

UC modules with respect to a torsion theory

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Abstract

In this paper, we give some characterizations of modules M over a ring R such that every submodule has a unique closure relative to a hereditary torsion theory on $\text{Mod-}R$ by using the concept of τ -closed submodule which was studied in [2]. We compare UC and τ -UC modules and examine the relationships between them. We also give some examples of τ -UC and UC modules and submodules which have a unique τ -closure and unique closure.

Key Words: Hereditary torsion theory, τ -closed submodule, UC module, τ -UC module

1. Introduction

In this paper, R will denote an associative ring with identity and all modules will be unitary right R -modules. In [8], Smith studied modules in which every submodule has a unique closure and called them UC modules. Then in [3], the authors called a submodule N of a R -module M τ -essential if M/N is τ -torsion and N is essential in M , where τ is a torsion theory on $\text{Mod-}R$. It is clear that a τ -essential submodule is essential. In [3], the authors defined τ -UC-module as follows: given a submodule N of M , by a τ -closure of N in M , they mean a τ -closed submodule K of M containing N such that N is τ -essential in K . A module M is called a τ -UC-module provided every submodule has a unique τ -closure in M and they studied the properties of this class. In [3], the concept of UC and τ -UC-modules are coincided if M is τ -torsion, in particular, these two concepts are equivalent in $\tau = (\text{Mod-}R, 0)$. On the other hand, in [6], Pardo introduced the concept of τ -large submodule which is not comparable with the essential submodule (see Example 2.4, 2.6). In this paper, by using the concept of τ -large submodule, we study τ -closed submodule and define τ -UC modules which are torsion theoretic analogue of UC modules and we give some examples (see Example 2.4, 2.6, 3.2 and 3.3) to show that the new module class is different from module class discussed in [3].

Now we recall some fundamental concepts of torsion theory from [1], [5] and [9]. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory on $\text{Mod-}R$. Modules in \mathcal{T} will be called τ -torsion and modules in \mathcal{F} will be called τ -torsion free modules. Let $M \in \text{Mod-}R$. The τ -torsion submodule of M is defined to be the sum of all τ -torsion submodules of M and denoted by $\tau(M)$. The τ -torsion class is $\mathcal{T} = \{M \in \text{Mod-}R : \tau(M) = M\}$ and the τ -torsion free class $\mathcal{F} = \{M \in \text{Mod-}R : \tau(M) = 0\}$. \mathcal{T} is closed under homomorphic images, direct sums, and extensions. \mathcal{F} is closed under isomorphisms, submodules, direct products and extensions.

2000 AMS Mathematics Subject Classification: 16S90.

In this paper, all torsion theories τ are assumed to be hereditary, that is we assume that submodules of τ -torsion modules are τ -torsion.

If I is an idempotent ideal of R , then it is well known that I determines a hereditary torsion theory τ_I with torsion class $\{M : MI = 0\}$. We refer to τ_I as the torsion theory corresponding to I . If τ_G is the Goldie torsion theory, then τ_G is hereditary and the τ_G -torsion submodule $\tau_G(M)$ of a module M is the second singular submodule of M . That is, $\tau_G(M)$ is the submodule $Z_2(M)$ of M such that $Z_2(M)/Z(M) = Z(M/Z(M))$, where for a module M , $Z(M)$ denotes the singular submodule of M .

A submodule N of a module M is called τ -dense (τ -pure) in M if M/N is τ -torsion (τ -torsion free). The set of all τ -dense right ideals of R will be denoted by $F_\tau(R)$. It is well known that $\tau(M) = \{m \in M : (0 : m) \in F_\tau(R)\}$, where $(0 : m) = \{r \in R : mr = 0\}$. We let $\mathcal{P}_\tau(M)$ denote the set of all τ -pure submodules of M . It is known that $\mathcal{P}_\tau(M)$ is closed under arbitrary intersections, $\tau(M) = \bigcap \{N : N \leq M, N \in \mathcal{P}_\tau(M)\}$ and if N is a τ -pure submodule of M , $\tau(N) = \tau(M)$. The τ -pure closure of N in M , denoted by N^c , is defined to be the intersection of all the τ -pure submodules of M that contain N . It is well known that $\tau(M/N) = N^c/N$.

2. Preliminaries

In [6], Pardo introduced the concept of τ -large submodule. A submodule N of an R -module M is called τ -**large** in M if, for $W \leq M$, $N \cap W \subseteq \tau(M)$ implies $W \subseteq \tau(M)$. We write $N \trianglelefteq_\tau M$ to denote that N is a τ -large submodule of M .

