Combinatorial results for order-preserving and order-decreasing transformations

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Abstract

Let $O_n$ and $C_n$ be the semigroup of all order-preserving transformations and of all order-preserving and order-decreasing transformations on the finite set $X_n = \{1, 2, \ldots, n\}$, respectively. Let $\text{Fix}(\alpha) = \{x \in X_n : x\alpha = x\}$ for any transformation $\alpha$. In this paper, for any $Y \subseteq X_n$, we find the cardinalities of the sets $O_{n,Y} = \{\alpha \in O_n : \text{Fix}(\alpha) = Y\}$ and $C_{n,Y} = \{\alpha \in C_n : \text{Fix}(\alpha) = Y\}$. Moreover, we find the numbers of transformations of $O_n$ and $C_n$ with $r$ fixed points.

Key Words: Order-preserving transformations, order-decreasing transformations, nilpotent, Catalan number

1. Introduction

Consider the finite set $X_n = \{1, 2, \ldots, n\}$ ordered in the standard way. Let $T_n$ be the full transformation semigroup on $X_n$. We shall call a transformation $\alpha : X_n \rightarrow X_n$ order-preserving if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X_n$, and decreasing (increasing) if $x\alpha \leq x$ ($x\alpha \geq x$) for all $x \in X_n$. Combinatorial properties of the semigroup $O_n$ of order-preserving transformations on $X_n$, and of its subsemigroup $C_n$, which consists of all decreasing and order-preserving transformations have been investigated over the last thirty years. (See, for example [2, 3, 4, 5, 6, 7].)

For $\alpha \in T_n$ we denote $\text{Fix}(\alpha) = \{x \in X_n : x\alpha = x\}$. For $Y \subseteq X_n$ we define

$$O_{n,Y} = \{\alpha \in O_n : \text{Fix}(\alpha) = Y\} \text{ and } C_{n,Y} = \{\alpha \in C_n : \text{Fix}(\alpha) = Y\}. $$

We write $O_{n,m}$ instead of $O_{n,Y}$ when $Y = \{m\}$. The $n$th Catalan number $C_n$ is $\frac{1}{n+1} \binom{2n}{n}$ (see, for example [3, 9]).

The numbers of transformations of $O_n$ and $C_n$ with $r$ fixed points have been computed by Higgins, and Laradji and Umar in [3, 7]. In both [3] and [7], there is no information about the cardinalities of the sets $O_{n,Y} = \{\alpha \in O_n : \text{Fix}(\alpha) = Y\}$ and $C_{n,Y} = \{\alpha \in C_n : \text{Fix}(\alpha) = Y\}$ for any non-empty subset $Y$ of $X_n$. The

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The aim of this paper we compute these cardinalities as follows:

$$|O_{n,Y}| = C_{m_1-1} (\prod_{j=2}^{r} C_{m_j-m_{j-1}}) C_{n-m_r}$$

for any $Y = \{m_1, m_2, \ldots, m_r\}$ with $m_1 < m_2 < \cdots < m_r$, and

$$|C_{n,Y}| = (\prod_{j=2}^{r-1} C_{m_j+1-m_j}) C_{n-m_r+1}$$

for any $Y = \{1, m_2, \ldots, m_r\}$ with $m_1 = 1 < m_2 < m_3 < \cdots < m_r$. Consequently, we also show that there are $\frac{r}{n} \binom{2n}{n+r}$ order-preserving transformations in $O_n$ with $r$ fixed points as in [3, 7], and that there are $\frac{r}{2n-r} \binom{2n-r}{n}$ order-preserving and order-decreasing transformations in $C_n$ with $r$ fixed points, as in [3].

2. Preliminaries

For any $\alpha \in T_n$ the equivalence relation $\equiv$ on $X_n$, defined by

$$x \equiv y \text{ if and only if } (\exists r, s \geq 0) x \alpha^r = y \alpha^s,$$

partitions $X_n$ into orbits $\Omega_1, \Omega_2, \ldots, \Omega_k$. The orbits are the connected components of the function graph, and provide valuable information about the structure of the transformation $\alpha$. Typically, an orbit consists of a cycle with some trees attached. If there are no attached trees, we say that the orbit $\Omega_i$ is cyclic, if the cycle consists of a single fixed point and $|\Omega_i| \geq 2$ we say that $\Omega_i$ is acyclic; if $\Omega_i$ consists of a single fixed point, we say that it is trivial (see [1, 3]). The following proposition was proved by Higgins in [3, Proposition 1.5]:

**Proposition 1** Each of the cycles of the components of $\alpha \in O_n$ consists of a unique fixed point. Each orbit of $\alpha \in O_n$ is convex in the ordered set $X_n$.

