Some results on g-frames in Hilbert spaces

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Abstract

In this paper we show that every g-frame for a Hilbert space $\mathcal{H}$ can be represented as a linear combination of two $g$-orthonormal bases if and only if it is a $g$-Riesz basis. We also show that every g-frame can be written as a sum of two tight g-frames with g-frame bounds one or a sum of a $g$-orthonormal basis and a $g$-Riesz basis for $\mathcal{H}$. We further give necessary and sufficient conditions on $g$-Bessel sequences $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ and $\{\Gamma_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ and operators $L_1$, $L_2$ on $\mathcal{H}$ so that $\{\Lambda_i L_1 + \Gamma_i L_2 : i \in J\}$ is a $g$-frame for $\mathcal{H}$. We next show that a $g$-frame can be added to any of its canonical dual $g$-frame to yield a new $g$-frame.

Key Words: Frame, g-frame, g-orthonormal basis, tight g-frame, g-Bessel sequence

1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer ([5]), reintroduced in 1986 by Daubechies, Grossman, and Meyer ([4]), and popularized from then on. In [11], a generalization of the frame concept was introduced. Sun introduced a g-frame and a g-Riesz basis in a complex Hilbert space and discussed some properties of them. A frame of subspaces ([1], [3]) and a system of bounded quasi-projectors ([6]) are a g-frame in a complex Hilbert space. From a g-frame, we may construct a frame for a complex Hilbert space ([11]). A natural question which immediately comes to mind is, “Which properties of the frame may be extended to the g-frame for a complex Hilbert space?” G-frames and g-Riesz bases in complex Hilbert spaces have some properties similar to those of frames, Riesz bases, but not all the properties are similar (see [11]). In this paper we generalize some results in [2], [7], [10] from frame theory to g-frames.

Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces and $\{\mathcal{H}_i\}_{i \in J} \subseteq \mathcal{K}$ is a sequence of separable Hilbert spaces, where $J$ is a subset of $\mathbb{Z}$, $\mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ is the collection of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}_i$. For each sequence $\{\mathcal{H}_i\}_{i \in J}$, we define the space $(\sum_{i \in J} \mathcal{H}_i)_{l_2}$ by

$$(\sum_{i \in J} \mathcal{H}_i)_{l_2} = \{\{f_i\}_{i \in J} : f_i \in \mathcal{H}_i, i \in J \text{ and } \sum_{i \in J} \|f_i\|^2 < \infty\}.$$  

With the inner product defined by

$$\langle\{f_i\}, \{g_i\}\rangle = \sum_{i \in J} \langle f_i, g_i \rangle,$$

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it is clear that \((\sum_{i \in J} \bigoplus \mathcal{H}_i)_\mathbb{R}\) is a Hilbert space.

A frame for a complex Hilbert space \(\mathcal{H}\) is a family of vectors \(\{f_i\}_{i \in J}\) so that there are two positive constants \(A\) and \(B\) satisfying

\[
A \|f\|^2 \leq \sum_{i \in J} |\langle f, f_i \rangle|^2 \leq B \|f\|^2, \quad f \in \mathcal{H}.
\]

The constants \(A\) and \(B\) are called lower and upper frame bounds.

A sequence \(\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}\) is called a generalized frame, or simply a g-frame, for \(\mathcal{H}\) with respect to \(\{\mathcal{H}_i\}_{i \in J}\) if there exist two positive constants \(A\) and \(B\) such that, for all \(f \in \mathcal{H}\),

\[
A \|f\|^2 \leq \sum_{i \in J} \|\Lambda_i f\|^2 \leq B \|f\|^2.
\]

The constants \(A\) and \(B\) are called the lower and upper g-frame bounds, respectively. The supremum of all such \(A\) and the infimum of all such \(B\) are called the optimal bounds. If \(A = B\) we call this g-frame a tight g-frame and if \(A = B = 1\), it is called a normalized tight g-frame. A g-frame is exact if it ceases to be a g-frame whenever any single element is removed from \(\{\Lambda_i\}_{i \in J}\).