It is clear that if M is a τ -torsion free module, then the concept of τ -large submodule coincides with the concept of essential submodule, in particular, if $\tau = (0, \text{Mod-}R)$, then these two concepts are equivalent. If $\tau = (\text{Mod-}R, 0)$, then every submodule of a module M is τ -large. More generally, every submodule of a τ -torsion module M is τ -large in M . On the other hand, the set of τ -large submodules and essential submodules of a module are unconnected. The submodule given in Example 2.4, is τ -large but not essential and the submodule given in Example 2.6 is essential but not τ -large. It is also easy to check that every τ -essential submodule is τ -large but the converse is not true (see Example 2.4).

We give the following proposition about the properties of τ -large submodules from Propositions 2.2, 2.3 and 2.4 in [6].

Proposition 2.1 *Let M be an R -module and $N, L \leq M$. Then*

- (1) N is τ -large in M if and only if N^c is τ -large in M .
- (2) If N is a τ -pure and τ -large submodule of M , then N is large in M .
- (3) If $N \subseteq L$ and $N \trianglelefteq_\tau M$, then $L \trianglelefteq_\tau M$.
- (4) $N \trianglelefteq_\tau L \trianglelefteq_\tau M$ if and only if $N \trianglelefteq_\tau M$.
- (5) If $f : M \rightarrow M'$ is an R -module homomorphism and $W \trianglelefteq_\tau M'$, then $f^{-1}(W) \trianglelefteq_\tau M$.

Proposition 2.2 [2, Proposition 2.2] (1) *Let L be a τ -large submodule of a module M . Then $N \cap L \trianglelefteq_\tau N$ for every $N \leq M$.*

(2) *Let $K \subseteq K'$ and $L \subseteq L'$ be submodules of a module M such that $K \trianglelefteq_\tau K'$ and $L \trianglelefteq_\tau L'$. Then $(K \cap L) \trianglelefteq_\tau (K' \cap L')$.*

Definition 2.3 Let M be a module and $K \leq M$. K is called a τ -**closed** submodule of M if whenever for any submodule L of M , $K \trianglelefteq_{\tau} L \leq M$ implies $K = L$. If N is a submodule of M such that $K \trianglelefteq_{\tau} N$ and N is τ -closed in M then N is called a τ -**closure** of K in M . We write $K \leq_{\tau-c} M$ to denote that K is a τ -closed submodule of M .

Clearly, if M is a τ -torsion module, then M is the unique τ -closure of a submodule of itself. But this result is not true for τ -closure of a submodule of M discussed in [3] (see Example 2.4). Moreover, a non- τ -torsion module cannot be a τ -closure of a τ -torsion submodule. Because a τ -torsion submodule of a non- τ -torsion module M cannot be τ -large in M . By Proposition 2.1-(2), it is also clear that if N is a τ -pure and closed submodule of M then N is τ -closed in M . But there exist closed submodules which are not τ -closed.

Example 2.4 Let $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2^3\mathbb{Z}$ and $N = \mathbb{Z}(1 + 2\mathbb{Z}, 2 + 2^3\mathbb{Z})$, then N is closed in M but it is not τ_G -closed in M since M is τ_G -torsion, where τ_G is the Goldie torsion theory. It is clear that N is not essential in M but it is τ_G -large in M .

Lemma 2.5 [2, Lemma 2.3] The τ -torsion submodule of a module M is τ -closed in M .

The following example can be found in [2].

Example 2.6 Let $R = \begin{bmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$, where F is a field. Consider the right R -module $M := R_R$. If τ_I is

the torsion theory on $\text{Mod-}R$ corresponding to the idempotent ideal $I = \begin{bmatrix} F & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, that is $\mathcal{T}_I = \{N \in \text{Mod-}$

$R : NI = 0\}$, then $\tau_I(M) = \begin{bmatrix} 0 & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$ is a τ -closed submodule of M by Lemma 2.5 and it is easy to check

that $\tau_I(M)$ is essential in M . Thus $\tau_I(M)$ is a τ -closed submodule of M which is not closed.

Lemma 2.7 [2, Lemma 2.4] Let N be a τ -closed submodule of a module M . Then N is τ -pure in M .

Proposition 2.8 [2, Proposition 2.5] Every submodule of a module M has a τ -closure in M .

