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On injective and subdirectly irreducible S-acts over left zero semigroups

Gholamreza Moghaddasi

Abstract

The aim of this paper is to characterize subdirectly irreducible S-acts over left zero semigroups. Also we compute the number of such acts and specify cogenerators acts over left zero semigroups. To do these we first take another look at the description of injective hulls of the separated S-acts over left zero semigroups.

Key Words: S-act, separated, injective, subdirectly irreducible, left zero semigroup

1. Preliminaries

Recall that, for a semigroup S, a (right) S-*act* (or S-*system*) is a set A together with a function $\alpha : A \times S \to A$, called the *action* of S (or the S-action) on A, such that for $a \in A$ and $s, t \in S$ (denoting $\alpha(a, s)$ by as), $a(st) = (as)t$, and if S is a monoid, with 1 as its identity, $a1 = a$.

An S-act A is called *separated* if for $a \neq b$ in A there exists an $s \in S \setminus \{1\}$ with $as \neq bs$.

A *homomorphism* $f : A \to B$ between S-acts is a function such that for each $a \in X$, $s \in S$, $f(as) = f(a)s$.

We denote the category of all (right) S-acts and homomorphisms between them by **Act-S**.

An element a of an S-act A is called a *fixed* or *zero* element if $as = a$ for all $s \in S$. We denote the set of all fixed elements of an S-act A by Fix$A$, which is in fact a subact of A. We will see that the set Fix$A$ plays an important role for acts over left zero semigroups.

Notice that for a given S-act A, the set $(\text{Fix}A)^S$ of all functions from S to Fix$A$ is an S-act. In fact, using the notation $(a_s)_{s\in S}$ for a given element of $(\text{Fix}A)^S$, $(a_s)_{s\in S} \cdot t$ is defined to be the constant family $(a_t)_{s\in S}$, for $t \in S$.

Recall that a semigroup S all whose elements is a left zero of S is called a left zero semigroup. This class of semigroups is important and useful since: 1) Every non empty set S can be turned into a left zero semigroup by defining $st = s$ for all $s, t \in S$. 2) The definition of rectangular semigroups and groups are based on the left zero semigroups. 3) This semigroup is applied in automata theory, theory of computation, Boolean algebra.

The following lemma is easily proved.

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Lemma 1.1 *Let S be a semigroup. Then the following are equivalent:*

(i) *S is left zero semigroup.*

(ii) *Considering S as a right S-system, every element of S is a zero.*

(iii) *For every S-system A, $AS \subseteq FixA$, where $AS = \{as : a \in A, s \in S\}$.*

So, we see that every nonempty act over a left zero semigroup has at least one fixed element. Also it is easy to see that for a left zero semigroup S there exist no separated S-act with at least two elements and only one fixed element.

Definition 1.2 An S-act B containing (an isomorphic copy of) an S-act A as a subact is called an *extension* of A. The S-act A is said to be a *retract* of B if there exists a homomorphism $f : B \to A$ such that $f\,|_A = id_A$, in which case f is said to be a *retraction.* The S-act A is called *absolute retract* if it is a retract of each of its extensions.

An S-act A is said to be *injective* if for every monomorphism (one-one homomorphism) $h : B \to C$ and every homomorphism $f : B \to A$ there exists a homomorphism $g : C \to A$ such that $g \circ h = f$.

A monomorphism $h : A \to B$ is called *essential* if any homomorphism $f : B \to C$ is a monomorphism whenever fh is a monomorphism.

An extension B of an S-act A is called an *injective hull* of A if it is an essential extension of A which is also injective.

2. Injective hulls

It is known (see [2]) that the injective hull of an act over an arbitrary semigroup S exists. In [4] we characterize injective S-acts over left zero semigroups and construct their injective hulls. In this section we take another look at the description of injective hulls of the separated S-acts over left zero semigroups.

We have the following result from [5].

Lemma 2.1 *For any semigroup S, in the category of S-acts, pushouts preserve monomorphisms.*

One can now use the above lemma and the results of Banaschewki (see [1]) to get the following:

Theorem 2.2 *Let S be an arbitrary semigroup. For an act A over S, we have:*

(1) *A is injective if and only if A is absolute retract, if and only if A has no proper essential extensions.*

(2) *B is an injective hull of A if and only if B is a maximal essential extension of A, if and only if B is a minimal injective extension of A.*

Now, considering the S-act $(\mathrm{Fix}A)^S$, we give a characterization theorem of injective S-acts over left zero semigroups.

