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Generalization of some properties of Banach algebras to fundamental locally multiplicative topological algebras

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Abstract

In this article we generalized some properties of Banach algebras, to a new class of topological algebras namely fundamental and fundamental locally multiplicative topological algebras (abbreviated by FLM). Also the new notion of sub-multiplicatively metrizable topological algebra is given and some well known spectral properties of Banach algebras are generalized to such kind of algebras.

Key Words: FLM algebras, fundamental topological algebras, holomorphic function, multiplicative linear functionals, semi-simple algebras, spectral radius

1. Introduction

The notion of fundamental topological spaces (also algebras) has been introduced in [1] in 1990 extending the meaning of both local convexity and local boundedness.

A topological linear space $\mathcal{A}$ is said to be fundamental one if there exists $b > 1$ such that for every sequence (x_n) of $\mathcal{A}$, the convergence of $b^n(x_n - x_{n-1})$ to zero in $\mathcal{A}$ implies that (x_n) is Cauchy.

A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental. The famous Cohen factorization theorem for complete metrizable fundamental topological algebras is proved in [1] and the n^{th} roots and quasi square roots in fundamental topological algebras are studied in [4].

The fundamental locally multiplicative topological algebras (abbreviated by FLM) with a property very similar to the normed algebras is also introduced in [2]. A fundamental topological algebra is called locally multiplicative if there exists a neighborhood U_0 of zero such that, for every neighborhood V of zero, the sufficiently large powers of U_0 lie in V.

Also in [2] a topological structure is defined on the algebraic dual space of an FLM algebra to make it a normed space, and some of the famous theorems of Banach algebras are extended for complete metrizable FLM algebras. In this paper we have studied the linear multiplicative functionals on FLM algebras and proved some results on them in section 2. In section 3, we introduced the new notion of sub-multiplicative metrizable topological algebras and by using it we generalized some properties of Banach algebras to FLM algebras.

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2. Multiplicative functional on FLM algebras

A version of the *Gleason, Kahane-Zelazko theorem* is proved for FLM algebras in [3]. In theorem 5.5 of [3], $T : \mathcal{A} \longrightarrow \mathbb{C}$ is assumed a non-zero linear functional on $\mathcal{A}$ and proved that T is multiplicative if and only if $T(a) \in Sp(a)$ for all $a \in \mathcal{A}$. Now by replacing $\mathbb{C}$ by a semi-simple complete metrizable FLM algebra $\mathcal{B}$ we generalized it as follow.

Theorem 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative complete metrizable FLM algebras, with unit elements, and let $\mathcal{B}$ be semi-simple. If $T : \mathcal{A} \longrightarrow \mathcal{B}$ is a linear mapping, such that $Sp(Tx) \subset Sp(x)$, for any $x \in \mathcal{A}$, then T is a multiplicative mapping.*

Proof. Let f be a multiplicative and linear functional on $\mathcal{B}$ and put $F(x) = f(Tx)$ for any $x \in \mathcal{A}$. So F is a linear functional on $\mathcal{A}$, and also by theorem 5.5 [3],

$$F(x) = f(Tx) \in Sp(Tx) \subset Sp(x),$$

and so by using again theorem 5.5 [3] F is multiplicative and linear functional on $\mathcal{A}$. It follows that

$$F(xy) = F(x)F(y),$$

or

$$f(Txy) = f(Tx)f(Ty) = f(TxTy).$$

Since f is arbitrary multiplicative linear functional on $\mathcal{B}$ and $\mathcal{B}$ is semi-simple, thus T is multiplicative. $\square$

Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative complete metrizable FLM algebras, with unit elements $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. If T is a multiplicative linear mapping from $\mathcal{A}$ into $\mathcal{B}$, such that $Te_{\mathcal{A}} \neq e_{\mathcal{B}}$, then it may be $Sp(Tx)$ is not a subset of $Sp(x)$. Furthermore $Sp(x) \subset Sp(Tx)$. For example, let A_1 and A_2 be commutative complete metrizable FLM algebras, $B = A_1 \oplus A_2$ and $T : A_1 \longrightarrow B$. Then we have the following theorem.