Since the orbits of $\alpha \in O_n$ are either acyclic or trivial, it follows that $\alpha \in O_n$ has a unique orbit if and only if $\alpha \in O_{n,m}$ for some $m \in X_n$.

A proof for the following result can be found in [3]:

**Lemma 2** $\sum_{k=1}^{n} C_{k-1} C_{n-k} = C_n$.

Let $C^+_{n}$ be the semigroups of all increasing and order-preserving full transformations on $X_n$. Then it is a well-known fact that $C_n$ and $C^+_{n}$ are “isomorphic”. Moreover, $|C_n| = |C^+_{n}| = C_n$ (see, for example [3, Theorem 3.1]). We denote the set of all nilpotent element of a semigroup $S$ with zero by $N(S)$. The following results were proved in [6, 7]:

**Lemma 3** $O_{n,1} = N(C_n)$, $O_{n,n} = N(C^+_{n})$ and $|O_{n,1}| = |O_{n,n}| = C_{n-1}$.
From [8, Ex 16b, p. 169] since
\[
\sum_{k=0}^{n} \frac{ac(p+qk)}{(a+bk)(c+bn-bk)} \left( \begin{array}{c} a+bk \\ k \end{array} \right) \left( \begin{array}{c} c+bn-bk \\ n-k \end{array} \right) = \frac{p(a+c) + aqn}{a+c+bn} \left( \begin{array}{c} a+c+bn \\ n \end{array} \right),
\]

it follows by replacing \( a, b, c, n, p \) and \( q \) with \( 2r, 2, 2, n-r-1, 1 \) and 0, and with \( r, 2, 2, n-r-2, 1 \) and 0, respectively that
\[
\sum_{k=0}^{n-r-1} \frac{r}{(r+k)(n-r-k)} \left( \begin{array}{c} 2r+2k \\ k \end{array} \right) \left( \begin{array}{c} 2n-2r-2k \\ n-r-1-k \end{array} \right) = \frac{r+1}{n} \left( \begin{array}{c} 2n \\ n-r-1 \end{array} \right) \tag{1}
\]
and
\[
\sum_{k=0}^{n-r-2} \frac{r}{(r+2k)(n-r-1-k)} \left( \begin{array}{c} r+2k \\ k \end{array} \right) \left( \begin{array}{c} 2n-2r-2-2k \\ n-r-2-k \end{array} \right) = \frac{r+2}{2n-(r+2)} \left( \begin{array}{c} 2n-(r+2) \\ n-(r+2) \end{array} \right). \tag{2}
\]

3. Order-preserving with fixed points

Proposition 4 Let \( \alpha \in O_{n,m} \). Then we have

(i) if \( 1 \leq x < m \leq n \) then \( x+1 \leq x\alpha \), and

(ii) if \( 1 \leq m < x \leq n \) then \( x\alpha \leq x-1 \).

Proof.  (i) Let \( \alpha \in O_{n,m} \). If \( 1 \leq x < m \leq n \) then either \( x+1 \leq x\alpha \) or \( x\alpha \leq x-1 \). If \( x = 1 \), then it is clear that \( 1\alpha \neq 1 \), and so \( 2 \leq 1\alpha \). Now suppose that \( 1 < x \), and that \( x\alpha \leq x-1 \). Since \( (x-1)\alpha \leq x\alpha \leq x-1 \), it follows that \( (x-1)\alpha \leq x-2 \). Similarly if we continue , then we have the following sequence
\[
(x-2)\alpha \leq x-3, \ (x-3)\alpha \leq x-4, \ldots , 2\alpha \leq 1.
\]
Thus we have \( 2\alpha = 1 \), and so \( 1\alpha = 1 \) which is a contradiction with \( \text{Fix}(\alpha) = \{m\} \neq \{1\} \), and hence \( x\alpha \geq x+1 \).

