Top generalized local cohomology modules

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Abstract

Let \((R, \mathfrak{m})\) be a commutative Noetherian local ring and \(M, N\) two non-zero finitely generated \(R\)-modules with \(\text{pd}(M) = n < \infty\) and \(\dim(N) = d\). In this paper, we show that if the top generalized local cohomology module \(H^{n+d}_\mathfrak{m}(M, N) \neq 0\), then the following statements are equivalent:

(i) \(\text{Ann}(0 : H^{d}_\mathfrak{m}(N)) = \mathfrak{p}\) for all \(\mathfrak{p} \in \text{Var}(\text{Ann}(H^{d}_\mathfrak{m}(N)))\);

(ii) \(\text{Ann}(0 : H^{n+d}_\mathfrak{m}(M, N)) = \mathfrak{p}\) for all \(\mathfrak{p} \in \text{Var}(\text{Ann}(H^{n+d}_\mathfrak{m}(M, N)))\).

Key Words: Artinian module, Generalized local cohomology.

1. Introduction

Throughout this paper, we assume that \((R, \mathfrak{m})\) is a commutative Noetherian local ring and \(M, N\) two non-zero finitely generated \(R\)-modules with \(\text{pd}(M) = n < \infty\) and \(\dim(N) = d\). For each \(i \geq 0\), the generalized local cohomology module \(H^i_\mathfrak{m}(M, N) = \lim_{\longrightarrow} \text{Ext}^i_R(M/\mathfrak{m}^n M, N)\), was introduced by Herzog [8] and studied further by Suzuki [16]. With \(M = R\), one clearly obtains the ordinary local cohomology module which was introduced by Grothendieck; see for example [4]. There are several well-known properties concerning the generalized local cohomology modules. It is well known that \(H^{n+d}_\mathfrak{m}(M, N)\) is Artinian and \(H^i_\mathfrak{m}(M, N) = 0\) for all \(i > n + d\); see for example [12] and [14].

An elementary property of finitely generated modules is that \(\text{Ann}(N/\mathfrak{p}N) = \mathfrak{p}\) for all \(\mathfrak{p} \in \text{Var}(\text{Ann}(N))\). For any Artinian \(R\)-module \(A\), the dual property is as follows:

\(\text{Ann}(0 : A/\mathfrak{p}) = \mathfrak{p}\) for all \(\mathfrak{p} \in \text{Var}(\text{Ann}(A))\).

If \(R\) is complete with respect to \(\mathfrak{m}\)-adic topology, it follows by Matlis duality that the property (*) is satisfied for all Artinian \(R\)-modules. However, there are Artinian modules which do not satisfy this property. For example, let \(R\) be the Noetherian local domain of dimension 2 constructed by Ferrand and Raynaud [7].
such that its $\mathfrak{m}$-adic completion $\hat{R}$ has an associated prime $\hat{q}$ of dimension 1. Then the Artinian $R$-module $A = H^1_{\mathfrak{m}}(R)$ does not satisfy the property (*); cf. Cuong and Nhan [6]. However, it seems to us that the property (*) is an important property of Artinian modules. For example, the property (*) is closely related to some questions on dimension for Artinian modules. In [6], it shown that $\text{N-dim}(A) = \dim(R/\text{Ann}(A))$ provided $A$ satisfies the property (*), where $\text{N-dim}(A)$ is the Noetherian dimension of $A$ defined by Roberts [15] (see also [9]). Note that this equality does not hold in general. Concretely, with the Artinian $A$ satisfies the property (*). Notice that the property (*) has been studied by many authors (see, for example, [5, 2, 3]).

The purpose of this paper is to prove the following theorem.

**Theorem 1.1** Let the generalized local cohomology module $H^{n+d}_{\mathfrak{m}}(M, N) \neq 0$. Then the following statements are equivalent:

1. $\text{Ann}(0 : H^d_{\mathfrak{m}}(N)) = \mathfrak{p}$ for all $\mathfrak{p} \in \text{Var}(\text{Ann}(H^d_{\mathfrak{m}}(N)))$;
2. $\text{Ann}(0 : H^{n+d}_{\mathfrak{m}}(M, N)) = \mathfrak{p}$ for all $\mathfrak{p} \in \text{Var}(\text{Ann}(H^{n+d}_{\mathfrak{m}}(M, N)))$.

**2. The results**

Following Macdonald [10], every Artinian module $A$ has a minimal secondary representation $A = A_1 + \ldots + A_n$, where $A_i$ is $\mathfrak{p}_i$-secondary. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the choice of the minimal secondary representation of $A$. This set is called the set of *attached prime ideals* of $A$, and denoted by $\text{Att}(A)$. The *cohomological dimension* of $M$ and $N$ with respect to $\mathfrak{m}$ is defined as

$$\text{cd}(\mathfrak{m}, M, N) = \sup\{i : H^i_{\mathfrak{m}}(M, N) \neq 0\}.$$ 

The following theorem extends [11, Theorem 2.2].