Theorem 2.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative complete metrizable FLM algebras, with unit elements $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, respectively. Let T be a linear multiplicative mapping from $\mathcal{A}$ to $\mathcal{B}$, such that $Te_{\mathcal{A}} = e_{\mathcal{B}}$. Then for any $x \in \mathcal{A}$, $Sp(Tx) \subset Sp(x)$.*

Proof. By assumption we have

$$e_{\mathcal{B}} = Te_{\mathcal{A}} = Txx^{-1} = TxTx^{-1},$$

for any invertible element $x \in \mathcal{A}$. This shows that, for any such x, Tx is invertible in $\mathcal{B}$ and

$$T(x^{-1}) = (Tx)^{-1}.$$

Now if $\lambda \notin Sp(x)$, then $x - \lambda e_{\mathcal{A}}$ is invertible in $\mathcal{A}$ and so $T(x - \lambda e_{\mathcal{A}}) = Tx - \lambda e_{\mathcal{B}}$ is invertible in $\mathcal{B}$. Therefore $\lambda \notin Sp(Tx)$. $\square$

3. New results on FLM algebras

In this section, by introducing the new notion of sub-multiplicative metrizable topological algebra, we generalize some well known spectral properties of Banach algebras to complete metrizable FLM algebras.

By $\Omega_{\mathcal{A}}$ we mean the set of all elements $a \in \mathcal{A}$ such that $\rho(a) < 1$, where $\rho(a)$ is the spectral radius of $a \in \mathcal{A}$. We denote the center of topological algebra $\mathcal{A}$, by $Z(\mathcal{A})$, such that

$$Z(\mathcal{A}) = \{a \in \mathcal{A} : ax = xa, \quad \text{for all } x \in \mathcal{A}\}.$$

Definition 3.1 *Let $(\mathcal{A}, d)$ be a metrizable topological algebra. We say $\mathcal{A}$ is a sub-multiplicative metrizable topological algebra if*

$$d(0, xy) \leq d(0, x)d(0, y)$$

for each $x, y \in \mathcal{A}$.

It is clear that, when $\mathcal{A}$ is a sub-multiplicatively metrizable topological algebra, the meter $d_{\mathcal{A}}$ is not a discrete meter; for example, if $d_{\mathcal{A}}$ is a Dirac meter on some ideal E of $\mathcal{A}$, the sub-multiplicatively of the meter fails. For abbreviation we denote $d_{\mathcal{A}}(0, x)$ by $D_{\mathcal{A}}(x)$ for any $x \in \mathcal{A}$.

The following lemma is proved for Banach algebras and has a similar proof for FLM algebras (see theorem 3.2.6, [5]). Therefore, we remove its proof, because it is well known and clear.

Lemma 3.2 *Let $\mathcal{A}$ be a complete metrizable FLM algebra and $x \in A$. Then for every nonconstant polynomial P with complex coefficients we have*

$$Sp(P(x)) = P(Sp(x)).$$

Let $\mathcal{A}$ be a complete metrizable fundamental topological algebra with unit e and $x \in \mathcal{A}$. If for some $b > 1$, $b^n x^n \longrightarrow 0$ in $\mathcal{A}$, then $e - x$ is invertible and

$$(e - x)^{-1} = \sum_{n=0}^{\infty} x^n$$

and also if for some $b > 1$, $b^n(e - x)^n \longrightarrow 0$, then x is invertible (theorem 4.1, [2]). If $\mathcal{A}$ is a complete metrizable FLM algebra with meter $d_{\mathcal{A}}$, then $(e - x)$ is invertible for $d_{\mathcal{A}}(0, x) = D_{\mathcal{A}}(x) < 1$. Now if $\mathcal{A}$ is a complete metrizable FLM algebra with sub-multiplicatively meter $d_{\mathcal{A}}$ and $\lambda \neq 0$, then $(e - \lambda x)$ is invertible for $d_{\mathcal{A}}(0, x) = D_{\mathcal{A}}(x) < |\lambda|$.

