On almost complex structures in the cotangent bundle

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Abstract

E. M. Patterson and K. Yano studied vertical and complete lifts of tensor fields and connections from a manifold $M_n$ to its cotangent bundle $T^*(M_n)$. Afterwards, K. Yano studied the behavior on the cross-section of the lifts of tensor fields and connections on a manifold $M_n$ to $T^*(M_n)$ and proved that when $\varphi$ defines an integrable almost complex structure on $M_n$, its complete lift $C \varphi$ is a complex structure. The main result of the present paper is the following theorem: Let $\varphi$ be an almost complex structure on a Riemannian manifold $M_n$. Then the complete lift $C \varphi$ of $\varphi$, when restricted to the cross-section determined by an almost analytic 1-form $\omega$ on $M_n$, is an almost complex structure.

Key word and phrases: Almost complex structure, cotangent bundle, cross-section, Nijenhuis tensor, analytic tensor field.

1. Preliminaries

Let $M_n$ be an n-dimensional manifold and $T^*(M_n)$ its cotangent bundle. We denote by $\mathfrak{X}_r^s(M_n)$ the set of all tensor fields of type $(r, s)$ on $M_n$. Similarly, we denote by $\mathfrak{X}_r^s(T^*(M_n))$ the corresponding set on $T^*(M_n)$.

In this section, we shall summarize all the basic definitions and results on cross-section in $T^*(M_n)$ that are needed later. Let $M_n$ be an n-dimensional manifold of class $C^\infty$ and $T^*(M_n)$ its cotangent bundle over $M_n$. If $x^i$ are local coordinates in a neighborhood $U$ of a point $x \in M_n$, then a covector $P$ at $x$ which is an element of $T^*(M_n)$ is expressible in the form $(x^i, p_i)$, where $p_i$ are components of $P$ with respect to the natural frame $\partial_i$. We may consider $(x^i, p_i) = (x^i, x^\tilde{i}) = x^J$, $i = 1, \ldots, n$; $\tilde{i} = n + 1, \ldots, 2n$; $J = 1, \ldots, 2n$ as local coordinates in a neighborhood $\pi^{-1}(U)$ ($\pi$ is the natural projection $T^*(M_n)$ onto $M_n$).

Now, consider $X \in \mathfrak{X}_1^1(M_n)$ and $\theta \in \mathfrak{X}_1^0(M_n)$, then $C X$ (complete lift) and $V \theta$ (vertical lift) have, respectively, components [5, p. 236], [6]

$$CX = \begin{pmatrix} X^h \\ -p_m \partial_h X^m \end{pmatrix}, \quad V \theta = \begin{pmatrix} 0 \\ \theta_h \end{pmatrix} \tag{1.1}$$

with respect to the coordinates $(x^h, x^\tilde{h})$ in $T^*(M_n)$, where $X^h$ and $\theta_h$ are local components of $X$ and $\theta$.

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For \( \varphi \in \mathcal{I}_1(M_n) \), we can define a vector field \( \gamma \varphi \in \mathcal{I}_0^0(T^*(M_n)) \) [5, p.232], [6]:

\[
\gamma \varphi = \left( \begin{array}{c}
0 \\
p_j \varphi^j
\end{array} \right)
\tag{1.2}
\]

where \( \varphi^j \) are local components of \( \varphi \) in \( M_n \). Clearly, we have \((\gamma \varphi)^V f = 0\) for any \( f \in \mathcal{I}_0^0(M_n) \), where \( V f = f \circ \pi \) is a vertical lift of \( f \). So that \( \gamma \varphi \) is a vertical vector field.

Suppose that there is given a 1-form \( \omega \in \mathcal{I}_0^0(M_n) \) whose local expression is \( \omega = \omega_i(x)dx^i \). Then the correspondence \( x \rightarrow \omega_x \), \( \omega_x \) being the value of \( \omega \) at \( x \in M_n \), determines a mapping \( \beta_\omega : M_n \rightarrow T^*(M_n) \), such that \( \pi \circ \beta_\omega = id_{M_n} \) and \( n \)-dimensional submanifold \( \beta_\omega(M_n) \) of \( T^*(M_n) \) is called the cross-section determined by \( \omega \) and its parametric representations are as follows:

\[
\begin{align*}
\begin{cases}
x^k &= x^k, \\
p_k &= \omega_k(x^1, \ldots, x^n),
\end{cases}
\tag{1.3}
\end{align*}
\]

with respect to the coordinates \( (x^k, p_k) \) in \( T^*(M_n) \). Differentiating (1.3) by \( x^j \), we see that \( n \) tangent vector fields \( B_j \) to \( \beta_\omega(M_n) \) have component

\[
B^K_j = \left( \frac{\partial x^K}{\partial x^j} \right) = \left( \begin{array}{c}
\delta^k_j \\
\partial_j \omega_k
\end{array} \right)
\tag{1.4}
\]

with respect to the natural frame \( \{ \partial_k, \partial_{\bar{k}} \} \) in \( T^*(M_n) \).