Theorem 2.3 *Let S be a left zero semigroup. Then an S-act A is injective if and only if the homomorphism $\varphi : A \to (FixA)^S$ with $\varphi(a) = (as)_{s\in S}$ is onto.*

Proof. Let A be injective and suppose that φ is not onto. Then there exists an element $(a_s)_{s\in S}$ of $(\mathrm{Fix}A)^S$ such that for all $a \in A$ there exists $t \in S$ with $at \neq a_t$. Consider the extension $A \cup \{(a_s)_{s\in S}\}$ of A, and define

$(a_s)_{s\in S}\cdot t = a_t$, for $t \in S$. Since A is injective, there exists a homomorphism $f : A \cup \{(a_s)_{s\in S}\} \to A$ such that $f\mid_A$ is the identity map. Then taking $f((a_s)_{s\in S}) = a$ we have

$$at = f((a_s)_{s\in S}.t) = f(a_t) = a_t,$$

which is a contradiction.

Conversely, let φ be onto. Then, using Theorem 2.2, we show that A is absolute retract. Let B be an extension of A and a_0 be a fixed element of A. Define a retraction $g : B \to A$ by $g\mid_A = id_A$ and for $b \in B - A$,

$$g(b) = \begin{cases} a_0 & \text{if } bS \subseteq B - A, \\ a_1 & \text{if } bS \subseteq A, \\ a_2 & \text{otherwise,} \end{cases}$$

where a_1 is a preimage of the element $(bs)_{s\in S}$ of $(\mathrm{Fix}A)^S$, for a given $b \in B$ with $bS \subseteq A$, and, $a_2 \in A$ is a preimage of $(a_s)_{s\in S}$ for a given $b \in B$ with $bS \cap A \neq \emptyset$ and $bS \cap (B - A) \neq \emptyset$, where $a_s = bs$ for all $s \in S$ with $bs \in A$, and $a_s = a_0$ for all $s \in S$ with $bs \notin A$.

Now, g is homomorphism. This is because, for $b \in B$, if $bS \subseteq B - A$, then for all $s \in S$, $bsS \subseteq B - A$ and so $g(bs) = a_0 = a_0 s = g(b)s$. Also, if $bS \subseteq A$, then $bsS \subseteq A$ and so $g(bs) = bs = a_1 s = g(b)s$. Finally, if $bS \cap A \neq \emptyset$, $bS \cap (B - A) \neq \emptyset$ then for $s \in S$ with $bs \in A$ we have $g(bs) = bs = a_2 s = g(b)s$ and for $s \in S$ with $bs \in B - A$ we have $g(bs) = a_0 = a_2 s = g(b)s$. □

Corollary 2.4 *Let S be a left zero semigroup. If an S-act A has only one fixed element then it is injective.*

Corollary 2.5 *Let S be a left zero semigroup. For every S-act A, $(FixA)^S$ is an injective S-act.*

Proof. We have

$$\begin{aligned} \mathrm{Fix}(\mathrm{Fix}A)^S &= \{(a_s)_{s\in S} : a_s \in \mathrm{Fix}A, (a_s)_{s\in S}\cdot t = (a_s)_{s\in S} \forall t \in S\} \\ &= \{(a_s)_{s\in S} : a_s \in \mathrm{Fix}A, a_t = a_s \forall s, t \in S\} \\ &= \{(a_0)_{s\in S} : a_0 \in \mathrm{Fix}A\} \end{aligned}$$

So, the elements of $(\mathrm{Fix}(\mathrm{Fix}A)^S)^S$ are of the form $((a_0)_{s\in S})_{t\in S}$, for a fixed element a_0 of A, which has clearly the preimage $(a_0)_{s\in S}$ in $(\mathrm{Fix}A)^S$ under $\phi : \mathrm{Fix}A)^S \to (\mathrm{Fix}(\mathrm{Fix}A)^S)^S$. □

The injective S-acts obtained in the above corollary, in fact, characterize all injective separated S-acts over left zero semigroups.

Theorem 2.6 *Let S be a left zero semigroup. A separated S-act A is injective if and only if it is a retract of $(FixA)^S$.*

Proof. The "only if" part is clear by the above corollary and the fact that the retracts of injective acts are injective. To prove the "if part", notice that the homomorphism $\varphi : A \to (\mathrm{Fix}A)^S$, given by $\varphi(a) = (as)_{s\in S}$, is one-one if (and only if) A is separated. So, if A is an injective separated S-act, the monomorphism φ has to have a left inverse which means that A is a retract of $(\mathrm{Fix}A)^S$. □

Now, applying the above theorem, we get the following result on injective hulls of separated S-acts over left zero semigroups.