Lemma 2.9 [2, Lemma 2.6] Let B be a τ -closed submodule of a module M . If $B \leq K \trianglelefteq_{\tau} M$, then $K/B \trianglelefteq_{\tau} M/B$.

Lemma 2.10 (1) If $K \leq L \leq_{\tau-c} M$ then $L/K \leq_{\tau-c} M/K$.

(2) If $K \leq L$, $K \leq_{\tau-c} M$ and $L/K \leq_{\tau-c} M/K$ then $L \leq_{\tau-c} M$.

Proof. (1) Let $K \leq L \leq_{\tau-c} M$. Suppose that $L \leq N \leq M$ such that $L/K \trianglelefteq_{\tau} N/K$. Let π denote the canonical map from N to N/K . Then $\pi^{-1}(L/K) = L \trianglelefteq_{\tau} N$ by Proposition 2.1-(5). By hypothesis $L = N$ and so $L/K = N/K$.

(2) Let $K \leq L$, $K \leq_{\tau-c} M$ and $L/K \leq_{\tau-c} M/K$. Suppose that $L \trianglelefteq_{\tau} N \leq M$. By Lemma 2.9 $L/K \trianglelefteq_{\tau} N/K$. By hypothesis, we have $L/K = N/K$ and so $L = N$. □

In [6], Pardo defined τ -complement submodules: A submodule B of a module M is said to be a τ -**complement** of A in M if B is maximal with respect to the property $A \cap B \subseteq \tau(M)$. A submodule C of M is called a τ -**complement submodule** of M if it is a τ -complement of some submodule S of M .

By using a simple application of Zorn's Lemma, we see that every submodule has τ -complement. It is clear that if M is a τ -torsion free module then a submodule N of M is a τ -complement submodule of M if and only if it is a complement submodule of M .

Proposition 2.11 [2, Proposition 2.11] *Let M be a module and $B \leq M$. Then B is a τ -complement submodule of M if and only if it is a τ -closed submodule of M .*

The following proposition can be found in [2]. But we give its proof here proof for completeness.

Proposition 2.12 *Let M be a module with the submodules $C \subseteq N$. If C is τ -closed in N and N is τ -closed in M , then C is τ -closed in M .*

Proof. It is enough to prove for τ -complement submodules by Proposition 2.11.

Let C be a τ -complement of S in N and N be a τ -complement of T in M . We claim that C is a τ -complement of $S + T$ in M .

By Lemma 2.7 and Proposition 2.11, we have that $N/C, M/N \in \mathcal{F}$ and so $M/C \in \mathcal{F}$. Then we get that $\tau(N) = \tau(M)$ and $\tau(M) \subseteq C$. Firstly, we show that $C \cap (S + T) \subseteq \tau(M)$. Take an element $x \in C \cap (S + T)$ and write $x = s + t$ where $s \in S$ and $t \in T$. Then $x - s = t \in N \cap T \subseteq \tau(M) \subseteq C$ and so $s \in C$. Hence $x = s + t \in (C \cap S) + (N \cap T) \subseteq \tau(M)$ and we get that $C \cap (S + T) \subseteq \tau(M)$.

Next, we want to show that C is a maximal submodule of M with respect to the property $C \cap (S + T) \subseteq \tau(M)$.

Let D be a submodule of M such that C is a proper submodule of D . In this case, we have two cases;

(i) Let $D \cap N \neq C$. Then since C is a τ -complement of S in N , we have that $D \cap N \cap S = D \cap S \not\subseteq \tau(N) = \tau(M)$ and so $D \cap (S + T) \not\subseteq \tau(M)$.

(ii) Let $D \cap N = C$. Then chose an element $d \in D \setminus N$ and so $(N + dR) \cap T \not\subseteq \tau(M)$ since N is a τ -complement of T in M . Thus we can chose an element $t = n + dr \notin \tau(M)$ where $t \in T, n \in N, r \in R$.

If $n \in C$, then $n + dr \in D$ and $D \cap T \not\subseteq \tau(M)$. Thus $D \cap (S + T) \not\subseteq \tau(M)$.

If $n \notin C$, then $(C + nR) \cap S \not\subseteq \tau(N) = \tau(M)$ and there are elements $c \in C, s \in S, l \in R$ such that $c + nl = s \notin \tau(M)$. Hence we get that $s - tl = (c + nl) - (n + dr)l \in D \cap (S + T)$ but $s - tl \notin \tau(M)$.