(ii) Let \( 1 \leq m < x \leq n \). If \( x = n \), then it is clear that \( n\alpha \neq n \), and so \( n\alpha \leq n-1 \). Now suppose that \( x < n \), and that \( x\alpha \geq x+1 \). Similarly, we have the following sequence
\[
(x+1)\alpha \geq x+2, \ (x+2)\alpha \geq x+3, \ldots , (n-1)\alpha \geq n.
\]
It follows that \( (n-1)\alpha = n \), and so \( n\alpha = n \) which is a contradiction with \( \text{Fix}(\alpha) = \{m\} \neq \{n\} \), and hence \( x\alpha \leq x-1 \).

We have the following corollary.

Corollary 5 For \( \alpha \in O_{n,m} \) if \( m \neq 1 \) then \( (m-1)\alpha = m \), and if \( m \neq n \) then \( (m+1)\alpha = m \).  \[ \square \]
Now consider the first special case $|Y| = 1$:

**Lemma 6** For every $m \in X_n$,

$$|O_{n,m}| = C_{m-1}C_{n-m}.$$  

**Proof.** For each $\alpha \in O_{n,m}$ we fix

$$\alpha_1 = \begin{pmatrix} 1 & \ldots & m-2 & m-1 & m \\ 1\alpha & \ldots & (m-2)\alpha & m & m \end{pmatrix}$$

and

$$\alpha_2 = \begin{pmatrix} 1 & 2 & \ldots & n-(m-1) \\ 1 & (m+2)\alpha - (m-1) & \ldots & n\alpha - (m-1) \end{pmatrix}.$$  

It follows from Proposition 4 that $\alpha_1 \in O_{m,m}$ and $\alpha_2 \in O_{n-m+1,1}$. Next consider the function

$$f : O_{n,m} \to O_{m,m} \times O_{n-m+1,1}$$

which maps each $\alpha \in O_{n,m}$ to the ordered pair $(\alpha_1, \alpha_2)$. Then it follows from Corollary 5 that $f$ is a bijection. Moreover, it follows from Lemma 3 that

$$|O_{n,m}| = |O_{m,m}| \cdot |O_{n-m+1,1}| = C_{m-1}C_{n-m},$$

as required.  

Next consider the second special case $|Y| = 2$.

**Lemma 7** If $Y = \{m, m+r\} \subseteq X_n \ (r \geq 1)$ then $|O_{n,Y}| = C_{m-1}C_{n-m-r}C_r$. In particular, $|O_{n,\{1,n\}}| = C_{n-1}$.

**Proof.** Let $Y = \{m, m+r\}$, and let $\alpha \in O_{n,Y}$. By Proposition 1 there exists a unique $0 \leq q \leq r-1$ such that

$$sa \leq m + q \quad \text{and} \quad ta \geq m + q + 1$$

for all $s \leq m + q$, and for all $t \geq m + q + 1$. Then we fix

$$\alpha_1 = \begin{pmatrix} 1 & 2 & \ldots & m+q \\ 1\alpha & 2\alpha & \ldots & (m+q)\alpha \end{pmatrix}$$

and

$$\alpha_2 = \begin{pmatrix} 1 \\ (m+q+1)\alpha - m - q & (m+q+2)\alpha - m - q & \ldots & n\alpha - m - q \end{pmatrix}$$

as above. Then it follows from Proposition 4 that $\alpha_1 \in O_{(m+q),m}$ and $\alpha_2 \in O_{(n-m-q),(r-q)}$. Next consider the function

$$f : O_{n,Y} \to \bigcup_{q=0}^{r-1} (O_{(m+q),m} \times O_{(n-m-q),(r-q)})$$
which maps each $\alpha \in O_{n,Y}$ to the ordered pair $(\alpha_1, \alpha_2)$. Since $f$ is a bijection, it follows from Lemmas 6 and 2 that

$$|O_{n,Y}| = \sum_{q=0}^{r-1} |O_{(m+q),m}| \cdot |O_{(n-m-q),(r-q)}|$$

$$= \sum_{q=0}^{r-1} (C_{m-1}C_q)(C_{r-q-1}C_{n-m-r}) = C_{m-1}C_{n-m-r} \sum_{q=0}^{r-1} C_{q}C_{r-q-1}$$

$$= C_{m-1}C_{n-m-r} \sum_{q=1}^{r} C_{q-1}C_{r-q} = C_{m-1}C_{n-m-r}C_r,$$

as required. \qedhere

Now we have the following theorem.

