q = s = 2$, the ideal $L_{2,2} = ((X_iY_j | i = 1, \ldots, n, \ j = 1, \ldots, m})$ doesn’t describe the edge ideal $I(G) = I_2(G)$ of a strong quasi-bipartite graph, but it is the edge ideal of a complete bipartite graph (with no loops) on the vertex set $\{x_1, \ldots, x_n; y_1, \ldots, y_m\}$. Then $L_{2,2}$ satisfies the characterization of unmixed bipartite graphs given in [7].

**3. The toric ideal of $K[L_{q,s}]$**

Let $R = K[x_1, \ldots, x_n; y_1, \ldots, y_m]$ and $L_{q,s} = (f_1, \ldots, f_p)$ be the ideal of Veronese bi-type. The monomial subring of $R$ spanned by $F = \{f_1, \ldots, f_p\}$ is the $K$-algebra $K[L_{q,s}] = K[F] = K[f_1, \ldots, f_p]$. Note that $K[F]$ is a graded algebra with the grading $K[F]_i = K[F] \cap R_i$. There is a graded epimorphism of $K$-algebras: $\varphi : S = K[T_1, \ldots, T_p] \rightarrow K[L_{q,s}]$ induced by $\varphi(T_i) = f_i$, where $S$ is a polynomial ring graded by $\deg(T_i) = \deg(f_i)$. Note that the map $\varphi$ is given by $\varphi(h(T_1, \ldots, T_p)) = h(f_1, \ldots, f_p)$ for all $h \in S$.

Let $I(L_{q,s})$ be the kernel of the $K$-algebra epimorphism, called the *toric ideal* of $K[L_{q,s}]$. It is known that the toric ideal of a monomial $K$-algebra is a graded prime ideal generated by a finite set of binomials.
Now we prove that \( I(L_{q,s}) \) has a quadratic Groebner basis. In order to formulate this result we have to recall the notion of sortability, introduced in [5], and we apply it to the monomial ideal \( L_{q,s} \).

Let \( A = K[z_1, \ldots, z_l] \) be a polynomial ring and \( L \) be a monomial ideal of \( A \) generated in degree \( q \). Let \( B \) be the set of the exponent vectors of the monomials of \( G(L) \). If \( u = (u_1, \ldots, u_l), v = (v_1, \ldots, v_l) \in B \), then \( \pi^u \pi^v \in L \). We write \( \pi^u \pi^v = z_1^{i_1} \cdots z_{2q}^{i_{2q}} \) with \( i_1 \leq i_2 \leq \cdots \leq i_{2q} \). Then we set \( \pi^u = \prod_{j=1}^q z_{2j-1} \) and \( \pi^v = \prod_{j=1}^q z_{2j} \). This defines a map

\[
sort : B \times B \rightarrow M_q \times M_q, \quad (u, v) \mapsto (u', v'),
\]

where \( M_q \) is the set of all integer vectors \((a_1, \ldots, a_l)\) such that \( \sum_{i=1}^l a_i = q \).

The set \( B \) is called sortable if \( \text{Im}(\text{sort}) \subseteq B \times B \).

The ideal \( L \) is called sortable if the set of exponent vectors of the monomials of \( G(L) \) is sortable. In other words, let \( \pi^u, \pi^v \in L \), then \( L \) is said sortable if \( \pi^u', \pi^v' \in L \), where \( (u', v') = \text{sort}(u, v) \).

**Theorem 3.1** Let \( R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m] \). \( L_{q,s} \) is sortable.

**Proof** Let \( L_{q,s} = \{(X_1^{a_1} \cdots X_n^{a_n} Y_1^{b_1} \cdots Y_m^{b_m}) | \sum_{i=1}^n a_i + \sum_{j=1}^m b_j = q, \ 0 \leq a_i, b_j \leq s \} \) and \( B \) be the set of the exponent vectors of the monomials of \( G(L_{q,s}) \).

Let \( f_i = X_1^{a_1} \cdots X_n^{a_n} Y_1^{b_1} \cdots Y_m^{b_m}, f_j = X_1^{c_1} \cdots X_n^{c_n} Y_1^{d_1} \cdots Y_m^{d_m} \in G(L_{q,s}) \), then \( u = (a_1, \ldots, a_n; b_1, \ldots, b_m) \), \( v = (c_1, \ldots, c_n; d_1, \ldots, d_m) \in B \). One has that \( f_i f_j = X_1^{a_1+c_1} \cdots X_n^{a_n+c_n} Y_1^{b_1+d_1} \cdots Y_m^{b_m+d_m} \) is a monomial of degree \( 2q \). If one replaces the set of variables \( \{X_1, \ldots, X_n\} \) with \( \{z_1, \ldots, z_{2q}\} \) and \( \{Y_1, \ldots, Y_m\} \) with \( \{z_{n+1}, \ldots, z_{n+m}\} \), then \( f_i f_j = z_i \cdots z_{i_{2q}} \) with \( i_1 \leq \cdots \leq i_{2q} \). Then we consider \( f'_i = \pi^u = \prod_{i=1}^q z_{2i-1} \) and \( f'_j = \pi^v = \prod_{i=1}^q z_{2i} \). We must prove that \( f'_i, f'_j \in L_{q,s} \). We have that \( f'_i \) is of degree \( q \) and we write \( f'_i = \prod_{i=1}^q z_{2i-1} = X_1^{a'_1} \cdots X_n^{a'_n} Y_1^{b'_1} \cdots Y_m^{b'_m} \). If \( a_i + c_i \) is even then \( a'_i = \frac{a_i+c_i}{2} \leq s \) and if \( a_i + c_i \) is odd then \( a'_i = \frac{a_i+c_i+1}{2} < s \). Similarly, if \( b_j + d_j \) is even then \( b'_j = \frac{b_j+d_j}{2} \leq s \). If \( b_j + d_j \) is odd then \( b'_j = \frac{b_j+d_j+1}{2} < s \). Moreover, because \( f'_i \) is of degree \( q \) and there exist \( a'_i \neq 0, b'_j \neq 0 \) with \( 0 \leq a'_i, b'_j \leq q \) for all \( i, j \), then \( X_1^{a'_1} \cdots X_n^{a'_n} \neq X_1^{\tilde{a}'_1} \cdots X_n^{\tilde{a}'_n} \) and \( Y_1^{b'_1} \cdots Y_m^{b'_m} \neq Y_1^{\tilde{b}'_1} \cdots Y_m^{\tilde{b}'_m} \). It follows that \( X_1^{a'_1} \cdots X_n^{a'_n} \in I_{q,s} \) and \( Y_1^{b'_1} \cdots Y_m^{b'_m} \in J_{r,s} \) with \( k + r = q \). Hence \( f'_i \in L_{q,s} \). In the same way the argument holds for \( f'_j \). Hence \( L_{q,s} \) is sortable.

**Corollary 3.2** Let \( R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m] \) and \( L_{q,s} \subseteq R \). Then:

1) \( I(L_{q,s}) \) has a quadratic Groebner basis.
2) \( K[L_{q,s}] \) is Koszul.

**Proof** 1) By Theorem 3.1 and [1](Lemma 5.2).
2) The conclusion follows by 1).

**References**