If $s - tl$ were in $\tau(M)$, then since $\tau(M) \subseteq N$, we have that $tl \in N \cap T \subseteq \tau(M)$. This is a contradiction with $s \notin \tau(M)$. Therefore, $D \cap (S + T) \not\subseteq \tau(M)$ and C is the maximal submodule with the property with respect to $C \cap (S + T) \subseteq \tau(M)$. This completes the proof. \square

3. τ -UC modules

Definition 3.1 *An R -module M is called a τ -UC module provided every submodule has a unique τ -closure in M .*

If M is a τ -torsion-free module, then τ -UC modules are precisely the UC modules. Every τ -torsion module M is a τ -UC module. Because M is the unique τ -closure of every submodule of itself.

The following example shows that a submodule may have different unique closure and unique τ -closure and so these two classes may be different. But they are equivalent when we consider the torsion theory $\tau = (0, \text{Mod-}R)$.

Example 3.2 Consider the R -module M in Example 2.6. We have stated that $\tau_I(M)$ is a τ -closed submodule of M and $\tau_I(M) \trianglelefteq M$. Thus the unique τ -closure of $\tau_I(M)$ is $\tau_I(M)$ but the unique closure of $\tau_I(M)$ is M .

The following example shows that a τ -UC module need not to be UC module discussed in [8] or τ -UC module discussed in [3].

Example 3.3 Let p be any prime number and $M = \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$. In [8], it was proved that M is not a UC module and hence not τ -UC module discussed in [3]. Because these two concepts, discussed in [8] and [3], are the same when the module is τ -torsion. But M is a τ_G -UC module as it is τ_G -torsion, where τ_G is the Goldie torsion theory.

Lemma 3.4 Let M be a τ -UC module and let U, V and V' be submodules of M with V the τ -closure of U in M and $U \trianglelefteq_\tau V'$. Then $V' \subseteq V$.

Proposition 3.5 The following statements are equivalent for a module M .

- (1) M is a τ -UC module.
- (2) If $K \trianglelefteq_\tau K'$ and $L \trianglelefteq_\tau L'$ then $K + L \trianglelefteq_\tau K' + L'$.
- (3) If K_i is a τ -large submodule of a submodule L_i of M for all i in an index set I , then $\sum_{i \in I} K_i$ is τ -large in $\sum_{i \in I} L_i$.
- (4) If $K \cap K' \trianglelefteq_\tau K$, $K \cap K' \trianglelefteq_\tau K'$ and $L \cap L' \trianglelefteq_\tau L$, $L \cap L' \trianglelefteq_\tau L'$, for submodules K, K', L, L' of M , then $(K + L) \cap (K' + L') \trianglelefteq_\tau K + L$ and $(K + L) \cap (K' + L') \trianglelefteq_\tau K' + L'$.
- (5) If $K \cap L \trianglelefteq_\tau L$ then $K \trianglelefteq_\tau K + L$.

Proof. (1) \Rightarrow (2) Let H denote the τ -closure of $K + L$ in M . Let G be a τ -closure of K in H . By Proposition 2.12, it follows that G is τ -closed in M . By Lemma 3.4, we get that $K' \subseteq G$ and so $K' \subseteq H$. Similarly $L' \subseteq H$. Since $K' + L' \subseteq H$ and $K + L \trianglelefteq_\tau H$, we get that $K + L \trianglelefteq_\tau K' + L'$.

(2) \Rightarrow (3) Let $(\sum_{i \in I} K_i) \cap W \subseteq \tau(\sum_{i \in I} L_i)$ for $W \leq \sum_{i \in I} L_i$. Let $x \in W$. There exists a finite subset F of I and $x_i \in L_i$ such that $x = \sum_{i \in F} x_i$. Then $xR \cap (\sum_{i \in F} K_i) \subseteq \tau(\sum_{i \in F} L_i)$. By (2), $\sum_{i \in F} K_i \trianglelefteq_\tau \sum_{i \in F} L_i$. Hence $xR \subseteq \tau(\sum_{i \in F} L_i)$ and $x \in \tau(\sum_{i \in I} L_i)$. This shows that $\sum_{i \in I} K_i \trianglelefteq_\tau \sum_{i \in I} L_i$.

(3) \Rightarrow (4) Let $N = (K \cap K') + (L \cap L')$. Then we get $N \trianglelefteq_\tau K + L$ and $N \trianglelefteq_\tau K' + L'$. Since $N \subseteq (K + L) \cap (K' + L')$, we get that $(K + L) \cap (K' + L') \trianglelefteq_\tau K + L$ and $(K + L) \cap (K' + L') \trianglelefteq_\tau K' + L'$ by Proposition 2.1.