**Theorem 2.1** Let the generalized local cohomology module $H^{n+d}_{\mathfrak{m}}(M, N) \neq 0$. Then $\text{Att}(H^{n+d}_{\mathfrak{m}}(M, N)) = \{\mathfrak{p} \in \text{Ass}(N) : \dim R/\mathfrak{p} = d\}$. In particular, $\text{Att}(H^{n+d}_{\mathfrak{m}}(M, N)) = \text{Att}(H^d_{\mathfrak{m}}(N))$.

**Proof.** We use induction on $d$. If $d = 0$, the module $N$ has finite length and so is annihilated by some power of $\mathfrak{m}$. Hence, by [13, Lemma 3.2], $H^n_{\mathfrak{m}}(M, N) \cong \text{Ext}^n_R(M, N)$ and so $$\text{Att}(H^n_{\mathfrak{m}}(M, N)) = \{\mathfrak{m}\} = \text{Ass}(N) = \{\mathfrak{p} \in \text{Ass}(N) : \dim R/\mathfrak{p} = 0\}.$$ The result has been proved in this case. Suppose, inductively, that $d \geq 1$ and that the result has been proved for non-zero, finitely generated $R$-modules of dimension $d-1$. Let $L$ be a largest submodule of $N$ with $\dim(L) < d$. (Note that, for two submodules $N_1$ and $N_2$ of $N$, and for any positive integer $t$, if $\dim(N_1) \leq t$ and $\dim(N_2) \leq t$, then $\dim(N_1 + N_2) \leq t$ and so $L$ is well defined.) Thus by the exact sequence $0 \rightarrow L \rightarrow N \rightarrow N/L \rightarrow 0$ we have $\dim(N) = \dim(N/L)$. It is easy to prove that $N/L$ has no non-zero submodule $K$ with $\dim(K) < d$ and so $\text{Ass}(N/L) \subseteq \{\mathfrak{p} \in \text{Supp}(N/L) : \dim R/\mathfrak{p} = d\}$. In addition, $\text{min}(\text{Supp}(N/L)) \subseteq \text{Ass}(N/L)$ and $\text{min}(\text{Supp}(N)) \subseteq \text{Ass}(N)$. Therefore $\text{Ass}(N/L) = \{\mathfrak{p} \in \text{Supp}(N/L) : \dim R/\mathfrak{p} = d\} \subseteq \{\mathfrak{p} \in \text{Ass}(N) : \dim R/\mathfrak{p} = d\}$.
dim \(R/p = d\). If \(p \in \text{Ass}(N)\) and \(\dim R/p = d\), then \(p \notin \text{Supp}(L)\), otherwise \(\dim R/p \leq \dim L < d\). Thus \(p \in \text{Supp}(N/L)\) and so \(\{p \in \text{Supp}(N/L) : \dim R/p = d\} = \{p \in \text{Ass}(N) : \dim R/p = d\}\). From the exact sequence

\[
H^{n+d}_m(M, L) \longrightarrow H^{n+d}_m(M, N) \longrightarrow H^{n+d}_m(M, N/L) \longrightarrow H^{n+d+1}_m(M, L),
\]
we have \(H^{n+d}_m(M, N) \cong H^{n+d}_m(M, N/L)\). Hence by this assumption our aim is to show that \(\text{Att}(H^{n+d}_m(M, N)) = \text{Ass}(N)\). Since \(d \geq 1\), from the exact sequence

\[
0 \longrightarrow \Gamma_m(N) \longrightarrow N \longrightarrow N/\Gamma_m(N) \longrightarrow 0,
\]
and [13, Lemma 3.2], we get that \(H^{n+d}_m(M, N) \cong H^{n+d}_m(M, N/\Gamma_m(N))\). Therefore we can assume that \(N\) is \(m\)-torsion free, and so \(\text{depth}(N) \geq 1\) by [4, Lemma 2.1.1]. Thus, for each \(x \in m\) which is a non-zero divisor on \(N\), we have \(\text{cd}(m, M, N/xN) \leq n + d - 1\), so that \(H^{n+d}_m(M, N/xN) = 0\), and the exact sequence

\[
0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0
\]
yields that \(H^{n+d}_m(M, N) = xH^{n+d}_m(M, N)\). It therefore follows form [4, Proposition 7.2.11] that \(\bigcup_{q \in \text{Att}(H^{n+d}_m(M, N))} q \subseteq \bigcup_{p \in \text{Ass}(N)} p\). Let \(q \in \text{Att}(H^{n+d}_m(M, N))\); it follows the above inclusion relation that \(q \subseteq p\) for some \(p \in \text{Ass}(N)\). Since \(H^{n+d}_m(-, -)\) is an \(R\)-linear functor, it follows that \(\text{Ann}(N) \subseteq \text{Ann}(H^{n+d}_m(M, N)) \subseteq q \subseteq p\). As \(d = \dim R/\text{Ann}(N) = \dim R/p\), it follows that \(q = p\). Therefore \(\text{Att}(H^{n+d}_m(M, N)) \subseteq \text{Ass}(N)\).

