On some new inequalities for convex functions

Mevlüt Tunç

Abstract

In the present paper we establish some new integral inequalities analogous to the well known Hadamard’s inequality by using a fairly elementary analysis.

Key Words: Hadamard’s inequality, convex function, means

1. Introduction

The inequality (see [1])

\[ f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}, \]

which holds for all convex functions \( f : [a, b] \rightarrow \mathbb{R} \), is known in the literature as Hadamard’s inequality. Since its discovery in 1893, Hadamard’s inequality [2] has been proven to be one of the most useful inequalities in mathematical analysis. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations and numerous applications; see [1, 8] and the references cited therein. The main purpose of this paper is to establish some new integral inequalities analogous to Hadamard’s inequality given in (1.1) involving two convex functions. The analysis used in the proof is elementary and we believe that the inequalities established here are of independent interest.

2. Main results

We need the following Lemma proved in [6] which deals with the simple characterization of convex functions.

Lemma 1 The following statements are equivalent for a mapping: \( f : [a, b] \rightarrow \mathbb{R} \);

i) \( f \) is convex on \([a, b]\),

ii) for all \( x, y \) in \([a, b]\) the mapping \( g : [0, 1] \rightarrow \mathbb{R} \), defined by \( g(t) = f(tx + (1 - t)y) \) is convex on \([0, 1]\).

For the proof of this Lemma, see [6].

Our main result is given in the following theorem.

2000 AMS Mathematics Subject Classification: 26A51, 26D15.
Theorem 2 Let \( f, g : [a, b] \to \mathbb{R} \) be two convex functions and \( fg \in L^1 ([a, b]) \). Then,

\[
\begin{align*}
\frac{1}{(b-a)^2} & \int_a^b (b-x) (f(a) g(x) + g(a) f(x)) \, dx \\
& + \frac{1}{(b-a)^2} \int_a^b (x-a) (f(b) g(x) + g(b) f(x)) \, dx \\
& \leq \frac{M(a, b)}{3} + \frac{N(a, b)}{6} + \frac{1}{b-a} \int_a^b f(x) g(x) \, dx,
\end{align*}
\]

where \( M(a, b) = f(a)g(a) + f(b)g(b), N(a, b) = f(a)g(b) + f(b)g(a) \).

Proof. Since \( f \) and \( g \) are convex functions, we have that

\[
\begin{align*}
f(ta + (1-t)b) & \leq tf(a) + (1-t)f(b) \\
g(ta + (1-t)b) & \leq tg(a) + (1-t)g(b)
\end{align*}
\]

for \( t \in [0, 1] \). Using the elementary inequality \( e \leq f \) and \( p \leq r \) then \( er + fp \leq cp + fr \) for \( e, f, p, r \in \mathbb{R} \), we get that

\[
\begin{align*}
f(ta + (1-t)b) [tg(a) + (1-t)g(b)] \\
+ g(ta + (1-t)b) [tf(a) + (1-t)f(b)] \\
\leq [tf(a) + (1-t)f(b)] [tg(a) + (1-t)g(b)] \\
+ f(ta + (1-t)b) g(ta + (1-t)b)
\end{align*}
\]

and we obtain

\[
\begin{align*}
g(a) f(ta + (1-t)b) + g(b) (1-t) f(ta + (1-t)b) \\
+ f(a) t g(ta + (1-t)b) + f(b) (1-t) g(ta + (1-t)b) \\
\leq t^2 f(a) g(a) + (1-t)^2 f(b) g(b) + t (1-t) f(a) g(b) \\
+ t (1-t) f(b) g(a) + f(ta + (1-t)b) g(ta + (1-t)b).
\end{align*}
\]

By the Lemma 1, \( f(ta + (1-t)b) \) and \( g(ta + (1-t)b) \) are convex on \([0, 1] \), they are integrable on \([0, 1]\) and consequently \( f(ta + (1-t)b)g(ta + (1-t)b) \) is also integrable on \([0, 1]\). Similarly, since \( f \) and \( g \) are convex on \([a, b] \), they are integrable on \([a, b]\) and hence \( fg \) is also integrable on \([a, b]\). Integrating both sides of the above inequality over \([0, 1]\), we get

\[
\begin{align*}
g(a) \int_0^1 t f(ta + (1-t)b) \, dt + g(b) \int_0^1 (1-t) f(ta + (1-t)b) \, dt \\
+ f(a) \int_0^1 t g(ta + (1-t)b) \, dt + f(b) \int_0^1 (1-t) g(ta + (1-t)b) \, dt \\
\leq f(a) g(a) \int_0^1 t^2 \, dt + f(b) g(b) \int_0^1 (1-t)^2 \, dt + f(a) g(b) \int_0^1 t (1-t) \, dt \\
+ f(b) g(a) \int_0^1 (1-t) \, dt + \int_0^1 f(ta + (1-t)b) g(ta + (1-t)b) \, dt.
\end{align*}
\]
By substituting \( ta + (1 - t)b = x \), \((a - b) \, dt = dx \) it is easy to observe that