**Theorem 8** Let $Y = \{m_1, m_2, \ldots, m_r\}$ with $m_1 < m_2 < \cdots < m_r$ be any subset of $X_n$. Then

$$|O_{n,Y}| = \prod_{j=1}^{r+1} C_{k_j},$$

where $k_1 = m_1 - 1$, $k_j = m_j - m_{j-1}$ $(2 \leq j \leq r)$ and $k_{r+1} = n - m_r$.

**Proof.** By Lemmas 6 and 7 we suppose that $r \geq 3$. Let $Y = \{m_1, m_2, \ldots, m_r\}$ with $m_1 < m_2 < \cdots < m_r$, and let

$$k_1 = m_1 - 1, \quad k_j = m_j - m_{j-1} \quad (2 \leq j \leq r)$$

and $k_{r+1} = n - m_r$.

Then, for each $\alpha \in O_{n,Y}$, we fix

$$\alpha_1 = \left( \begin{array}{cccc}
1 & \cdots & k_1 & 1 + 1 \\
\alpha & \cdots & k_{1/\alpha} & 1 + 1
\end{array} \right),$$

$$\alpha_j = \left( \begin{array}{cccc}
1 & \cdots & 2 & k_j \\
1 & (m_{j-1} + 1) & \cdots & k_{j-1} + 1
\end{array} \right),$$

$$\alpha_{r+1} = \left( \begin{array}{cccc}
1 & \cdots & k_{r+1} & k_{r+1} + 1 \\
1 & \cdots & n - 1 & n - 1
\end{array} \right),$$

where $2 \leq j \leq r$. Then it follows from Proposition 4 that $\alpha_1 \in O_{k_1+1,k_1+1}$, $\alpha_j \in O_{k_j+1,\{1,k_j+1\}}$ $(2 \leq j \leq r)$ and $\alpha_{r+1} \in O_{k_{r+1}+1,1,1}$. Next, define the set

$$O^*_{n,Y} = O_{k_1+1,k_1+1} \times O_{k_2+1,\{1,k_2+1\}} \times \cdots \times O_{k_{r+1}+1,\{1,k_{r+1}+1\}} \times O_{k_{r+1}+1,1,1},$$

as the cartesian product of the $r+1$ sets. Now consider the function $f : O_{n,Y} \rightarrow O^*_{n,Y}$ which maps $\alpha \in O_{n,Y}$ to the ordered $(r+1)$-pair $(\alpha_1, \alpha_2, \ldots, \alpha_{r+1})$. Since $f$ is a bijection, it follows from Lemmas 3 and 7 that

$$|O_{n,Y}| = |O_{k_1+1,k_1+1}| \prod_{j=2}^{r} |O_{k_j+1,\{1,k_j+1\}}||O_{k_{r+1}+1,1,1}|$$

$$= \prod_{j=1}^{r+1} C_{k_j},$$

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For any \( r \in X_n \) we define
\[
F(n, r) = |\{ \alpha \in O_n : |\text{Fix}(\alpha)| = r \}|
\]
as the number of order-preserving transformations which have exactly \( r \) fixed points. Let \( Y = \{m_1, m_2, \ldots, m_r\} \) with \( m_1 < m_2 < \cdots < m_r \) be any subset of \( X_n \). Now take \( k_1 = m_1 \), \( k_j = m_j - m_{j-1} \) (2 \( \leq j \leq r \)) and \( k_{r+1} = n + 1 - m_r \). Then it is clear that \( (k_1, k_2, \ldots, k_{r+1}) \) is a positive integer solution of the equation
\[
x_1 + x_2 + \cdots + x_{r+1} = n + 1.
\]
(3)

Conversely, every positive integer solution of Equation (3) gives a subset of \( X_n \) with \( r + 1 \) elements. If we denote the set of all positive integer solutions of Equation (3) by \( \mathcal{P}_{r+1}(n+1) \), then we have
\[
F(n, r) = \sum_{(k_1, k_2, \ldots, k_{r+1}) \in \mathcal{P}_{r+1}(n+1)} C_{k_1-1}C_{k_2}C_{k_3} \cdots C_{k_r}C_{k_{r+1}-1}.
\]

Moreover, we have the following result.

