B. Y. Chen inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature

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Abstract

In this paper, we prove B. Y. Chen inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature, i.e., relations between the mean curvature, scalar and sectional curvatures, Ricci curvatures and the sectional curvature of the ambient space. The equality cases are considered.

Key Words: Riemannian manifold of quasi-constant curvature, B. Y. Chen inequality, Ricci curvature

1. Introduction

In [11], B. Y. Chen and K. Yano introduced the notion of a Riemannian manifold \((M, g)\) of quasi-constant curvature as a Riemannian manifold with the curvature tensor satisfying the condition

\[
R(X, Y, Z, W) = a [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] +
+b [g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W) +
g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z)],
\]

(1.1)

where \(a, b\) are scalar functions and \(T\) is a 1-form defined by

\[
g(X, P) = T(X),
\]

(1.2)

and \(P\) is a unit vector field. It can be easily seen that, if the curvature tensor \(R\) is of the form (1.1), then the manifold is conformally flat. If \(b = 0\) then the manifold reduces to a space of constant curvature.

A non-flat Riemannian manifold \((M^n, g)\) \((n > 2)\) is defined to be a quasi-Einstein manifold [4] if its Ricci tensor satisfies the condition

\[
S(X, Y) = ag(X, Y) + b A(X)A(Y),
\]

where \(a, b\) are scalar functions such that \(b \neq 0\) and \(A\) is a non-zero 1-form such that \(g(X, U) = A(X)\) for every vector field \(X\) and \(U\) is a unit vector field. If \(b = 0\) then the manifold reduces to an Einstein manifold. It can be easily seen that every Riemannian manifold of quasi-constant curvature is a quasi-Einstein manifold.

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One of the basic problems in submanifold theory is to find simple relations between the extrinsic and intrinsic invariants of a submanifold. In [6], [7], [9] and [10], B. Y. Chen established some inequalities in this respect. They are called B. Y. Chen inequalities.

Afterwards, many geometers studied similar problems for different submanifolds in various ambient spaces, for example see [1]–[3], [12] and [13].

Motivated by the studies of the above authors, in the present paper, we study B. Y. Chen inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature.

2. Preliminaries

Let $M$ be an $n$-dimensional submanifold of an $(n + m)$-dimensional Riemannian manifold $N^{n+m}$. The Gauss and Weingarten formulas are given respectively by

$$
\tilde{\nabla} X = \nabla X + h(X, Y) \quad \text{and} \quad \tilde{\nabla} N = -A_N X + \nabla_\perp X
$$

for all $X, Y \in TM$ and $N \in T^\perp M$, where $\tilde{\nabla}$, $\nabla$ and $\nabla_\perp$ are the Riemannian, induced Riemannian and normal connections in $\tilde{M}$, $M$ and the normal bundle $T^\perp M$ of $M$, respectively, and $h$ is the second fundamental form related to the shape operator $A$ by $g(h(X, Y), N) = g(A_N X, Y)$. The Gauss equation is given by

$$
\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(X, Z), h(Y, W))
$$

(2.1) for all $X, Y, Z, W \in TM$, where $R$ is the curvature tensor of $M$.

The mean curvature vector $H$ is given by $H = \frac{1}{n} \text{trace}(h)$. The submanifold $M$ is totally geodesic in $N^{m+n}$ if $h = 0$, and minimal if $H = 0$ [5].

Using (1.1), the Gauss equation for the submanifold $M^n$ of a Riemannian manifold of quasi-constant curvature is

$$
R(X, Y, Z, W) = a [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + 
+b [g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W)] + 
g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z)] + 
+g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)).
$$

(2.2)

Let $\pi \subset T_x M^n$, $x \in M^n$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of $M^n$. For any orthonormal basis $\{e_1, \ldots, e_m\}$ of the tangent space $T_x M^n$, the scalar curvature $\tau$ at $x$ is defined by

$$
\tau(x) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j).
$$

We recall the following algebraic Lemma:

Lemma 2.1 [6] Let $a_1, a_2, \ldots, a_n, b$ be $(n + 1)$ ($n \geq 2$) real numbers such that

$$
\left( \sum_{i=1}^{n} a_i \right)^2 = (n - 1) \left( \sum_{i=1}^{n} a_i^2 + b \right).
$$

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Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3 = \ldots = a_n$.

Let $M^n$ be an $n$-dimensional Riemannian manifold, $L$ a $k$-plane section of $T_xM^n$, $x \in M^n$, and $X$ a unit vector in $L$.

We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of $L$ such that $e_1 = X$.