We say \(\{\Lambda_i\}_{i \in J}\) is a g-frame sequence, if it is a g-frame for \(\mathcal{H}\) whenever the space sequence \(\mathcal{H}_i\) is clear. We say \(\{\Lambda_i\}_{i \in J}\) is g-complete, if \(\{f : \Lambda_i f = 0, \forall i \in J\} = \{0\}\); and is called g-orthonormal basis for \(\mathcal{H}\), if

\[
\langle \Lambda^*_i g_i, \Lambda^*_j g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle, \quad i, j \in J, \quad g_i \in \mathcal{H}_i, \quad g_j \in \mathcal{H}_j,
\]

and

\[
\sum_{i \in J} \|\Lambda_i f\|^2 = \|f\|^2.
\]

We say that \(\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}\) is a g- Riesz basis for \(\mathcal{H}\), if it is g-complete and there exist constants \(0 < A \leq B < \infty\), such that for any finite subset \(I \subseteq J\) and \(g_i \in \mathcal{H}_i, \ i \in I\),

\[
A \sum_{i \in I} \|g_i\|^2 \leq \|\sum_{i \in I} \Lambda_i^* g_i\|^2 \leq B \sum_{i \in I} \|g_i\|^2.
\]

Recall that a unitary operator \(K : \mathcal{H} \to \mathcal{H}\) is an onto isometry, a partial isometry is an operator that is an isometry on the orthogonal complement of its kernel, a co-isometry is an operator whose adjoint is an into isometry, and a maximal partial isometry is either an isometry or a co-isometry.

In order to present the main results of this paper, we need the following Theorems and Propositions which can be found in [11], [2] and [9]

**Proposition 1.1** ([2]) Let \(K : \mathcal{H} \to \mathcal{H}\) be a bounded linear operator. Then the following hold:

i) \(K = a(U_1 + U_2 + U_3)\), where each \(U_j, j = 1, 2, 3\), is a unitary operator and \(a\) is a constant.

ii) If \(K\) is onto, then it can be written as a linear combination of two unitary operators if and only if \(K\) is invertible.
Theorem 1.2 ([11]) Let \( \{ \Lambda_i \in \mathcal{L}(H, \mathcal{H}_i) : i \in J \} \) be a g-frame for \( H \) with respect to \( \{ \mathcal{H}_i \}_{i \in J} \). The operator

\[
S : H \to H, \quad Sf = \sum_{i \in J} \Lambda_i^* \Lambda_i f,
\]
is a positive invertible operator and every \( f \in H \) has an expansion

\[
f = \sum_{i \in J} S^{-1} \Lambda_i^* \Lambda_i f = \sum_{i \in J} \Lambda_i^* \Lambda_i S^{-1} f.
\]

So \( \{ \tilde{\Lambda}_i = \Lambda_i S^{-1} \in \mathcal{L}(H, \mathcal{H}_i) : i \in J \} \) is a g-frame for \( H \) with respect to \( \{ \mathcal{H}_i \}_{i \in J} \) and is called canonical dual g-frame of \( \{ \Lambda_i \} \). The operator \( S \) is called the g-frame operator of \( \{ \Lambda_i \} \).

Definition 1.3 Let \( \{ \Lambda_i \in \mathcal{L}(H, \mathcal{H}_i) : i \in J \} \) be a g-frame for \( H \). Then the synthesis operator for \( \{ \Lambda_i \} \) is the operator

\[
T : (\bigoplus_{i \in J} \mathcal{H}_i)_{l^2} \to H,
\]
defined by

\[
T(f_i) = \sum_{i \in J} \Lambda_i^*(f_i).
\]

We call the adjoint \( T^* \) of the synthesis operator the analysis operator.

Proposition 1.4 ([9]) Let \( \{ \Lambda_i \in \mathcal{L}(H, \mathcal{H}_i) : i \in J \} \) be a g-frame for \( H \). Then the analysis operator for \( \{ \Lambda_i \in \mathcal{L}(H, \mathcal{H}_i) : i \in J \} \) is the operator

\[
T^* : H \to (\bigoplus_{i \in J} \mathcal{H}_i)_{l^2},
\]
defined by

\[
T^*(f) = \{ \Lambda_i(f) \}_{i \in J}.
\]

Proposition 1.5 ([9]) Let \( \{ \Lambda_i \}_{i \in J} \) be a sequence in \( \mathcal{L}(H, \mathcal{H}_i) \). Then the following are equivalent:

i) \( \{ \Lambda_i \}_{i \in J} \) is a g-frame for \( H \);

ii) The operator \( T : (\bigoplus_{i \in J} \mathcal{H}_i)_{l^2} \to \sum_{i \in J} \Lambda_i^*(f_i) \) is well-defined and bounded from \( (\bigoplus_{i \in J} \mathcal{H}_i)_{l^2} \) onto \( H \);

iii) The operator \( S : f \mapsto f \sum_{i \in J} \Lambda_i^* \Lambda_i f \) is well-defined and bounded from \( H \) onto \( H \).