(4) \Rightarrow (5) Suppose that $K \cap L \trianglelefteq_\tau L$. Since, $K \cap L \trianglelefteq_\tau K \cap L$ and $K \cap K \trianglelefteq_\tau K$, we get that $(K \cap L) + K = K \trianglelefteq_\tau K + L$ by (4).

(5) \Rightarrow (1) Let K and K' be τ -closures of a submodule N of M . Then $K \cap K' \trianglelefteq_\tau K'$ and so $K \trianglelefteq_\tau K + K'$ and hence $K = K + K'$. Thus $K' \subseteq K$. Similarly $K \subseteq K'$ and hence this shows that M is a τ -UC module. \square

Theorem 3.6 *The following statements are equivalent for a module M .*

- (1) M is a τ -UC module.
- (2) If $K \subseteq L$ are submodules of M and K' is a τ -closure of K in M , then there exists a τ -closure L' of L in M such that $K' \subseteq L'$.
- (3) If K is a τ -closed submodule of M then $K \cap N$ is a τ -closed submodule of N , for every submodule N of M .
- (4) If K and L are τ -closed submodules of M then $K \cap L$ is a τ -closed submodule of M .

Proof. (1) \Rightarrow (2) By Proposition 3.5 we get that $L = K + L \leq_{\tau} K' + L$. Let L' be a τ -closure of $K' + L$ in M . Then L' is a τ -closure of L in M and $K' \subseteq L'$.

(2) \Rightarrow (3) Let L be a τ -closure of $K \cap N$ in N and L' be a τ -closure of L in M . Then L' is a τ -closure of $K \cap N$ in M . Since K is τ -closed in M we have $L' \subseteq K$ by (2). Hence $L \subseteq K \cap N$ and so $K \cap N = L$.

(3) \Rightarrow (4) Suppose that K and L are τ -closed submodules of M . By (3), $K \cap L$ is τ -closed in L . By Proposition 2.12, it follows that $K \cap L$ is τ -closed in M .

(4) \Rightarrow (1) Let K and L be τ -closures of a submodule N of M . Then $N \leq_{\tau} K \cap L$. By Proposition 2.1 we get that $K \cap L \leq_{\tau} K$ and $K \cap L \leq_{\tau} L$. By (4), $K \cap L$ is τ -closed in M . Hence $K \cap L = K = L$ and so M is a τ -UC module. \square

Theorem 3.7 *M is a τ -UC module if and only if every submodule of M is a τ -UC module.*

Proof. First suppose that M is τ -UC module. Let $N \leq M$ and $K \leq N$. Suppose that $K \leq_{\tau} T \leq_{\tau-c} N$ and $K \leq_{\tau} T' \leq_{\tau-c} N$. By Proposition 3.5, we get that $K \leq_{\tau} T + T'$. By Proposition 2.1-(3), $T \leq_{\tau} T + T'$ and $T' \leq_{\tau} T + T'$. Thus we get that $T + T' = T = T'$.

Conversely suppose that every submodule of M is a τ -UC module. Let $K, L \leq M$, $K \leq_{\tau} K' \leq M$ and $L \leq_{\tau} L' \leq M$. Let $W \leq K' + L'$ and $W \cap (K + L) \subseteq \tau(K' + L')$. Take an element $x = y + z \in W$ where $y \in K'$ and $z \in L'$. Let $N = yR + zR$. Note that $K \cap N \leq_{\tau} K' \cap N$ and $L \cap N \leq_{\tau} L' \cap N$. Since N is τ -UC we get that $K \cap N + L \cap N \leq_{\tau} K' \cap N + L' \cap N$ by Proposition 3.5. Since $xR \cap (K \cap N + L \cap N) \subseteq \tau(K' + L') \cap (K' \cap N + L' \cap N) = \tau(K' \cap N + L' \cap N)$ we have $xR \subseteq \tau(K' \cap N + L' \cap N) \subseteq \tau(K' + L')$. Hence $x \in \tau(K' + L')$ and so $W \subseteq \tau(K' + L')$. It follows that $K + L \leq_{\tau} K' + L'$. By Proposition 3.5, M is a τ -UC module. \square

Proposition 3.8 *If M is a τ -UC module then M/K is a UC module for every $K \leq_{\tau-c} M$.*