One defines [8] the Ricci curvature (or $k$-Ricci curvature) of $L$ at $X$ by

$$Ric_L(X) = K_{12} + K_{13} + \ldots + K_{1k},$$

where $K_{ij}$ denotes, as usual, the sectional curvature of the 2-plane section spanned by $e_i, e_j$. For each integer $k$, $2 \leq k \leq n$, the Riemannian invariant $\Theta_k$ on $M^n$ is defined by:

$$\Theta_k(x) = \frac{1}{k-1} \inf_{L, X} Ric_L(X), \quad x \in M^n,$$

where $L$ runs over all $k$-plane sections in $T_xM^n$ and $X$ runs over all unit vectors in $L$.

Decomposing the vector field $P$ on $M$ uniquely into its tangent and normal components $P^T$ and $P^\perp$, respectively, we have

$$P = P^T + P^\perp. \quad (2.3)$$

3. Chen First Inequality

Recall that the Chen first invariant is given by

$$\delta_{M^n}(x) = \tau(x) - \inf \{K(\pi) \mid \pi \subset T_xM^n, x \in M^n, \dim \pi = 2\},$$

(see for example [10]), where $M^n$ is a Riemannian manifold, $K(\pi)$ is the sectional curvature of $M^n$ associated with a 2-plane section, $\pi \subset T_xM^n, x \in M^n$ and $\tau$ is the scalar curvature at $x$.

Let us define

$$P_\pi = pr_\pi P, \quad (3.1)$$

where $\pi$ is a 2-plane section of $T_xM^n, x \in M^n$.

For submanifolds of a Riemannian manifold of quasi-constant curvature we establish the following optimal inequality, which will call Chen first inequality.

**Theorem 3.1** Let $M^n, n \geq 3$, be an $n$-dimensional submanifold of an $(n + m)$-dimensional Riemannian manifold of quasi-constant curvature $N^{n+m}$. Then we have

$$\delta_{M^n}(x) \leq (n - 2) \left( \frac{n^2}{2(n-1)} \|H\|^2 + (n+1) \frac{a}{2} \right)$$

$$+ b \left( (n - 1) \|P^T\|^2 - \|P_x\|^2 \right), \quad (3.2)$$

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where $\pi$ is a 2-plane section of $T_xM^n, x \in M^n$. The equality case of inequality (3.2) holds at a point $x \in M^n$ if and only if there exists an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of $T_xM^n$ and an orthonormal basis $\{e_{n+1}, ..., e_{n+m}\}$ of $T_x^\perp M^n$ such that the shape operators of $M^n$ in $N^{n+m}$ at $x$ have the forms

$$A_{e_{n+1}} = \begin{pmatrix}
a & 0 & 0 & \cdots & 0 \\
b & 0 & 0 & \cdots & 0 \\
0 & \mu & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \mu
\end{pmatrix}, \quad a + b = \mu,$$

$$A_{e_{n+1}} = \begin{pmatrix}
h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\
h_{21}^r & -h_{11}^r & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad 2 \leq i \leq m,$$

where we denote by $h_{ij}^r = g(h(e_i, e_j), e_r), 1 \leq i, j \leq n$ and $n + 1 \leq r \leq n + m$.

**Proof.** Let $x \in M^n$ and $\{e_1, e_2, ..., e_n\}$ and $\{e_{n+1}, ..., e_{n+m}\}$ be orthonormal basis of $T_xM^n$ and $T_x^\perp M^n$, respectively. For $X = W = e_i, Y = Z = e_j, i \neq j$, from the equations (2.2), (2.3) and (1.2) it follows that

$$a + b \left[ g(P^T, e_j)^2 + g(P^T, e_i)^2 \right] = R(e_i, e_j, e_j, e_i) +
+ g(h(e_i, e_j), h(e_i, e_j)) - g(h(e_i, e_i), h(e_j, e_j)).$$

By summation after $1 \leq i, j \leq n$, it follows from the previous relation that

$$2\tau + \|h\|^2 - n^2 \|H\|^2 = 2b(n - 1) \|P^T\|^2 + (n^2 - n)a, \tag{3.3}$$

where we denote by

$$\|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)).$$

One takes

$$\varepsilon = 2\tau - \frac{n^2(n - 2)}{n - 1} \|H\|^2 - (n^2 - n)a - 2b(n - 1) \|P^T\|^2. \tag{3.4}$$

Then, from (3.3) and (3.4) we get

$$n^2 \|H\|^2 = (n - 1) \left(\|h\|^2 + \varepsilon\right). \tag{3.5}$$

Let $x \in M^n, \pi \subset T_xM^n$, dim $\pi = 2, \pi = sp \{e_1, e_2\}$. We define $e_{n+1} = \frac{H}{\|H\|}$ and from the relation (3.5) we obtain

$$\left(\sum_{i=1}^n h_{ii}^{n+1}\right)^2 = (n - 1)(\sum_{i,j=1}^n \sum_{r=n+1}^{n+m} (h_{ij}^r)^2 + \varepsilon).$$