To establish the reverse inclusion, let \(p \in \text{Ass}(N)\), so that \(\text{cd}(m, M, R/p) = n + d\). By the theory of primary decomposition, there exists a \(p\)-primary submodule \(T\) of \(N\); thus \(N/T\) is a non-zero finitely generated \(R\)-module with \(\text{Ass}(N/T) = \{p\}\). By [1, Theorem B], we get that \(n + d = \text{cd}(m, M, R/p) \leq \text{cd}(m, M, N/T) \leq \text{cd}(m, M, N) = n + d\) and so \(H^{n+d}_m(M, N/T) \neq 0\). Therefore \(\emptyset \notin \text{Att}(H^{n+d}_m(M, N/T)) \subseteq \text{Ass}(N/T) = \{p\}\) and hence \(\text{Att}(H^{n+d}_m(M, N/T)) = \{p\}\). The exact sequence

\[
0 \longrightarrow T \longrightarrow N \longrightarrow N/T \longrightarrow 0
\]
induces an epimorphism \(H^{n+d}_m(M, N) \longrightarrow H^{n+d}_m(M, N/T) \longrightarrow 0\). Hence \(\{p\} \subseteq \text{Att}(H^{n+d}_m(M, N))\) and so \(\text{Ass}(N) \subseteq \text{Att}(H^{n+d}_m(M, N))\). This complete the proof that \(\text{Att}(H^{n+d}_m(M, N)) = \text{Ass}(N)\).

The following consequence immediately follows by Theorem 2.1 and [4, Proposition 7.2.11].

**Corollary 2.2** Let the situations be as in Theorem 2.1. Then

\[
\text{Var}(\text{Ann}(H^d_m(N))) = \text{Var}(\text{Ann}(H^{n+d}_m(M, N))).
\]

Following [5], let \(U_N(0)\) be the largest submodule of \(N\) of dimension less than \(d\). Note that if \(0 = \bigcap_{p \in \text{Ass}(N)} N(p)\) is a reduced primary decomposition of the zero submodule of \(N\), then \(U_N(0) = \bigcap_{\dim R/p = d} N(p)\). Therefore we have \(\text{Ass} N/U_N(0) = \{p \in \text{Ass}(N) : \dim R/p = d\}\). Hence \(\text{Supp} N/U_N(0) = \bigcup_{p \in \text{Ass}(N), \dim R/p = d} \text{Var}(p)\). The set \(\text{Supp} N/U_N(0)\) is called the unmixed support of \(N\) and denoted by \(U\text{Supp}(N)\).

**Corollary 2.3** Let the situations be as in Theorem 2.1. Then

\[
U\text{Supp}(N) = \text{Var}(\text{Ann}(H^{n+d}_m(M, N))).
\]
Proof. By [10] the set of all minimal prime ideals containing $\text{Ann}(H_{m}^{n+d}(M,N))$ and the set of all minimal elements of $\text{Att}(H_{m}^{n+d}(M,N))$ are the same. Therefore $\text{Var}(\text{Ann}(H_{m}^{n+d}(M,N))) = \bigcup_{p \in \text{Ass}(N), \dim R/p = d} \text{Var}(p) = \text{Usupp}(N)$. □

\textbf{Theorem 2.4} Let the situations be as in Theorem 2.1. Then the following statements are equivalent:

(i) $\text{Usupp}(N)$ is catenary;

(ii) $H_{m}^{d}(N)$ satisfies the property $(\ast)$;

(iii) $H_{m}^{n+d}(M,N)$ satisfies the property $(\ast)$.

Proof. By Corollaries 2.2, 2.3, we have $\text{Var}(\text{Ann}(H_{m}^{d}(N))) = \text{Var}(\text{Ann}(H_{m}^{n+d}(M,N))) = \text{Usupp}(N)$ and $\text{Var}(\text{Ann}_{R}(H_{m}^{d}(N))) = \text{Var}(\text{Ann}_{R}(H_{m}^{n+d}(M,N))) = \text{Usupp}_{R}(\hat{N})$. Hence the equivalence follows by [5, Proposition 2.2 and Theorem 3.4]. □

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References


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