Proof. Let $K \leq L \leq M$, $K \leq N \leq M$ such that $L/K \leq_{\tau-c} M/K$ and $N/K \leq_{\tau-c} M/K$. By Lemma 2.10, $L \leq_{\tau-c} M$ and $N \leq_{\tau-c} M$. By Theorem 3.6, it follows that $N \cap L \leq_{\tau-c} M$ and hence by Lemma 2.10, we obtain that $(N \cap L)/K = (N/K) \cap (L/K) \leq_{\tau-c} M/K$ and so M/K is a τ -UC module by Theorem 3.6. Since M/K is τ -torsion-free, it is a UC module. \square

Definition 3.9 *A submodule K of a module M is called τ -R-closed if $K \cap mR$ is not τ -large in mR for all $m \in M \setminus K$. We write $K \leq_{\tau-r} M$ to denote that K is a τ -R-closed submodule of M .*

Lemma 3.10 (1) If $K \leq_{\tau-r} M$, then $K \leq_{\tau-c} M$.

(2) If $K_i \leq_{\tau-r} M$ ($i \in I$), then $\bigcap_{i \in I} K_i \leq_{\tau-r} M$.

Proof. (1) Let $K \leq_{\tau-r} M$. Suppose that $K \not\leq_{\tau} L \leq M$. Let $x \in L$. Then $K \cap xR \not\leq_{\tau} xR$ by Proposition 2.2 and so $x \in K$. Thus $L = K$.

(2) Let $K = \bigcap_{i \in I} K_i$ and $m \in M \setminus K$. Then $m \notin K_i$ for some $i \in I$. By hypothesis, $K_i \cap mR \not\leq_{\tau} mR$ and so $K \cap mR \not\leq_{\tau} mR$. It follows that $K = \bigcap_{i \in I} K_i \leq_{\tau-r} M$. \square

Proposition 3.11 The following statements are equivalent for a module M .

(1) M is a τ -UC module.

(2) If $K \leq_{\tau-c} M$ then $K \leq_{\tau-r} M$.

(3) If $K_i \leq_{\tau-c} M$ ($i \in I$) then $\bigcap_{i \in I} K_i \leq_{\tau-c} M$.

Proof. (1) \Rightarrow (2) Suppose that M is a τ -UC module. Let $K \leq_{\tau-c} M$ and $m \in M \setminus K$. Then $K \not\leq_{\tau} K + mR$. By Proposition 3.5 (1) \Rightarrow (5), we get that $K \cap mR \not\leq_{\tau} mR$ and so $K \leq_{\tau-r} M$.

(2) \Rightarrow (3) By Lemma 3.10 (1), (2).

(3) \Rightarrow (1) By (4) \Rightarrow (1) of Theorem 3.6. \square

Corollary 3.12 M is a τ -UC module if and only if the following condition (*) holds:

(*) If $K_i \leq_{\tau-c} L_i \leq M$ ($i \in I$) then $\bigcap_{i \in I} K_i \leq_{\tau-c} \bigcap_{i \in I} L_i$.

Proof. First suppose that M is a τ -UC module. Let $K_i \leq_{\tau-c} L_i \leq M$ ($i \in I$) and $L = \bigcap_{i \in I} L_i$. Note that L and L_i ($i \in I$) are all τ -UC modules by Theorem 3.7. By (1) \Rightarrow (3) of Theorem 3.6, we get that $K_i \cap L \leq_{\tau-c} L$ ($i \in I$) and hence $\bigcap_{i \in I} K_i = \bigcap_{i \in I} (K_i \cap L) \leq_{\tau-c} L$ by Proposition 3.11. The converse is obtained by Theorem 3.6. \square

Proposition 3.13 M is a τ -UC module if and only if for each submodule N of M , $N_{\tau}^* = \{m \in M : N \cap mR \leq_{\tau} mR\}$ is a submodule of M . In this case, $N_{\tau}^* = \{m \in M : N \leq_{\tau} (N + mR)\}$ and N_{τ}^* is the unique τ -closure of N .

Proof. Suppose that M is a τ -UC module. Let $N \leq M$. By (1) \Rightarrow (5) of Proposition 3.5, $m \in N_{\tau}^*$ if and only if $N \leq_{\tau} N + mR$. Let $m_1, m_2 \in N_{\tau}^*$ and $r \in R$. Then $N \leq_{\tau} N + m_1R$ and $N \leq_{\tau} N + m_2R$. By (1) \Rightarrow (2) of Proposition 3.5, we get that $N \leq_{\tau} N + m_1R + m_2R$. and so $N \leq_{\tau} N + (m_1 + m_2r)R$ by Proposition 2.1-(4). Hence $m_1 + m_2r \in N_{\tau}^*$ and then $N_{\tau}^* \leq M$.