Theorem 3.3 *Let $\mathcal{A}$ be a complete metrizable FLM algebra with sub-multiplicative meter $d_{\mathcal{A}}$. Then $\rho(x) = \lim_{n\to\infty} D_{\mathcal{A}}(x^n)^{1/n}$.*

Proof. From above discussion, we have $\rho(x) \leq D_{\mathcal{A}}(x)$, for any $x \in \mathcal{A}$. Now if applied the previous lemma to x^n, we have $\rho(x)^n \leq D_{\mathcal{A}}(x^n)$. Let f be a linear functional on $\mathcal{A}$, then the map from $\mathbb{C} \setminus Sp(x)$ to $\mathbb{C}$, which $\lambda \mapsto f((\lambda e - x)^{-1})$ is holomorphic. By theorem 4.1 of [2], we have

$$f((\lambda e - x)^{-1}) = \frac{1}{\lambda}(f(e) + \frac{f(x)}{\lambda} + \ldots + \frac{f(x^n)}{\lambda^n} + \ldots).$$

Fix λ, such that $|\lambda| > \rho(x)$. Then for every linear functional f on $\mathcal{A}$, we have $\sup_n |\frac{f(x^n)}{\lambda^n}| < \infty$. By applying the Banach-Steinhaus theorem (theorem 2.8, [7]) to the space of all continuous linear functional on $\mathcal{A}$ and to the sequence of T_n from that to $\mathbb{C}$, defined by $T_n(f) = \frac{f(x^n)}{\lambda^n}$, we conclude that there exists a constant C, depending to λ, such that $D_{\mathcal{A}}(x^n) \leq C|\lambda|^n$ for all $n \geq 1$. Then

$$\lim_{n \longrightarrow \infty} \sup D_{\mathcal{A}}(x^n)^{1/n} \leq |\lambda|,$$

for all $|\lambda| \geq \rho(x)$. Hence we conclude

$$\rho(x) \leq \lim_{n \longrightarrow \infty} \inf D_{\mathcal{A}}(x^n)^{1/n} \leq \lim_{n \longrightarrow \infty} \sup D_{\mathcal{A}}(x^n)^{1/n} \leq \rho(x),$$

therefore $\rho(x) = \lim_{n\to\infty} D_{\mathcal{A}}(x^n)^{1/n}$. □

Let $E(\mathcal{A})$ be the set of all elements $x \in \mathcal{A}$ for which $E(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$, can be defined. If $\mathcal{A}$ be a complete metrizable FLM algebra, then $E(\mathcal{A}) = \mathcal{A}$ (theorem 5.4, [3]).

The following theorem is a version of Zemánek theorem (theorem 5.3.1, [5]) for FLM algebras:

Theorem 3.4 *Let $\mathcal{A}$ be a complete metrizable FLM algebra with sub-multiplicatively meter $d_{\mathcal{A}}$. Then the following statements are equivalent:*

(i) a is in the Jacobson radical of $\mathcal{A}$;

(ii) $Sp(a+x) = Sp(x)$, for all $x \in \mathcal{A}$;

(iii) $\rho(a+x) = 0$, for all quasi-nilpotent elements x in $\mathcal{A}$;

(iv) $\rho(a+x) = 0$, for all quasi-nilpotent elements x in a neighborhood 0 in $\mathcal{A}$;

(v) there exists $C > 0$ such that $\rho(x) \leq CD_{\mathcal{A}}(x-a)$, for all $x \in \mathcal{A}$ in a neighborhood of a in $\mathcal{A}$.

Proof. Straightforward. □

Theorem 3.5 *Let $\mathcal{A}$ be a complete semi-simple metrizable FLM algebra with sub-multiplicative meter. If $g : \Omega_{\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}}$ be a holomorphic map satisfying $g(0) = 0$ and $g'(0) = I$, then $g(c) = c$ for all $c \in \Omega_{\mathcal{A}} \cap Z(\mathcal{A})$.*

To prove this theorem we need to the following lemma.

Lemma 3.6 *Let $\mathcal{A}$ be a complete metrizable FLM algebra with sub-multiplicative meter $d_{\mathcal{A}}$. Suppose that $x, y \in \mathcal{A}$ satisfy $xy = yx$. Then $\rho(x+y) \leq \rho(x) + \rho(y)$ and $\rho(xy) \leq \rho(x)\rho(y)$.*

Proof. Since $xy = yx$, then $(xy)^n = x^n y^n$ for each integer $n \geq 1$. By theorem 3.3, we have

$$\begin{aligned}\rho(xy) &= \lim_{n\to 0} D_{\mathcal{A}}((xy)^n)^{1/n} = \lim_{n\to 0} D_{\mathcal{A}}(x^n y^n)^{1/n} \\ &\leq \lim_{n\to 0} D_{\mathcal{A}}(x^n)^{1/n} \lim_{n\to 0} D_{\mathcal{A}}(y^n)^{1/n} \\ &= \rho(x)\rho(y).\end{aligned}$$