Proposition 1.6 ([9]) Let \( \{ \Theta_i \in \mathcal{L}(H, \mathcal{H}_i) : i \in J \} \) be a g-orthonormal basis for \( H \) with respect to \( \{ \mathcal{H}_i \}_{i \in J} \) and \( \{ \Lambda_i \in \mathcal{L}(H, \mathcal{H}_i) : i \in J \} \) be a g-frame for \( H \) with respect to \( \{ \mathcal{H}_i \}_{i \in J} \). Then there is a bounded and onto operator \( K : H \to H \) such that \( \Lambda_i = \Theta_i K^* \) for all \( i \in J \). Furthermore, \( K \) is invertible if \( \{ \Lambda_i \}_{i \in J} \) is a g-Riesz basis for \( H \) and \( K \) is unitary if \( \{ \Lambda_i \}_{i \in J} \) is a g-orthonormal basis for \( H \).
2. Some g-frame representations

In this section we show that every g-frame for a Hilbert space \( \mathcal{H} \) can be written as a sum of three g-orthonormal bases for \( \mathcal{H} \). We next show that a g-frame can be represented as a linear combination of two g-orthonormal bases if and only if it is a g-Riesz basis. We further show that every g-frame can be written as a sum of two tight g-frames with g-frame bounds one or a sum of a g-orthonormal basis and a g-Riesz basis for \( \mathcal{H} \).

**Proposition 2.1** If \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-frame for a Hilbert space \( \mathcal{H} \), and \( \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-orthonormal basis for \( \mathcal{H} \), there are g-orthonormal bases \( \{ \Upsilon_i \}, \{ \Gamma_i \}, \{ \Psi_i \} \) for \( \mathcal{H} \) and a constant \( a \) so that \( \Lambda_i = a(\Upsilon_i + \Gamma_i + \Psi_i) \) for all \( i \in J \).

**Proof.** By Proposition 1.6 there is a bounded and onto operator \( K : \mathcal{H} \to \mathcal{H} \) such that \( \Lambda_i = \Theta_iK^* \) and by Proposition 1.1 we have \( K^* = a(U_1 + U_2 + U_3) \), where each \( U_j \) is a unitary operator and \( a \) is a constant. So \( \Lambda_i = \Theta_iK^* = a(\Theta_iU_1 + \Theta_iU_2 + \Theta_iU_3) \). Since \( \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-orthonormal basis and for each \( r = 1, 2, 3, U_r \) is a unitary operator, we have

\[
\langle (\Theta_iU_r^*)g_i, (\Theta_jU_r)^*g_j \rangle = \langle \Theta_i^*g_i, \Theta_j^*g_j \rangle = \delta_{i,j}\langle g_i, g_j \rangle,
\]

and

\[
\sum_{i \in J} \|\Theta_iU_rf\|^2 = \|U_rf\|^2 = \|f\|^2.
\]

So \( \{\Theta_iU_r\}_i \) is a g-orthonormal basis and the proof is complete by putting \( \Upsilon_i = \Theta_iU_1, \Gamma_i = \Theta_iU_2 \) and \( \Psi_i = \Theta_iU_3 \). \( \square \)

**Proposition 2.2** ([12]) For the family \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) the following two statements are equivalent:

i) The sequence \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-Riesz basis for \( \mathcal{H} \).

ii) The sequence \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-frame for \( \mathcal{H} \), and if \( \sum_{i \in J} \Lambda_i^*g_i = 0 \) then \( g_i = 0 \) for all \( i \in J \).

**Proposition 2.3** If \( \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-orthonormal basis for \( \mathcal{H} \) then we have a g-frame \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) which can be written as a linear combination of two g-orthonormal bases for \( \mathcal{H} \) if and only if \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-Riesz basis for \( \mathcal{H} \).