Conversely suppose that $N_{\tau}^* \leq M$ for any $N \leq M$. Clearly $N \leq N_{\tau}^*$. We claim that $N \leq_{\tau} N_{\tau}^*$. Let $W \leq N_{\tau}^*$ and $W \cap N \subseteq \tau(N_{\tau}^*)$. Take $m \in W$. We have $N \cap mR \leq_{\tau} mR$ and so $(N \cap mR) \cap mR \subseteq \tau(N_{\tau}^*) \cap mR = \tau(mR)$. It follows that $mR = \tau(mR)$ and $m \in \tau(N_{\tau}^*)$ and hence $N \leq_{\tau} N_{\tau}^*$. Suppose that $N \leq_{\tau} K \leq_{\tau-c} M$. Let $k \in K$. Then $N \cap kR \leq_{\tau} kR$ by Proposition 2.2-(1) and so $k \in N_{\tau}^*$. Thus $N \leq K \leq N_{\tau}^*$. We get that $K \leq_{\tau} N_{\tau}^*$ by Proposition 2.1. It follows that $K = N_{\tau}^*$. Thus N_{τ}^* is the unique τ -closure of N . \square

A module M is called an **extending** module if every closed submodule is a direct summand of M . Various properties of extending modules can be found in [4]. In [2] the authors introduced the concept of

τ -extending module which is a generalization of extending module. A module M is said to be τ -**extending** if every τ -closed submodule of M is a direct summand of M . It is easily seen that M is τ -extending if and only if every submodule of M is τ -large in a direct summand of M .

Lemma 3.14 [2, Lemma 3.2] *Any direct summand of a τ -extending module is τ -extending.*

Lemma 3.15 *Let $M = A \oplus B$ for some submodules A, B of M . If $A \trianglelefteq_{\tau} M$ then $B \subseteq \tau(M)$.*

Proof. Since $A \trianglelefteq_{\tau} M$, $A \cap B = 0 \trianglelefteq_{\tau} B$. Thus $B \subseteq \tau(M)$. □

The following theorem is a generalization of [7, Theorem 3.1].

Theorem 3.16 *Let M be a τ -UC module such that $M = \bigoplus_{i \in I} M_i$ is the direct sum of modules M_i ($i \in I$), for some non-empty index set I . Then the following statements are equivalent.*

- (1) M is τ -extending.
- (2) There exists $i \in I$ such that M_i is τ -extending and every τ -closed submodule K of M with $K \cap M_i \subseteq \tau(M)$ is a direct summand.
- (3) There exists $i \in I$ such that M_i is τ -extending and every τ -complement of M_i in M is a τ -extending module and a direct summand of M .
- (4) The module M_i is τ -extending for each $i \in I$ and every τ -closed submodule L of M with $L \cap M_i \subseteq \tau(M)$ ($i \in I$) is a direct summand of M .

Proof. (1) \Rightarrow (2) It is clear by Lemma 3.14 and the hypothesis.

(2) \Rightarrow (3) Let L be a τ -complement of M_i in M . It follows that L is a direct summand of M . Let N be a τ -closed submodule of L . By Proposition 2.12, we get that N is a direct summand of M , and hence also of L . Thus L is τ -extending.

(3) \Rightarrow (1) Let H be a τ -closed submodule of M . By Theorem 3.6, we get that $H \cap M_i$ is a τ -closed submodule of M_i . By (3), $H \cap M_i$ is a direct summand of M_i and hence also of M . Thus $M = (H \cap M_i) \oplus H'$ for some $H' \leq M$. Now $H = (H \cap M_i) \oplus (H \cap H')$ and there exists a τ -closed submodule K of H such that $H \cap H' \trianglelefteq_{\tau} K$. It is easily seen that $K = (H \cap H') \oplus (K \cap H \cap M_i)$. We have $K \cap M_i = (K \cap H \cap M_i) \oplus (H \cap H' \cap M_i) = K \cap H \cap M_i$. By Lemma 3.15, it follows that $K \cap M_i \subseteq \tau(M)$. By Zorn's Lemma, there exists a τ -complement L of M_i in M such that $K \leq L$. Moreover K is τ -closed in M and hence K is τ -closed in L by Proposition 2.12. Applying (3), we see that K is a direct summand of the τ -extending module L and L is a direct summand of M . Hence K is a direct summand of M and so $H \cap H'$ is a direct summand of M . It follows that H is a direct summand of M .