Let $\rho(x) < \alpha$, $\rho(y) < \beta$ and $a = x/\alpha$, $b = y/\beta$. Then $\rho(a) < 1$ and $\rho(b) < 1$. Therefore there exists some integer N such that for $n \geq N$, we have $\max(D_{\mathcal{A}}(a^{2^n}), D_{\mathcal{A}}(b^{2^n})) < 1$. Now let $\gamma_n = \max_{0\leq k\leq 2^n} D_{\mathcal{A}}(a^k)D_{\mathcal{A}}(b^{2^n-k})$, then we have

$$\begin{aligned}
D_{\mathcal{A}}((a+b)^{2^n})^{1/2^n} &= D_{\mathcal{A}}(\sum_{k=0}^{2^n}\left(\begin{array}{c}2^n\\ k\end{array}\right)x^k y^{2^n-k})^{1/2^n}\\
&\leq (\sum_{k=0}^{2^n}\left(\begin{array}{c}2^n\\ k\end{array}\right)\alpha^k\beta^{2^n-k}D_{\mathcal{A}}(a^k)D_{\mathcal{A}}(b^{2^n-k}))^{1/2^n}\\
&\leq (\alpha+\beta)\gamma_n^{1/2^n}.
\end{aligned}$$

The sequence (γ_n) is decreasing and therefore

$$\begin{aligned}
\rho(x+y) &= \lim_{n\longrightarrow\infty}(D_{\mathcal{A}}(x+y)^{2^n})^{1/2^n}\\
&\leq (\alpha+\beta)\lim_{n\longrightarrow\infty}\sup\gamma_n^{1/2^n}\\
&\leq (\alpha+\beta)\lim_{n\longrightarrow\infty}\sup\gamma_N^{1/2^n}\\
&= \alpha+\beta,
\end{aligned}$$

for arbitrary $\rho(x) < \alpha$, $\rho(y) < \beta$. The proof is complete. □

Proof. [Proof of theorem 3.5]

Fix $c \in \Omega_A \cap Z(A)$. Define $f : \mathbb{C} \longrightarrow \Omega_A$ with $f(\lambda) = g(\lambda c)$. f is holomorphic on

$$\{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{\rho(c)}\}.$$

Then g has Taylor expansion about 0 and we have

$$g(\lambda c) = \lambda c + \sum_{j=2}^{\infty}\lambda^j a_j \qquad (|\lambda| < \frac{1}{\rho(c)}).$$

Now we have to prove that $a_j = 0$, for all j; if not the case, suppose for contradiction, that there is some j with $a_j \neq 0$ and let k be the smallest integer such that $a_k \neq 0$. Take $q \in A$ with $\rho(q) = 0$ and let $n \geq 1$. Then, writing g^n for the n-fold composition $g\circ\cdots\circ g$, we have

$$g^n(\lambda c + \lambda^k nq) = \lambda c + \lambda^k n(ak+q) + O(\lambda^{k+1}) \qquad (\lambda \to 0).$$

Now as c and q commute, it follows that $\rho(\lambda c + \lambda^k nq) \leq \rho(\lambda c) + \rho(\lambda^k nq) = |\lambda|\rho(c)$ (lemma 3.4), and so we can define a holomorphic function $h : \{0 < |\lambda| < 1/\rho(c)\} \longrightarrow \mathcal{A}$ by

$$h(\lambda) = \frac{g^n(\lambda c + \lambda^k nq) - \lambda c}{n\lambda^k} \qquad (0 < |\lambda| < 1/\rho(c)),$$

isolated singularity at $\lambda = 0$ can be removed by setting $h(0) = a_k + q$. By Vesentini's theorem (theorem 3.4.7, [5]), the composition $\rho \circ h$ is a subharmonic function on $\{0 < |\lambda| < 1/\rho(c)\}$, and so by the maximum principle

$$\rho(h(0)) \leq \max_{|\lambda|=1} \rho(h(\lambda)).$$

Making use of lemma 3.4 again to estimate the right-hand side, it follows that

$$\rho(a_k + q) \leq 2/n.$$

As this is true for each n, we can let $n \longrightarrow \infty$ deduce that $\rho(a_k + q) = 0$. And as this holds for each $q \in \mathcal{A}$ with $\rho(q) = 0$, Zemánek's characterization of the radical (theorem 3.5), implies that a_k belongs to the radical of $\mathcal{A}$, which is zero since $\mathcal{A}$ is semi-simple. Thus $a_k = 0$, and we have arrived at a contradiction. We conclude that indeed $a_j = 0$ for all $j \geq 2$, and hence from (1) that $g(c) = c$. □