**Proof.** If \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-Riesz basis, by Proposition 1.6, there is an invertible operator \( K : \mathcal{H} \to \mathcal{H} \) such that \( \Lambda_i = \Theta_iK^* \) and by Proposition 1.1 we have \( K^* = aU_1 + bU_2 \) for some constants \( a, b \), and unitary operators \( U_1 \) and \( U_2 \). So \( \Lambda_i = \Theta_iK^* = a\Theta_iU_1 + b\Theta_iU_2 \). Since \( \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-orthonormal basis and \( U_1 \) and \( U_2 \) are unitary operators, for \( g_i \in \mathcal{H}_i \), \( g_j \in \mathcal{H}_j \) and \( f \in \mathcal{H} \), we have

\[
\langle (\Theta_iU_r^*)g_i, (\Theta_jU_r)^*g_j \rangle = \langle \Theta_i^*g_i, \Theta_j^*g_j \rangle = \delta_{i,j}\langle g_i, g_j \rangle
\]

and

\[
\sum_{i \in J} \|\Theta_iU_rf\|^2 = \|U_rf\|^2 = \|f\|^2.
\]
Proposition 2.4  If \( \{ \Theta_i \}_{i \in \mathcal{I}} \) is a g-orthonormal basis, then \( \Theta_i \) is a bounded and onto operator for \( \mathcal{H} \) and constants \( a, b \) such that \( \Lambda_i = a \Theta_i T + b \Gamma_i \) for all \( i \in J \). By Proposition 1.6, there is an onto operator \( T \), and unitary operators \( K \) and \( R \) such that \( \Lambda_i = \Theta_i T^* \), \( \Gamma_i = \Theta_i R^* \). Since \( \Lambda_i = a \Theta_i + b \Gamma_i \) and \( \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-orthonormal basis for \( \mathcal{H} \), we have \( T = aK + bR \) and so, by Proposition 1.1, \( T \) is an invertible operator. If \( \sum_{i \in J} \Lambda_i g_i = 0 \) then \( T \sum_{i \in J} \Theta_i^* g_i = 0 \), and so \( \sum_{i \in J} \Theta_i^* g_i = 0 \). Therefore, by Proposition 2.2, \( g_i = 0 \) which implies that the family \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-Riesz basis.

\[ \sum_{i \in J} \| \Theta_i K^* f \|^2 = \| K^* f \|^2 = \| f \|^2. \]

Every operator \( K \) on a Hilbert space can be written in the form \( K = VP = \| T \| V(W + W^*) \), where \( W \) is unitary and \( V \) is a maximal partial isometry. It follows that \( VW \) and \( VW^* \) are maximal partial isometries. That is, each of these operators is either an isometry or a co-isometry. However, if \( K \) has dense range, \( V \) must be a co-isometry (see [2]).

By using the above facts we have the following propositions.

Proposition 2.5  If \( \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-orthonormal basis for \( \mathcal{H} \) then every g-frame is the sum of two normalized tight g-frames for \( \mathcal{H} \).

\[ \Lambda_i = \Theta_i K^* = \Theta_i \left( \frac{\| T \|^2}{2} ((VW)^* + (VW^*)^*) \right), \]

and, by Proposition 2.4, \( \Theta_i(VW)^* \) and \( \Theta_i(VW^*)^* \) are normalized tight g-frames.

Proposition 2.6  If \( \{ \Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-orthonormal basis for \( \mathcal{H} \), then every g-frame for a Hilbert space \( \mathcal{H} \) is the sum of a g-orthonormal basis for \( \mathcal{H} \) and a g-Riesz basis for \( \mathcal{H} \).

\[ L = \frac{3}{4} I + \frac{1}{4} (1 - \epsilon) \frac{K^*}{\| K^* \|}. \]
Then we have $\| I - L \| < 1$ and $\| L \| \leq 1$. So $L$ is an invertible operator and, as in the proof of proposition 1.1, (see [2]) we can write

$$L = \frac{1}{2}(W + W^*),$$

where $W$ is a unitary operator. We also have the relation

$$K^* = \frac{4\| K^* \|}{(1 - \epsilon)}\left[\frac{1}{2}(W + W^*) - \frac{3}{4}I\right]$$