On the other hand, the fibre being represented by

\[
\begin{align*}
\begin{cases}
x^k &= \text{const.}, \\
p_k &= p_k.
\end{cases}
\tag{1.5}
\end{align*}
\]

On differentiating (1.5) by \( p_j \), we see that \( n \) tangent vector fields \( C_j \) to the fibre have components

\[
C^K_j = \left( \frac{\partial x^K}{\partial p_j} \right) = \left( \begin{array}{c}
0 \\
\delta^k_j
\end{array} \right)
\tag{1.6}
\]

with respect to the natural frame \( \{ \partial_k, \partial_{\bar{k}} \} \) in \( T^*(M_n) \). \( 2n \) local vector fields \( B_j \) and \( C_j \), being linearly independent, form a frame along the cross-section. We call this the adapted \((B,C)\)-frame along the cross-section [4]. Taking account of (1.1) and (1.2) on the cross-section, we can see that \( C X, V \theta \) and \( \gamma \varphi \) have along \( \beta_\omega(M_n) \) components of the form [4], (see also [5])

\[
C X = \left( \begin{array}{c}
X^j \\
-L_X \omega_j
\end{array} \right), \quad V \theta = \left( \begin{array}{c}
0 \\
\theta_j
\end{array} \right), \quad \gamma \varphi = \left( \begin{array}{c}
0 \\
\omega_h \varphi^h_j
\end{array} \right)
\tag{1.7}
\]

with respect to the adapted \((B,C)\)-frame. Similarly, if \( N \in \mathcal{I}_1^2(M_n) \), then \( \gamma N \in \mathcal{I}_1^1(T^*(M_n)) \) is an affinor field along \( \beta_\omega(M_n) \) with components [5, p. 232]

\[
\gamma N = \left( \begin{array}{c}
0 \\
N^h_{ij} \omega_h \\
0
\end{array} \right)
\tag{1.8}
\]

with respect to the adapted \((B,C)\)-frame, where \( S^h_{ij} \) are local components of \( S \) in \( M_n \) (For applications of \( \gamma N \), see the formula (2.8)).
2. Main results

Let \( \varphi \in \mathcal{I}^1(M_n) \) and \( \omega \in \mathcal{I}^0(M_n) \). We define an operator

\[
\Phi_\varphi : \mathcal{I}^0(M_n) \to \mathcal{I}^2(M_n)
\]

associated with \( \varphi \) and applied to the 1-form \( \omega \) by

\[
(\Phi_\varphi \omega)(X; Y) = (L_{\varphi X} \omega - L_X \tilde{\omega})(Y) = (\varphi X)(\omega(Y)) - X(\omega(\varphi Y)) + \omega((L_Y \varphi)X),
\]

where \( \tilde{\omega}(Y) = (\omega \circ \varphi)(X) = \omega(\varphi Y) \) for any \( X, Y \in \mathcal{I}^1(M_n) \).

When \( \varphi \) is an almost complex structure, a 1-form satisfying \( \Phi_\varphi \omega = 0 \) is said to be almost analytic [5, p. 309].

In a Riemannian connection \( \nabla \), the equation of almost analytic 1-form \( \omega \):

\[
(\varphi X)(\omega(Y)) - X(\omega(\varphi Y)) + \omega((L_Y \varphi)X) = 0
\]

may be written as

\[
(\nabla_{\varphi X} \omega)(Y) - (\nabla_X \omega)(\varphi Y) - \omega((\nabla_X \varphi)Y) + \omega((\nabla_Y \varphi)X) = 0,
\]

which is equivalent to the condition for the almost analyticity. Thus, the equation (2.1) is an expression of the condition for the 1-form \( \omega \) to be almost analytic in terms a Riemannian connection \( \nabla \).

**Remark:** A tensor field \( \eta \in \mathcal{I}^0(M_n) \) which satisfies

\[
\eta(\varphi X, Y) = \eta(X, \varphi Y)
\]

_for any \( X, Y \in \mathcal{I}^1(M_n) \) is said to be pure._ Applications of this type tensor fields are studied by many authors (for example see [1–3]).

From (2.1), taking the alternation with respect to \( X \) and \( Y \), we find that

\[
(\nabla_{\varphi X} \omega)(Y) - (\nabla_X \omega)(\varphi Y) + (\nabla_Y \omega)(\varphi X) - (\nabla_X \varphi)(\varphi Y) = 0,
\]

i.e. \( (\nabla_X \omega)Y - (\nabla_Y \omega)X = (\wedge \nabla \omega)(X, Y) \) is the pure 2-form with respect to the structure \( \varphi \) for an almost analytic 1-form \( \omega \) on a Riemannian manifold.