(1) \Rightarrow (4) By Lemma 3.14.

(4) \Rightarrow (1) Let P be a τ -closed submodule of M . By Theorem 3.6, we get that $P \cap M_i$ is a τ -closed submodule of M_i for each $i \in I$. As M_i is τ -extending, it follows that $M_i = (P \cap M_i) \oplus M'_i$ for some $M'_i \leq M_i$. Let $M' = \bigoplus_{i \in I} M'_i$, $P' = \bigoplus_{i \in I} (P \cap M_i)$. Then $M = P' \oplus M'$, $P' \leq P$ and so $P = P' \oplus (P \cap M')$. Let K be a τ -closed submodule of P such that $P \cap M' \trianglelefteq_{\tau} K$. Then $K = (P \cap M') \oplus (K \cap P')$ and so $K \cap P' \subseteq \tau(M)$ by Lemma 3.15.

We claim that $K \cap M_i \subseteq \tau(M)$. Take $x \in K \cap M_i$ and write $x = y + z$, where $y \in P \cap M'$, $z \in K \cap P'$. Then $x - y = z \in M_i + P \cap M'$. Since $K \cap P' \subseteq \tau(M)$ and $M_i \cap P \cap M' = 0$ we get that

$(0 : z) = (0 : x - y) = (0 : x) \cap (0 : -y) \in F_\tau(R)$. It follows that $(0 : x) \in F_\tau(R)$ and so $x \in \tau(M)$. Thus $K \cap M_i \subseteq \tau(M)$ and K is τ -closed by Proposition 2.12. By (4), K is a direct summand of M . It follows that $P \cap M'$ is a direct summand of M' . Thus P is a direct summand of M . \square

Finally we give the following theorem to sum up our results about τ -UC modules.

Theorem 3.17 *The following statements are equivalent for a module M .*

- (1) M is a τ -UC module.
- (2) If $K \trianglelefteq_\tau K'$ and $L \trianglelefteq_\tau L'$, for submodules K, K', L, L' of M , then $K + L \trianglelefteq_\tau K' + L'$.
- (3) If K_i is a τ -large submodule of a submodule L_i of M for all i in an index set I , then $\sum_{i \in I} K_i$ is τ -large in $\sum_{i \in I} L_i$.
- (4) If $K \cap K' \trianglelefteq_\tau K$, $K \cap K' \trianglelefteq_\tau K'$ and $L \cap L' \trianglelefteq_\tau L$, $L \cap L' \trianglelefteq_\tau L'$, for submodules K, K', L, L' of M , then $(K + L) \cap (K' + L') \trianglelefteq_\tau K + L$ and $(K + L) \cap (K' + L') \trianglelefteq_\tau K' + L'$.
- (5) If $K \cap L \trianglelefteq_\tau L$ for submodules K, L of M , then $K \trianglelefteq_\tau K + L$.
- (6) If $K \subseteq L$ are submodules of M and K' is a τ -closure of K in M , then there exists a τ -closure L' of L in M such that $K' \subseteq L'$.
- (7) If K is a τ -closed submodule of M , then $K \cap N$ is a τ -closed submodule of N , for every submodule N of M .
- (8) If K and L are τ -closed submodules of M , then $K \cap L$ is a τ -closed submodule of M .
- (9) The intersection of any collection of τ -closed submodules of M is τ -closed.
- (10) If $K_\lambda \subseteq L_\lambda$ ($\lambda \in \Lambda$) are submodules of M such that K_λ is τ -closed in L_λ for all λ in Λ , then $\bigcap_{\lambda \in \Lambda} K_\lambda$ is τ -closed in $\bigcap_{\lambda \in \Lambda} L_\lambda$.
- (11) Every submodule of M is a τ -UC module.
- (12) Every τ -closed submodule of M is τ - R -closed.
- (13) For every submodule N of M , $N_\tau^* = \{m \in M : N \cap mR \trianglelefteq_\tau mR\}$ is a submodule of M .

In this case, for each submodule N of M , $N_\tau^* = \{m \in M : N \trianglelefteq_\tau N + mR\}$ and N_τ^* is the unique τ -closure of N .

Acknowledgement

The authors thank referees for their comments and suggestions to improve this note.

The first author thanks the Scientific Technological Research Council of Turkey (TUBITAK) for the financial support. The authors were supported by the Scientific Research Project Administration of Akdeniz University.

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Received: 14.09.2010