$$= \frac{2\| K^* \|}{(1 - \epsilon)}[W + R],$$

where $R = W^* - \frac{3}{4}I$. Since $W$ is unitary, $\{\Theta_i \mid W : i \in J\}$ is a $g$-orthonormal basis, and $W^*$ is unitary which implies that $R$ is an isomorphism (possibly into). But, it is easily checked that $R$ is onto, since

$$\| I - \frac{1}{2}R \| = \| \frac{1}{4}I + \frac{1}{2}W^* \| < 1.$$ 

Thus, $\frac{1}{2}R$ is an invertible operator and hence $R$ is an invertible operator. We also have

$$\sum_{i \in J}(\Theta_i R)^* g_i = \sum_{i \in J} R^*\Theta_i^* g_i = R^* (\sum_{i \in J} \Theta_i^* g_i).$$

Since $R$ is an invertible operator, if $\sum_{i \in J}(\Theta_i R)^* g_i = 0$ then $\sum_{i \in J} \Theta_i^* g_i = 0$ and since $\{\Theta_i : i \in J\}$ is a $g$-orthonormal basis we conclude $g_i = 0$ for all $i \in J$. Therefore, by Proposition 2.2, $\Theta_i R$ is a $g$-Riesz basis for $\mathcal{H}$. 

3. Sums of $g$-bessel sequences

In this section we give necessary and sufficient conditions on $g$-Bessel sequences $\{A_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ and $\{\Gamma_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ and operators $L_1, L_2$ on $\mathcal{H}$ so that $\{\Lambda_i L_1 + \Gamma_i L_2 : i \in J\}$ is a $g$-frame for $\mathcal{H}$, and we show that a $g$-frame can be added to any of its canonical dual $g$-frame to yield a new $g$-frame.

**Proposition 3.1** Let $\{A_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ and $\{\Gamma_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ be $g$-Bessel sequences in $\mathcal{H}$ with analysis operators $T_1, T_2$ and $g$-frame operators $S_1, S_2$, respectively. For the given operators $L_1, L_2 : \mathcal{H} \to \mathcal{H}$ the following are equivalent:

i) $\{\Lambda_i L_1 + \Gamma_i L_2 : i \in J\}$ is a $g$-frame for $\mathcal{H}$.

ii) $T_1 L_1 + T_2 L_2$ is a bounded and one-to-one operator on $\mathcal{H}$.

iii) The operator $S = L_1^* T_1^* L_1 + L_1^* T_2^* L_2 + L_2^* T_1^* L_1 + L_2^* T_2^* L_2$ is a well-defined and bounded mapping from $\mathcal{H}$ onto $\mathcal{H}$. Moreover, in this case, $S$ is the $g$-frame operator for $\{\Lambda_i L_1 + \Gamma_i L_2 : i \in J\}$. 

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Theorem 3.2 Let \{\Lambda_i L_1 + \Gamma_i L_2 : i \in J\} be a g-frame if and only if its analysis operator \( T \) which is defined by

\[
T(f) = \{(\Lambda_i L_1 + \Gamma_i L_2)(f)\}_{i \in J} = \{\Lambda_i L_1(f)\}_{i \in J} + \{\Gamma_i L_2(f)\}_{i \in J}
\]

is a bounded and one-to-one operator on \( \mathcal{H} \), and this happens if and only if the g-frame operator for our family

\[
S = (T_1 L_1 + T_2 L_2)^* T_1 L_1 + T_2 L_2
\]

is well defined and bounded.

Proof. Let \( T_1, T_2 \) be the analysis operators for \( \{\Lambda_i : i \in J\} \) and \( \{\Gamma_i : i \in J\} \), respectively. By letting \( L_1 = I = L_2 \) in Proposition 3.1, we see that the g-frame operator for \( \{\Lambda_i + \Gamma_i : i \in J\} \) is

\[
S = T_1^* T_1 + T_2^* T_2 + T_1^* T_1 + T_2^* T_2 = S_1 + R + R^* + S_2.
\]

\( \square \)

Corollary 3.3 If \( \{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\} \) is a g-frame with g-frame operator \( S \) and \( \{\Gamma_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\} \) is a g-Bessel sequence in \( \mathcal{H} \), such that \( f = \sum_{i \in J} \Lambda_i^* \Gamma_i f \), for all \( f \in \mathcal{H} \), then \( \{\Lambda_i S^a + \Gamma_i S^b : i \in J\} \) is a g-frame, for all real numbers \( a \) and \( b \).