**Theorem 9** \( F(n, r) = \frac{r}{n} \left( \frac{2n}{n+r} \right) \).

**Proof.** For this we use induction on \( r \). If \( r = 1 \) then it follows from Lemmas 6 and 2 that
\[
F(n, 1) = \sum_{m=1}^{n} |O_{n,m}| = \sum_{m=1}^{n} C_{m-1}C_{n-m} = C_n.
\]

Suppose that \( \alpha \in O_n \) has \( r + 1 \) fixed points, say \( m_1 < \cdots < m_r < m_{r+1} \). Then consider the orbit of \( \alpha \) which contains \( m_{r+1} \). Since, by Proposition 1, this orbit is convex, there exists a unique \( m_r < k \leq m_{r+1} \) such that the restricted transformation \( \alpha \big|_{Y_k} : Y_k = \{k, k+1, \ldots, n\} \rightarrow Y_k \) of \( \alpha \) has unique fixed point, and that the restricted transformation \( \alpha \big|_{X_{n-Y_k}} \) has \( r \) fixed points. Similarly, the transformations \( \alpha \big|_{Y_k} : Y_k \rightarrow Y_k \) with a unique fixed point can be put into one-to-one correspondence with \( \beta : X_{n-k+1} \rightarrow X_{n-k+1} \) with a unique fixed point. Since the number of such transformations is \( C_{n-k+1} \), and since \( k \in \{r + 1, \ldots, n\} \), it follows from the inductive hypothesis that
\[
F(n, r + 1) = \sum_{k=r+1}^{n} F(k-1, r)C_{n-k+1} = \sum_{k=0}^{n-r-1} F(r + k, r)C_{n-r-k}.
\]

Therefore, it follows from Equation (1) that
\[
F(n, r + 1) = \sum_{k=0}^{n-r-1} \frac{r}{r+k} \left( \frac{2r+2k}{2r+k} \right) \frac{1}{n-r-k} \left( \frac{2n-2r-2k}{n-r-1-k} \right)
\]
\[
= \frac{r+1}{n} \left( \frac{2n}{n-r-1} \right) = \frac{r+1}{n} \left( \frac{2n}{n+r+1} \right),
\]
as required. \( \square \)
4. Order-decreasing with fixed points

Finally, we consider the order-decreasing subsemigroup $C_n$ of $O_n$. Recall that $1 \in \text{Fix}(\alpha)$ for all $\alpha \in C_n$. For any $Y = \{1, m_2, m_3, \ldots, m_r\} \subseteq X_n$ we define

$$C_{n,Y} = \{\alpha \in C_n : \text{Fix(}\alpha) = Y\}. $$

Since $C_{n,\{1\}} = N(C_n)$, it follows from Lemma 3 that $|C_{n,\{1\}}| = C_{n-1}$. Next we have the following theorem.

**Theorem 10** Let $Y = \{1, m_2, \ldots, m_r\}$ with $m_1 = 1 < m_2 < \cdots < m_r$ ($r \geq 1$) be a subset of $X_n$. Then

$$|C_{n,Y}| = \prod_{j=1}^{r} C_{k_j-1},$$

where $k_j = m_{j+1} - m_j$ $(1 \leq j \leq r-1)$ and $k_r = n - m_r + 1$.

**Proof.** Since $|C_{n,\{1\}}| = C_{n-1}$, we suppose that $r \geq 2$. For each $\alpha \in C_{n,Y}$, we similarly fix

$$\alpha_j = \begin{pmatrix} 1 & 2 & \cdots & m_{j+1} - m_j \\ 1 & (m_j + 1)\alpha - m_j + 1 & \cdots & (m_{j+1} - 1)\alpha - m_j + 1 \end{pmatrix},$$

$$\alpha_r = \begin{pmatrix} 1 & 2 & \cdots & n - m_r + 1 \\ 1 & (m_r + 1)\alpha - m_r + 1 & \cdots & n\alpha - m_r + 1 \end{pmatrix},$$

where $1 \leq j \leq r-1$. Let $k_j = m_{j+1} - m_j$ $(1 \leq j \leq r-1)$ and $k_r = n - m_r + 1$. Similarly, we have $\alpha_j \in N(C_{k_j})$ for each $1 \leq j \leq r$. Now consider the function