Theorem 2.7 *Let S be a left zero semigroup. If A is a separated S-act then $(FixA)^S$ is the injective hull of A.*

Proof. By Corollary 2.5, $(\mathrm{Fix}A)^S$ is injective. Also, as we mentioned in the proof of the above theorem, $\varphi : A \to (\mathrm{Fix}A)^S$ is a monomorphism. So, it is enough to prove that φ is an essential monomorphism. Let B be an S-act, and $f : (\mathrm{Fix}A)^S \to B$ be a homomorphism such that $f\circ\varphi$ be one-one. Let $(a_s)_{s\in S}, (a'_s)_{s\in S} \in (\mathrm{Fix}A)^S$ be such that $f((a_s)_{s\in S}) = f((a'_s)_{s\in S})$. Then for every $t \in S$, $f((a_s)_{s\in S} \cdot t) = f((a'_s)_{s\in S} \cdot t)$, that is $f(a_t)_{s\in S} = f(a'_t)_{s\in S}$. This means $f \circ \varphi(a_t) = f \circ \varphi(a'_t)$, for all $t \in S$. But $f \circ \varphi$ is one-one, so $a_t = a'_t$, for all $t \in S$. Hence $(a_s)_{s\in S} = (a'_s)_{s\in S}$. □

Corollary 2.8 *Let S be a left zero semigroup. Every S-act A with $|A| > |(FixA)^S|$ is not separated.*

Proof. If A is separated, since $(\mathrm{Fix}A)^S$ is its injective hull, $A \subseteq (\mathrm{Fix}A)^S$ and therefore $|A| \leq |(\mathrm{Fix}A)^S|$ which is a contradiction. □

Remark: For a left zero semigroup S, let n be any finite number and F_n be the class of all S-acts with n fixed elements. In any class F_n there exists only one separated injective S-act, up to isomorphism. Since for $A \in F_n$, $(\mathrm{Fix}A)^S$, the injective hull of A, has n fixed elements, by the proof of Corollary 2.5. So, if there exists a separated injective S-act B with n fixed elements then $B \simeq (\mathrm{Fix}B)^S$. But $(\mathrm{Fix}A)^S \simeq (\mathrm{Fix}B)^S$, by the isomorphism $(a_s)_{s\in S} \mapsto (b_s)_{s\in S}$, where $b_s = \psi(a_s)$ and ψ is an isomorphism $\mathrm{Fix}(A) \to \mathrm{Fix}(B)$. Therefore $B \simeq (\mathrm{Fix}A)^S$; that is $(\mathrm{Fix}A)^S$ is the unique (up to isomorphism) separated injective S-act in F_n.

Finally we count the number of separated S-acts with two fixed elements that yield the number of all subdirectly irreducible S-acts in the next section.

Theorem 2.9 *Let S be a left zero semigroup with $|S| = n$. The number of all non isomorphic separated S-acts with two fixed elements is 2^{2^n-2}.*

Proof. For a separated S-act A with $\mathrm{Fix}A = \mathbf{2} = \{a_0, b_0\}$, by Theorem 2.7, $\mathbf{2}^S$ is its injective hull, and by the proof of Corollary 2.5, $\mathrm{Fix}(\mathbf{2}^S) = \{(a_0)_{s\in S}, (b_0)_{s\in S}\}$. Therefore $\mathbf{2}^S$ has $2^n - 2$ non fixed elements. Therefore the cardinality of $\mathbf{2}^S - \mathrm{Fix}(\mathbf{2}^S)$ is $2^n - 2$, and the cardinality of $\mathcal{E} = \mathcal{P}(\mathbf{2}^S - \mathrm{Fix}(\mathbf{2}^S))$ is 2^{2^n-2}. But there is clearly a one-one correspondence between $\mathcal{E}$ and

$$\mathcal{D} = \{D | D = C \cup \{(a_0)_{s\in S}, (b_0)_{s\in S}\},\ C \subseteq \mathbf{2}^S - \{(a_0)_{s\in S}, (b_0)_{s\in S}\}\}.$$

Also, it is easy to see that there is a one-one correspondence between $\mathcal{D}$ and the set $\mathcal{B}$ of all (non isomorphic) separated S-acts with two fixed elements. This yields that $|\mathcal{B}| = 2^{2^n-2}$. □