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or equivalently,
\[
\left( \sum_{i=1}^{n} h_{ii}^{n+1} \right)^2 = (n - 1) \left\{ \sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 \right\} + \sum_{i,j=1}^{n} \sum_{r=n+2}^{n+m} (h_{ij}^{r})^2 + \varepsilon. \tag{3.6}
\]

By using Lemma 2.1 we have from (3.6),
\[
2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n} \sum_{r=n+2}^{n+m} (h_{ij}^{r})^2 + \varepsilon. \tag{3.7}
\]

Gauss equation for \( X = W = e_1, Y = Z = e_2 \) gives
\[
K(\pi) = R(e_1, e_2, e_2, e_1) = a + b \left[ g(P^T, e_1)^2 + g(P^T, e_2)^2 \right] + \sum_{r=n+1}^{m} \left[ h_{11}^r h_{22}^r - (h_{12}^r)^2 \right] \geq
\]
\[
\geq a + b \left[ g(P^T, e_1)^2 + g(P^T, e_2)^2 \right] + \frac{1}{2} \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n} \sum_{r=n+2}^{n+m} (h_{ij}^{r})^2 + \varepsilon + \frac{1}{2} \sum_{r=n+2}^{n+m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{n+m} (h_{12}^r)^2 =
\]
\[
= a + b \left[ g(P^T, e_1)^2 + g(P^T, e_2)^2 \right] + \frac{1}{2} \sum_{i \neq j} (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{n+m} \sum_{i,j>2} (h_{ij}^r)^2 + \frac{1}{2} \sum_{r=n+2}^{n+m} \sum_{j>2} (h_{ij}^r h_{22}^r)^2 + \sum_{j>2} \left( (h_{11}^r)^2 + (h_{22}^r)^2 \right) + \frac{1}{2} \varepsilon \geq
\]
\[
\geq a + b \left[ g(P^T, e_1)^2 + g(P^T, e_2)^2 \right] + \frac{\varepsilon}{2},
\]

which implies
\[
K(\pi) \geq a + b \left[ g(P^T, e_1)^2 + g(P^T, e_2)^2 \right] + \frac{\varepsilon}{2}. \tag{3.8}
\]

From (3.1) it follows that
\[
g(P^T, e_1)^2 + g(P^T, e_2)^2 = \left\| P_\pi \right\|^2.
\]

Using (3.4) we get from (3.8)
\[
K(\pi) \geq \tau - (n - 2) \left[ \frac{n^2}{2(n + 1)} \left\| H \right\|^2 + (n + 1) \frac{q}{2} \right] + b \left[ \left\| P_\pi \right\|^2 - (n - 1) \left\| P^T \right\|^2 \right],
\]

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which represents the inequality to prove.

The equality case holds at a point \( x \in M^n \) if and only if it achieves the equality in all the previous inequalities and we have the equality in the Lemma.

\[
\begin{align*}
    h_{ij}^{n+1} &= 0, \quad \forall i \neq j, i, j > 2, \\
    h_{ij}^r &= 0, \quad \forall i \neq j, i, j > 2, r = n + 1, \ldots, n + m, \\
    h_{11}^r + h_{22}^r &= 0, \quad \forall r = n + 2, \ldots, n + m, \\
    h_{ij}^{n+1} &= h_{ij}^{n+1} = 0, \quad \forall j > 2, \\
    h_{11}^{n+1} + h_{22}^{n+1} &= h_{33}^{n+1} = \ldots = h_{nn}^{n+1}.
\end{align*}
\]

We may choose \( \{e_1, e_2\} \) such that \( h_{12}^{n+1} = 0 \) and we denote by \( a = h_{11}^r, b = h_{22}^r, \mu = h_{33}^{n+1} = \ldots = h_{nn}^{n+1} \).

It follows that the shape operators take the desired forms.

\[\square\]

**Corollary 3.2** Under the same assumptions as in Theorem 3.1, if \( P \) is tangent to \( M^n \), we have

\[
\delta_{M^n}(x) \leq (n - 2) \left[ \frac{n^2}{2(n-1)} \|H\|^2 + \left( n + 1 \right) \frac{a}{2} \right] + b \left[ n - 1 - \|P\|^2 \right].
\]

If \( P \) is normal to \( M^n \), we have

\[
\delta_{M^n}(x) \leq (n - 2) \left[ \frac{n^2}{2(n-1)} \|H\|^2 + \left( n + 1 \right) \frac{a}{2} \right].
\]

4. \( k \)-Ricci curvature

We first state a relationship between the sectional curvature of a submanifold \( M^n \) of a space of quasi-constant curvature and the associated squared mean curvature \( \|H\|^2 \). Using this inequality, we prove a relationship between the \( k \)-Ricci curvature of \( M^n \) (intrinsic invariant) and the squared mean curvature \( \|H\|^2 \) (extrinsic invariant), as another answer of the basic problem in submanifold theory which we have mentioned in the introduction.