$$f : C_{n,Y} \to N(C_{k_1}) \times N(C_{k_2}) \times \cdots \times N(C_{k_r})$$

which maps $\alpha \in C_{n,Y}$ to the ordered $r$-pair $(\alpha_1, \alpha_2, \ldots, \alpha_r)$. Since $f$ is a bijection, it follows from Lemma 3 that

$$|C_{n,Y}| = \prod_{j=1}^{r} C_{k_j-1},$$

as required. \hfill \Box

For every $r \in X_n$ we define

$$N(n, r) = |\{\alpha \in C_n : |\text{Fix(}\alpha)| = r\}|$$

as the number of order-decreasing and order-preserving transformations which have exactly $r$ fixed points. Let $Y = \{1, m_2, \ldots, m_r\}$ with $m_1 = 1 < m_2 < \cdots < m_r$ be a subset of $X_n$. Now take $k_j = m_{j+1} - m_j$ $(1 \leq j \leq r-1)$ and $k_r = n - m_r + 1$. Then it is clear that $(k_1, k_2, \ldots, k_r)$ is a positive integer solution of the equation

$$x_1 + x_2 + \cdots + x_r = n.$$  (4)
Conversely, every positive integer solution of Equation (4) gives a subset, which contains 1, of $X = 1$. Suppose that $r \in \mathbb{N}$.

We use induction on $m$ as required.

\[ N(n, r) = \sum_{(k_1, k_2, \ldots, k_r) \in P_r(n)} C_{k_1-1} C_{k_2-1} \cdots C_{k_r-1}. \]

Moreover, we have the following result.

**Theorem 11** $N(n, r) = \frac{r}{2n-r} \binom{2n-r}{n}$.

**Proof.** We use induction on $r$ as before. From Lemma 3 the equation holds for $r = 1$. Suppose that $\alpha \in C_n$ has $r + 1 \geq 2$ fixed points, say $1 < m_2 < \cdots < m_r < m_{r+1}$. Then consider the orbit of $\alpha$ which contains $m_{r+1}$. Since $\alpha \in C_n$, the restricted transformation $\alpha_{|Y_{r+1}} : Y_{r+1} = \{m_{r+1}, m_{r+1} + 1, \ldots, n\} \to Y_{r+1}$ of $\alpha$ has unique fixed point (namely $m_{r+1}$), and that the restricted transformation $\alpha_{|X_{m_{r+1}}} \in \mathcal{C}$ has $r$ fixed points. Similarly, the transformations $\alpha_{|Y_{r+1}} : Y_{r+1} \to Y_{r+1}$ with a unique fixed point can be put into one-to-one correspondence with $\beta : X_{n-m_{r+1}} \to X_{n-m_{r+1}}$ with a unique fixed point. Since the number of such transformations is $C_{n-m_{r+1}}$, and since $m_{r+1} \in \{r + 1, \ldots, n\}$, it follows from the inductive hypothesis that

\[
N(n, r + 1) = \sum_{k=r+1}^{n} N(k-1, r) C_{n-k} = \left( \sum_{k=0}^{n-r-2} N(r+k, r) C_{n-r-1-k} \right) + N(n-1, r) C_0.
\]

Therefore, it follows from Equation (2) that

\[
N(n, r + 1) = \sum_{k=0}^{n-r-2} \frac{r}{2n-r-2} \left( \frac{2n-r-2}{n} \right) \frac{1}{n-r-1-k} \left( \frac{2n-2r-2-2k}{n-r-2-2k} \right) + N(n-1, r) \\
= \frac{r+2}{2n-r-2} \binom{2n-r-2}{n} + \frac{r}{2n-r-2} \binom{2n-r-2}{n-1} \\
= (r+2) \cdot \frac{(2n-r-3)!}{n! \cdot (n-r-2)!} + r \cdot \frac{(2n-r-3)!}{(n-1)! \cdot (n-r-1)!} \\
= \frac{(2n-r-3)!}{n! \cdot (n-r-1)!} \cdot (r+1)(n-r-1) + \frac{2n-r-3}{2n-r-1} \\
= \frac{(2n-r-1)}{2n-r-1} \binom{2n-r-1}{n},
\]

as required. \qed
Notice that we also have the recurrence relation

\[ N(n, r + 1) = N(n, r) - N(n - 1, r - 1) \]

from Equation (5) as in [3, Equation 3.5].

References