3. Subdirectly irreducible

An equivalence relation ρ on an S-act A is called a congruence on A, if $a\rho a'$ implies $(as)\rho(a's)$ for $a, a' \in A$, $s \in S$. We denote by $Con(A)$ the set of all congruence on A. For $a, b \in A$, $\rho_{a,b}$ denote the smallest congruence on A containing (a,b). It is in fact, the equivalence relation generated by $\{(as,bs) : s \in S \cup \{1\}\}$. Its elements are as follows: $x\rho y \Leftrightarrow \exists s_1, s_2, ..., s_n \in S \cup \{1\}$, $p_1, p_2, ..., p_n, q_1, q_2, ..., q_n \in A$ such that

$$x = p_1 s_1 \qquad q_2 s_2 = q_3 s_3 \qquad ......q_n s_n = y$$
$$q_1 s_1 = p_2 s_2 \qquad q_3 s_3 = p_4 s_4 \; ....$$

where $(p_i, q_i) = (as, bs)$ or $(q_i, p_i) = (as, bs)$ for some $s \in S \cup \{1\}$.

Recall that a right S-act A is subdirectly irreducible if $\bigcap_{i\in I} \rho_i \neq \Delta$ for every family of congruences ρ_i on A with $\rho_i \neq \Delta$ (see [7]). Also recall that a right S-act A is simple if Δ and $\bigtriangledown$ are the only congruences of A (see [3]).

Notice that for each semigroup S and every S-act A with $|A| = 2$ there exist only two congruences Δ and $\bigtriangledown$ on A. Therefore these S-acts are subdirectly irreducible and are also simple.

Lemma 3.1 (1) *For any semigroup S, every S-act with at least three fixed elements (in particular every S-act with trivial action and at least three elements) is Subdirectly reducible.*

(2) *For a left zero semigroup S, every S-act A with $|A| > 2$ and $|FixA| = 1$ is subdirectly reducible.*

Proof. (1) Let a, b, c be three different fixed elements of an S-act A. Then $\rho_{a,b} \cap \rho_{a,c} = (\Delta \cup \{(a,b),(b,a)\}) \cap (\Delta \cup \{(a,c),(c,a)\}) = \Delta$; that is, A is subdirectly reducible.

(2) Let a_0 be the only fixed element and b, c be two different and non fixed elements of A. We have $\rho_{a_0,b} \cap \rho_{b,c} = (\Delta \cup \{(a_0,b),(b,a_0)\}) \cap (\Delta \cup \{(b,c),(c,b)\}) = \Delta$; that is, A is subdirectly reducible. □

Now we characterize the subdirectly irreducible S-acts over left zero semigroups.

Theorem 3.2 *Let S be a left zero semigroup. An S-act A with $|A| > 2$ is subdirectly irreducible if and only if A is separated and $|FixA| = 2$.*

Proof. Let A be subdirectly irreducible. By the above lemma we have $|\text{Fix}A| = 2$, say $\text{Fix}A = \{a_0, b_0\}$. To prove separatedness, on the contrary, let $x \neq y \in A$ be such that $xs = ys$, for every $s \in S$. Then $\rho_{a_0,b_0} \cap \rho_{x,y} = \Delta$ which is a contradiction. Conversely let A be separated, $\text{Fix}A = \{a_0, b_0\}$, and θ be a nontrivial congruence on A. Then there exist $x \neq y \in A$ such that $(x,y) \in \theta$. Thus for every $s \in S$, we have $(xs, ys) \in \theta$ and $xs, ys \in \text{Fix}A = \{a_0, b_0\}$, also since A is separated, there exists $s \in S$ such that $xs \neq ys$. This means $(a_0,b_0),(b_0,a_0) \in \theta$. Therefore $\bigcap_{\Delta\neq\theta} \theta \supseteq \Delta \cup \{(a_0,b_0),(b_0,a_0)\}$, and hence A is subdirectly irreducible. □

Now by above theorem and Theorem 2.7, we get the characterization of the injective hulls of subdirectly irreducible S-acts over a left zero semigroup S.

Corollary 3.3 *Let S be a left zero semigroup. Then $\mathbf{2}^S$ is the injective hull of each subdirectly irreducible S-act A with $|A| > 2$.*

Note: Let S be a left zero semigroup. Notice that

(1) Every S-act A with $|A| = 2$ and the trivial actions is separated, so by Theorem 2.7, $(\text{Fix}A)^S = \mathbf{2}^S$ is its injective hull.

(2) Every S-act A with one or two elements and $|\text{Fix}A| = 1$ is injective, by Corollary 2.4, and hence its injective hulls is itself.