\[
\int_0^1 tg(ta + (1 - t)b) \, dt
\]

\[
= \frac{1}{b - a} \int_a^b \frac{x - b}{a - b} g(x) \, dx = \frac{1}{(b - a)^2} \int_a^b (b - x) g(x) \, dx,
\]

\[
\int_0^1 (1 - t) g(ta + (1 - t)b) \, dt
\]

\[
= \frac{1}{b - a} \int_a^b \frac{a - x}{a - b} g(x) \, dx = \frac{1}{(b - a)^2} \int_a^b (x - a) g(x) \, dx,
\]

and

\[
\int_0^1 tf(ta + (1 - t)b) \, dt = \frac{1}{(b - a)^2} \int_a^b (b - x) f(x) \, dx
\]

\[
\int_0^1 (1 - t) f(ta + (1 - t)b) \, dt = \frac{1}{(b - a)^2} \int_a^b (x - a) f(x) \, dx.
\]

It can be easily checked that

\[
\int_0^1 t^2 \, dt = \int_0^1 (1 - t)^2 \, dt = \frac{1}{3}, \quad \int_0^1 t (1 - t) \, dt = \frac{1}{6}
\]

\[
\int_0^1 f(ta + (1 - t)b) g(ta + (1 - t)b) \, dt = \frac{1}{b - a} \int_a^b f(x) g(x) \, dx.
\]

When the above expressions are taken into account, the proof is complete. \( \square \)

**Theorem 3** Let \( f, g : [a, b] \to \mathbb{R} \) be two convex functions and \( fg \in L^1 ([a, b]) \). Then

\[
\frac{1}{b - a} \int_a^b \left( f \left( \frac{a + b}{2} \right) g(x) + g \left( \frac{a + b}{2} \right) f(x) \right) \, dx \leq \frac{1}{2(b - a)} \int_a^b f(x) g(x) \, dx
\]

\[
+ \frac{1}{12} M(a, b) + \frac{1}{6} N(a, b) + f \left( \frac{a + b}{2} \right) g \left( \frac{a + b}{2} \right),
\]

where \( M(a, b) = f(a)g(a) + f(b)g(b) \), \( N(a, b) = f(a)g(b) + f(b)g(a) \).

**Proof.** Since \( f \) and \( g \) are convex on \([a, b]\), we have that

\[
f \left( \frac{a + b}{2} \right) = f \left( \frac{ta + (1 - t)b}{2} + \frac{(1 - t)a + tb}{2} \right) \leq \frac{f(ta + (1 - t)b) + f((1 - t)a + tb)}{2}
\]
\[ g\left(\frac{a+b}{2}\right) = g\left(\frac{ta + (1-t)b}{2} + \frac{(1-t)a + tb}{2}\right) \leq g\left(\frac{ta + (1-t)b + g((1-t)a + tb)}{2}\right) \]

for \( t \in [0,1] \). Again as explained, in the proof of inequality (2.2) given above, we multiply by one under the other and by one across the other of the above inequality and then we add these inequalities, so we obtain

\[
\begin{align*}
&f\left(\frac{a+b}{2}\right) g\left(\frac{ta + (1-t)b + g((1-t)a + tb)}{2}\right) \\
&+ g\left(\frac{a+b}{2}\right) f\left(\frac{ta + (1-t)b + f((1-t)a + tb)}{2}\right) \\
\leq& f\left(\frac{ta + (1-t)b + f((1-t)a + tb)g(ta + (1-t)b) + g((1-t)a + tb)}{2}\right) \\
&+ f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)
\end{align*}
\]

then

\[
\begin{align*}
&\frac{1}{4} f\left(\frac{a+b}{2}\right) g(ta + (1-t)b) + g((1-t)a + tb) \\
&+ \frac{1}{4} g\left(\frac{a+b}{2}\right) f(ta + (1-t)b + f((1-t)a + tb)) \\
\leq& \frac{1}{4} f(ta + (1-t)b)g(ta + (1-t)b) \\
&+ \frac{1}{4} f\left(\frac{ta + (1-t)b + f((1-t)a + tb)}{2}\right) g\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
&+ \frac{1}{4} f\left(\frac{ta + (1-t)b + g((1-t)a + tb)}{2}\right) g\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
&+ \frac{1}{4} \left[ g(ta + (1-t)b) + g((1-t)a + tb) \right] f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
&+ \frac{1}{4} \left[ f(ta + (1-t)b) + f((1-t)a + tb) \right] g\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)
\end{align*}
\]