**Theorem 4.1** Let \( M^n, n \geq 3, \) be an \( n \)-dimensional submanifold of an \((n + m)\)-dimensional space of quasi-constant curvature \( N^{n+m} \). Then we have

\[
\|H\|^2 \geq \frac{2\tau}{n(n-1)} - a - \frac{2b}{n} \|P^T\|^2.
\]  \hspace{1cm} \text{(4.1)}

**Proof.** Let \( x \in M^n \) and \( \{e_1, e_2, \ldots, e_n\} \) and orthonormal basis of \( T_x M^n \). The relation (3.3) is equivalent with

\[
n^2 \|H\|^2 = 2\tau + \|h\|^2 - (n^2 - n)a - 2b(n - 1) \|P^T\|^2.
\]  \hspace{1cm} \text{(4.2)}

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We choose an orthonormal basis \( \{ e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m} \} \) at \( x \) such that \( e_{n+1} \) is parallel to the mean curvature vector \( H(x) \) and \( e_1, \ldots, e_n \) diagonalize the shape operator \( A_{e_{n+1}} \). Then the shape operators take the forms

\[
A_{e_{n+1}} = \begin{pmatrix}
  a_1 & 0 & \cdots & 0 \\
  0 & a_2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & a_n \\
\end{pmatrix},
\]

(4.3)

\[ A_{e_r} = (h^r_{ij}), \ i, j = 1, \ldots, n; r = n + 2, \ldots, n + m, \text{trace} \ A_r = 0. \] (4.4)

From (4.2), we get

\[
n^2 \| H \|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{n+m} \sum_{i,j=1}^{n} (h^r_{ij})^2 - n(n-1)a - 2b(n-1) \| P^T \|^2. \] (4.5)

On the other hand, since

\[ 0 \leq \sum_{i<j} (a_i - a_j)^2 = (n-1) \sum_i a_i^2 - 2 \sum_{i<j} a_i a_j, \]

we obtain

\[
n^2 \| H \|^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + 2 \sum_{i<j} a_i a_j \leq n \sum_{i=1}^{n} a_i^2, \] (4.6)

which implies

\[
\sum_{i=1}^{n} a_i^2 \geq n \| H \|^2.
\]

We have from (4.5)

\[
n^2 \| H \|^2 \geq 2\tau + n \| H \|^2 - n(n-1)a - 2b(n-1) \| P^T \|^2 \] (4.7)

or, equivalently,

\[
\| H \|^2 \geq \frac{2\tau}{n(n-1)} - a - \frac{2b}{n} \| P^T \|^2,
\]

this proves the theorem. \( \square \)

**Corollary 4.2** Under the same assumptions as in Theorem 4.1, if \( P \) is tangent to \( M^n \), we have

\[
\| H \|^2 \geq \frac{2\tau}{n(n-1)} - a - \frac{2b}{n}.
\]
If $P$ is normal to $M^n$, we have

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - a.$$ 

Using Theorem 4.1, we obtain the following:

**Theorem 4.3** Let $M^n, n \geq 3$, be an $n$-dimensional submanifold of an $(n + m)$-dimensional Riemannian manifold of quasi-constant curvature $N^{n+m}$. Then, for any integer $k, 2 \leq k \leq n$, and any point $x \in M^n$, we have

$$\|H\|^2(p) \geq \Theta_k(p) - a - \frac{2b}{n} \|P^T\|^2.$$  

(4.8)

**Proof.** Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_x M$. Denote by $L_{i_1 \ldots i_k}$ the $k$-plane section spanned by $e_{i_1}, \ldots, e_{i_k}$. By the definitions, one has

$$\tau(L_{i_1 \ldots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \ldots, i_k\}} \text{Ric}_{L_{i_1 \ldots i_k}}(e_i),$$

$$\tau(x) = \frac{1}{C_{n-2}} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \tau(L_{i_1 \ldots i_k}).$$

From (4.1) and the above relations, one derives

$$\tau(x) \geq \frac{n(n-1)}{2} \Theta_k(p),$$

which implies (4.8). $\Box$

**Corollary 4.4** Under the same assumptions as in Theorem 4.3, if $P$ is tangent to $M^n$, we have

$$\|H\|^2(p) \geq \Theta_k(p) - a - \frac{2b}{n}.$$ 

If $P$ is normal to $M^n$, we have

$$\|H\|^2(p) \geq \Theta_k(p) - a.$$ 

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