In theorem 3.5 the property of sub-multiplicativity of FLM algebras is essential but in the next theorem we do not need it.

Theorem 3.7 *Let $\mathcal{A}$ be a semi-simple complete metrizable FLM algebra. Given $a \in \Omega_A \backslash Z_{\mathcal{A}}$, then there exists a holomorphic map $g : \Omega_{\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}}$ satisfying $g(0) = 0$ and $g'(0) = I$ such that $g(a) \neq a$.*

Proof. Let $a \in \Omega_A \backslash Z_{\mathcal{A}}$. Then there exists $u \in \mathcal{A}$, such that $au \neq ua$. Suppose that $d_{\mathcal{A}}(0, u) < 1$, where $d_{\mathcal{A}}$ is a meter on $\mathcal{A}$. Then $v := \log(e - u)$ satisfies $e^{-v}ae^{v} \neq a$. Define $g : \Omega_{\mathcal{A}} \longrightarrow \Omega_{\mathcal{A}}$ by

$$g(x) = e^{-\frac{xv}{a}} x e^{\frac{xv}{a}} \qquad (x \in \Omega_{\mathcal{A}}).$$

Then g is a holomorphic function, $g(0) = 0$ and $g'(0) = I$, but $g(a) = e^{-v}ae^{v} \neq a$. □

By combination of theorems 3.5 and 3.7, we have the following theorem.

Theorem 3.8 *Let $\mathcal{A}$ and $\mathcal{B}$ be semi-simple complete metrizable FLM algebras with sub-multiplicatively meter. If $f : \Omega_{\mathcal{A}} \longrightarrow \Omega_{\mathcal{B}}$ is a biholomorphic map, then $f(\Omega_{\mathcal{A}} \cap Z_{\mathcal{A}}) = \Omega_{\mathcal{B}} \cap Z_{\mathcal{B}}$.*

Proof. Let $c \in \Omega_{\mathcal{A}} \cap Z_{\mathcal{A}}$ be an arbitrary. Without loss of generality suppose that $c \neq 0$. Then from assumption about f, $f(c) \neq f(0)$. Take $b \in \mathcal{B}$, and define $h : \Omega_{\mathcal{B}} \longrightarrow \Omega_{\mathcal{B}}$ by

$$h(y) = e^{-(\frac{y-f(0)}{f(c)-f(0)})^2 b} y e^{(\frac{y-f(0)}{f(c)-f(0)})^2 b} \qquad (y \in \Omega_{\mathcal{B}}).$$

By above definition h is a holomorphic function, $h(f(0)) = f(0)$ and $h'(f(0)) = I$. Now set $g = f^{-1} \circ h \circ f$, then g is a holomorphic function from $\Omega_{\mathcal{A}}$ into $\Omega_{\mathcal{A}}$, such that $g(0) = 0$ and $g'(0) = I$. Therefore $g(c) = c$ (theorem 2.1), and from definition of g, we have $h(f(c)) = f(c)$. By proof of theorem 2.2, $f(c) \in Z_{\mathcal{B}}$. In case $c = 0$, by continuity of f, the proof remains true. Hence $f(\Omega_{\mathcal{A}} \cap Z_{\mathcal{A}}) \subset \Omega_{\mathcal{B}} \cap Z_{\mathcal{B}}$.

Let $c \in \Omega_{\mathcal{B}} \cap Z_{\mathcal{B}}$, with applying the same argument to f^{-1}, we have $c \in f(\Omega_{\mathcal{A}} \cap Z_{\mathcal{A}})$. Therefore $\Omega_{\mathcal{B}} \cap Z_{\mathcal{B}} \subset f(\Omega_{\mathcal{A}} \cap Z_{\mathcal{A}})$. □

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