Proof. If \( T_1 \) and \( T_2 \) are the analysis operators for \( \{\Lambda_i S^a : i \in J\} \) and \( \{\Gamma_i S^b : i \in J\} \), respectively, then for \( R = T_1^* T_2 \) we have

\[
R(f) = T_1^* T_2(f)
\]

\[
= T_1^* (\{\Gamma_i S^b f\})
\]

\[
= \sum_{i \in J} (\Lambda_i S^a)^* \Gamma_i S^b f
\]

\[
= \sum_{i \in J} S^a \Lambda_i^* \Gamma_i S^b f
\]

\[
= S^{a+b} f.
\]

Since \( S \) is invertible, \( \{\Lambda_i S^a + \Gamma_i S^b : i \in J\} \) is a g-frame, by Theorem 3.2.

\( \square \)
Corollary 3.4 If \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a g-frame with g-frame operator \( S \) and \( \{ \tilde{\Lambda}_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) is a canonical dual g-frame then \( \{ \Lambda_i S^a + \tilde{\Lambda}_i S^b : i \in J \} \) is a g-frame for all real numbers \( a, b \).

Proposition 3.5 Let \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J \} \) be a g-frame for a Hilbert space \( \mathcal{H} \) with g-frame operator \( S \) and g-frame bounds \( A \) and \( B \). Let \( \{ I_1, I_2 \} \) be a partition of \( J \) and let \( S_j \) be the g-frame operator for the g-Bessel sequences \( \{ \Lambda_i : i \in I_j \} \), \( j = 1, 2 \). Then \( \{ \Lambda_i + \Lambda_i S^a_i : i \in I_1 \} \bigcup \{ \Lambda_i + \Lambda_i S^b_i : i \in I_2 \} \), is a g-frame for any real numbers \( a, b \) that the operator \( S_1(I + S^a_1)^2 + S_2(I + S^b_2)^2 \) is onto.

Proof. Note that, for each \( f \in \mathcal{H} \)

\[
\left( \sum_{i \in I_1} \| \Lambda_i f + \Lambda_i S^a_i f \|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i \in I_1} \| \Lambda_i f \|^2 \right)^{\frac{1}{2}} + \left( \sum_{i \in I_1} \| \Lambda_i S^a_i f \|^2 \right)^{\frac{1}{2}} \leq \sqrt{B} \| f \| + \sqrt{B} \| S^a_i f \| \leq \sqrt{B} \| f \|.
\]

Similarly, we have

\[
\left( \sum_{i \in I_2} \| \Lambda_i f + \Lambda_i S^b_i f \|^2 \right)^{\frac{1}{2}} \leq \sqrt{B} \| f \|. 
\]

Thus

\[
\big\{ \Lambda_i + \Lambda_i S^a_i : i \in I_1 \big\} \bigcup \big\{ \Lambda_i + \Lambda_i S^b_i : i \in I_2 \big\},
\]

is a g-Bessel sequence. On the other hand, the frame operator for \( \{ \Lambda_i + \Lambda_i S^a_i : i \in I_1 \} \) is

\[
\sum_{i \in I_1} (\Lambda_i + \Lambda_i S^a_i)^*(\Lambda_i + \Lambda_i S^a_i) = \sum_{i \in I_1} \Lambda_i^* \Lambda_i + S^a_i \sum_{i \in I_1} \Lambda_i^* \Lambda_i
\]

\[
+ \sum_{i \in I_1} \Lambda_i^* \Lambda_i S^a_i + S^a_i \sum_{i \in I_1} \Lambda_i^* \Lambda_i S^a_i
\]

\[
= S_1 + S^a_1 S_1 + S_1 S^a_1 + S^a_1 S_1 S^a_1
\]

\[
= S_1 (I + S^a_1)^2,
\]

Similarly for \( \{ \Lambda_i + \Lambda_i S^b_i : i \in I_2 \} \) the frame operator is \( S_2(I + S^b_2)^2 \). Hence, the g-frame operator \( S_0 \) for our family is an onto and bounded operator and hence, by Proposition 1.5, \( \{ \Lambda_i + \Lambda_i S^a_i : i \in I_1 \} \bigcup \{ \Lambda_i + \Lambda_i S^b_i : i \in I_2 \} \) is a g-frame.