\[ = \frac{1}{4} f(ta + (1-t)b)g(ta + (1-t)b) \\
+ \frac{1}{4} f\left(\frac{ta + (1-t)b + f((1-t)a + tb)}{2}\right) g\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
+ \frac{1}{4} \left[2t(1-t)\right] f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
+ \frac{1}{4} \left[ t^2 + (1-t)^2 \right] f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
+ \frac{1}{4} \left[ f(ta + (1-t)b) + f((1-t)a + tb) \right] g\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right). \]
Again as explained, in the proof of inequality (2.2) given above we integrate both sides of the above inequality over \([0, 1]\) and obtain

\[
\frac{1}{2} f \left( \frac{a + b}{2} \right) \int_0^1 [g (ta + (1 - t)b) + g ((1 - t)a + tb)] \, dt \\
+ \frac{1}{2} g \left( \frac{a + b}{2} \right) \int_0^1 [f (ta + (1 - t)b) + f ((1 - t)a + tb)] \, dt \\
\leq \frac{1}{4} \int_0^1 f (ta + (1 - t)b) g (ta + (1 - t)b) \, dt \\
+ \frac{1}{4} \int_0^1 f ((1 - t)a + tb) g ((1 - t)a + tb) \, dt \\
+ \frac{1}{4} [f (a) g (a) + f (b) g (b)] \int_0^1 [2t (1 - t)] \, dt \\
+ \frac{1}{4} [f (a) g (b) + f (b) g (a)] \int_0^1 [t^2 + (1 - t)^2] \, dt \\
+ f \left( \frac{a + b}{2} \right) g \left( \frac{a + b}{2} \right) \int_0^1 \, dt.
\]

By substituting \(ta + (1 - t)b = x\), it is easy to observe that

\[
\int_0^1 f (ta + (1 - t)b) g (ta + (1 - t)b) \, dt = \frac{1}{b - a} \int_a^b f (x) g (x) \, dx \\
\int_0^1 f (ta + (1 - t)b) \, dt = \int_0^1 f ((1 - t)a + tb) \, dt = \frac{1}{b - a} \int_a^b f (x) \, dx.
\]

From the above inequality it is easy to observe that

\[
\frac{f \left( \frac{a + b}{2} \right)}{b - a} \int_a^b g (x) \, dx + \frac{g \left( \frac{a + b}{2} \right)}{b - a} \int_a^b f (x) \, dx \\
\leq \frac{1}{2} \frac{b - a}{b - a} \int_a^b f (x) g (x) \, dx \\
+ \frac{1}{12} M (a, b) + \frac{1}{6} N (a, b) + f \left( \frac{a + b}{2} \right) g \left( \frac{a + b}{2} \right)
\]

The proof is complete.

3. Applications for special means

As in [8], we shall consider the means as arbitrary real numbers \(a, b, a \neq b\). In the sources there includes

\(A (a, b) = \frac{a + b}{2}, \quad a, b \in \mathbb{R}, \) (arithmetic mean);
Now, using the results of Section 2, we illustrate some applications of special means of real numbers.

Proposition 4 Let $0 < a < b$. Then,

$$\frac{4A(a, b) - 2L(a, b)}{L(a, b)G^2(a, b)} \leq \frac{2K^2(a, b) + 4G^2(a, b)}{3G^3(a, b)}$$

(3.4)

or

$$\frac{4A(a, b)}{L(a, b)} \leq \frac{2K^2(a, b)}{3G^2(a, b)} + \frac{10}{3}.$$

Proof. The assertion from Theorem 2 is applied to the convex mapping $f(x) = g(x) = \frac{1}{x}, x \in [a, b]$. □

Proposition 5 Let $a, b \in I^o \subset \mathbb{R}$ ($I^o$ is the interior of $I$), $a < b, 0 \notin [a, b]$ and $n \in \mathbb{Z} \setminus \{-1, 0\}, a \neq b$. Then,

$$2A^n(a, b) L_n^o(a, b) \leq \frac{1}{2} L_{2n}^o(a, b) + \frac{K^2(a^n, b^n)}{6} + \frac{G^{2n}(a, b)}{3} + A^{2n}(a, b).$$

(3.5)

Proof. The assertion from Theorem 3 is applied to the convex mapping $f(x) = g(x) = x^n, x \in [a, b]$ and $n \in \mathbb{Z} \setminus \{-1, 0\}, a \neq b$. □

References


Mevlüt TUNÇ
Department of Mathematics,
Faculty of Arts and Sciences,
Kilis 7 Aralik University;
Kilis, 79000, TURKEY
e-mail: mevluttunc@kilis.edu.tr

Received: 26.05.2010