Corollary 3.4 *Let S be a left zero semigroup. An S-act A with $|A| > 2$ is subdirectly irreducible if and only if A is isomorphic to a subact of $\mathbf{2}^S$.*

Proof. Let A be subdirectly irreducible, by Corollary 3.3, $\mathbf{2}^S$ is the injective hull of A, and hence A is a subact of $\mathbf{2}^S$. Conversely, let A be a subact of $\mathbf{2}^S$. Then, since $|A| > 2$ and $\mathbf{2}^S$ is separated, A is separated and has two fixed elements. Therefore, by Theorem 3.2, A is subdirectly irreducible. □

Now using Birkhoff's Subdirect Representation Theorem for S-acts, we get the following result.

Theorem 3.5 *Let S be a left zero semigroup. Every S-act A can be embedded into a product of $\mathbf{2}^S$.*

In the next theorem we compute the number of subdirectly irreducible S-acts over a finite left zero semigroup S.

Theorem 3.6 *Let S be a left zero semigroup with $|S| = n$. Then there are $2^{2^n-2} + 2$ subdirectly irreducible S-acts.*

Proof. We know that the two fixed element act $\mathbf{2}$ is just separated S-act which has two elements. This together with Theorem 3.2 and Theorem 2.9 imply that the number of subdirectly irreducible S-acts A with $|A| > 2$ is $2^{2^n-2} - 1$. Also, computing the subdirectly irreducible S-acts A with $|A| \leq 2$ we get that there are (up to isomorphism) only three such acts: the singleton S-act $\{a\}$, the two fixed element act $\mathbf{2}$, and the two element act $\{a, b\}$ with one fixed element. In all, the number of subdirectly irreducible S-acts is $(2^{2^n-2} - 1) + 3 = 2^{2^n-2} + 2$. □

Recall that an S-act A is called θ-simple if it contain no subact other than A and the one element subact (see [7]).

Corollary 3.7 *Let S be a left zero semigroup. The only subdirectly irreducible θ-simple S-act is $\mathbf{2} = \{0, 1\}$ with trivial actions.*

Theorem 3.8 *Let S be a left zero semigroup. An S-act C is cogenerator if and only if it contains $\mathbf{2}^S$.*

Proof. By Theorem III.7.3 of [7], an S-act C is a cogenerator if and only if C contains the injective hull of every subdirectly irreducible θ-simple S-act. Now, applying Corollaries 3.7 and 3.3, we get the result. □

Corollary 3.9 *Let S be a left zero semigroup. Every cogenerator S-act has at least two fixed elements.*

There is another proof for the above corollary for an arbitrary semigroup S (see Proposition II.4.17 of [7]).

Theorem 3.10 *Let S be a finite left zero semigroup with at least two elements. Then the only subdirectly irreducible cogenerator S-act is $\mathbf{2}^S$.*

Proof. If K is a subdirectly irreducible cogenerator then, by Theorem 3.8, it contains $\mathbf{2}^S$. Also, by Theorem 3.2, K is separated and has only two fixed elements. Therefore, by Corollary 2.8, $|K| \leq |\mathbf{2}^S|$. So, by finiteness of S, and hence $\mathbf{2}^S$, $K = \mathbf{2}^S$. □

We close the paper by characterizing simple S-acts. Recall that an S-act A is called simple if $\mathrm{Con}A = \{\triangle, \nabla\}$. It is easy to check that every S-act A with $|A| \leq 2$ is simple but no S-act A with trivial action and $|A| > 2$ is simple.

Theorem 3.11 *For a left zero semigroup S, there exists no simple S-act A with $|A| > 2$.*

Proof. Let $a \neq b$ be elements of A. Then in the case where $a, b \in \mathrm{Fix}A$ we have $\rho_{a,b} \neq \nabla$, since $|A| > 2$, hence there exists $(a, b \neq) x \in A$ and $(a, x) \notin \rho_{a,b}$. Therefore $\rho_{a,b}$ is a non trivial congruence on A. Also in the case where one of a, b is not fixed, taking $a \notin \mathrm{Fix}A$, then $\rho_{a,b} \neq \nabla$. Since otherwise if $\rho_{a,b} = \nabla$ then for each $x \neq y \in A$, we have $(x, y) \in \rho_{a,b}$. Therefore there exist $s, t \in S$ such that $as = x$, $bt = y$. Thus $x, y \in \mathrm{Fix}A$, by Lemma 1.1. Thus $(a, x) \notin \rho_{x,y}$, and so $\rho_{x,y}$ is a non trivial congruence on A. □

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Gholamreza MOGHADDASI
Department of Mathematics
Sabzevar Tarbiat Moallem University
P.O. Box 397, Sabzevar-IRAN
e-mail: moghaddasimath@yahoo.com

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