4. Subsequence of g-frames

A g-frame for Hilbert space \( \mathcal{H} \) has been decomposed into two infinite subsequences, if one of the subsequence is a g-frame for \( \mathcal{H} \) a necessary and sufficient condition under which the other subsequence is a g-frame for \( \mathcal{H} \) has been given.
Theorem 4.1 Let \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{N} \} \) be a g-frame for a Hilbert space \( \mathcal{H} \) and let \( \{ m_k \} \) and \( \{ n_k \} \) be two infinite increasing sequences with \( \{ m_k \} \cup \{ n_k \} = \mathbb{N} \). Also let \( \{ \Lambda_{m_k} : k \in \mathbb{N} \} \) be a g-frame for \( \mathcal{H} \). Then \( \{ \Lambda_{n_k} : k \in \mathbb{N} \} \) is a g-frame for \( \mathcal{H} \) if and only if there exists a bounded linear operator \( U : (\sum_{k \in \mathbb{N}} \oplus \mathcal{H}_{n_k})_\Pi \rightarrow (\sum_{k \in \mathbb{N}} \oplus \mathcal{H}_{m_k})_\Pi \) such that \( U(\{ \Lambda_{n_k}f \}_{k \in \mathbb{N}}) = \{ \Lambda_{m_k}f \}_{k \in \mathbb{N}} \), \( f \in \mathcal{H} \).

Proof. Let \( A \) be a lower bound of the g-frame \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{N} \} \). Since

\[
\sum_{k \in \mathbb{N}} \|\Lambda_{m_k}f\|^2 = \|U(\{ \Lambda_{n_k}f \}_{k \in \mathbb{N}})\| \leq \|U\| \|\{ \Lambda_{n_k}f \}_{k \in \mathbb{N}}\|
\]

we have

\[
\sum_{k \in \mathbb{N}} \|\Lambda_{n_k}f\|^2 \geq \sum_{k \in \mathbb{N}} \|\Lambda_{m_k}f\|^2 \geq \frac{A}{\|U\|} \|f\|^2,
\]

which implies that \( \{ \Lambda_{n_k} : k \in \mathbb{N} \} \) is a g-frame. Conversely, let \( \{ \Lambda_{n_k} : k \in \mathbb{N} \} \) be a g-frame. Let \( T_1, T_2 \) be the analysis operators for \( \{ \Lambda_{n_k} : k \in \mathbb{N} \} \) and \( \{ \Lambda_{m_k} : k \in \mathbb{N} \} \), respectively. Put \( U = T_2 S_1^{-1} T_1^* \). Then \( U \) is a bounded linear operator with the desired properties. \( \square \)

In the following Theorem, we give a sufficient condition for a g-frame of nonzero elements in terms of g-frame sequences for its exactness.

Theorem 4.2 Let \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{N} \} \) be a g-frame for a Hilbert space \( \mathcal{H} \) with optimal bounds \( A \) and \( B \) such that \( \Lambda_i \neq 0 \), for all \( i \in \mathbb{N} \). If for every infinite increasing sequence \( \{ n_k \} \) in \( \mathbb{N} \), \( \{ \Lambda_{n_k}f \}_{k \in \mathbb{N}} \) is a g-frame sequence with optimal bounds \( A \) and \( B \), then \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{N} \} \) is an exact g-frame.

Proof. Suppose \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{N} \} \) is not exact. Then there exists a positive integer \( m \in \mathbb{N} \) such that \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \neq m, i \in \mathbb{N} \} \) is a g-frame. Let \( \{ n_k \} \) be an increasing sequence given by \( n_k = k, k = 1, 2, ..., m-1 \) and \( n_k = k+1, k = m, m+1, ... \). Since \( \{ \Lambda_{n_k}f \}_{k \in \mathbb{N}} \) is a g-frame sequence with optimal bounds \( A \) and \( B \), we have

\[
A\|f\|^2 \leq \sum_{i \neq m} \|\Lambda_if\|^2 \leq B\|f\|^2.
\]

Therefore, by g-frame inequality for the frame \( \{ \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{N} \} \), for all \( f \in \mathcal{H} \) we have

\[
\|\Lambda_mf\|^2 = 0,
\]

This given \( \Lambda_m = 0 \), which is a contradiction. \( \square \)

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References