We calculate

\[
\begin{align*}
-\omega((\nabla_X \varphi)Y) + \omega((\nabla_Y \varphi)X) &= -\omega((\nabla_X \varphi)Y) \\
(\nabla_X \omega)(\varphi Y) - (\nabla_X \omega)(\varphi Y) + \omega((\nabla_Y \varphi)X) &= 0 \\
(\nabla_Y \omega)(\varphi X) - (\nabla_Y \omega)(\varphi X) &= 0 \\
(\nabla_X \omega)(\varphi Y) + (\nabla_Y \omega)(\varphi X) &= 0
\end{align*}
\]

(2.2)
By virtue of (2.2), the equation (2.1) is written as

\[(\nabla Y \tilde{\omega})X - (\nabla X \tilde{\omega})Y = (\nabla Y \omega)(\varphi X) - (\nabla \varphi X \omega)(Y). \tag{2.3}\]

If we substitute \(\varphi X\) into \(X\), then the equation (2.3) may also be written as

\[-((\nabla Y \omega)X - (\nabla X \omega)Y) = (\nabla Y \tilde{\omega})\varphi X - (\nabla \varphi X \tilde{\omega})Y\]

or

\[(\nabla Y \tilde{\omega})X - (\nabla X \tilde{\omega})Y = (\nabla Y \tilde{\omega})\varphi X - (\nabla \varphi X \tilde{\omega})Y, \tag{2.4}\]

where \(\tilde{\omega} = \tilde{\omega} \circ \varphi\). The equation (2.4) is condition that \(\tilde{\omega} \in \mathfrak{A}_1^0(M_n)\) be almost analytic.

From equations (2.3) and (2.4), we have

**Theorem 1** If a 1-form \(\omega\) on a Riemannian manifold with an almost complex structure \(\varphi\) is almost analytic, then the 1-form \(\tilde{\omega} = \omega \circ \varphi\) is also almost analytic.

We shall now prove the following proposition.

**Proposition** In a Riemannian manifold, the condition

\[\Phi_\varphi \tilde{\omega} = (\Phi_\varphi \omega) \circ \varphi + \omega \circ N_\varphi\]

holds, where \(N_\varphi\) is the Nijenhuis tensor of \(\varphi\).

**Proof.** We shall now apply the operator \(\Phi_\varphi\) to the 1-form \(\tilde{\omega} = \omega \circ \varphi\)

\[(\Phi_\varphi \tilde{\omega})(X; Y) = (L_{\varphi X} \tilde{\omega} - L_X (\tilde{\omega} \circ \varphi))(Y) = (L_{\varphi X} (\omega \circ \varphi) - L_X ((\omega \circ \varphi) \circ \varphi))(Y)\]

\[= ((L_{\varphi X} \omega) \circ \varphi + \omega \circ (L_{\varphi X} \varphi) - (L_X (\omega \circ \varphi)) \circ \varphi - (\omega \circ \varphi) \circ (L_X \varphi))(Y)\]

\[= (L_{\varphi X} \omega - L_X (\omega \circ \varphi))(\varphi Y) + (\omega \circ (L_{\varphi X} \varphi) - (\omega \circ \varphi) \circ (L_X \varphi))(Y)\]

\[= (L_{\varphi X} \omega - L_X (\omega \circ \varphi))(\varphi Y) + \omega((L_{\varphi X} \varphi)Y) - \omega(\varphi(L_X \varphi)Y)\]

\[= (\Phi_\varphi \omega)(\varphi Y) + \omega([\varphi, \varphi]X, Y] - \varphi(X, \varphi Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y])\]

\[= (\Phi_\varphi \omega)(X; \varphi Y) + \omega(N_\varphi(X, Y)). \tag{2.5}\]

Thus, the proof is complete.

We note that the 1-form \(\omega\) in Proposition is not necessary to be almost analytic, in general. In particular, if the 1-form \(\omega\) is almost analytic, then from Theorem 1 and Proposition, we have

**Theorem 2** For an almost analytic 1-form \(\omega\) on a Riemannian manifold with an almost complex structure \(\varphi\), we have the following equation.

\[\omega \circ N_\varphi = 0.\]
Let \( \varphi \in \mathcal{S}_1(M_n) \). Then, the complete lift \( C\varphi \) of \( \varphi \) along the cross-section \( \omega \) to \( T^*(M_n) \) has local components of the form

\[
C\varphi = \begin{pmatrix}
\varphi_h^i - \partial_h \varphi_i^a \partial_a \omega - \partial_h \varphi_i^a + \partial_a \varphi_h^i + \partial_i \omega_h & 0 \\
0 & \varphi_h^i
\end{pmatrix}
\]

with respect to the adapted \((B,C)\)-frame [4]. We consider that the local vector fields

\[
C X_{(i)} = C \left( \frac{\partial}{\partial x^i} \right) = C \left( \delta_h^i \frac{\partial}{\partial x^h} \right) = \begin{pmatrix} X^i \\ 0 \end{pmatrix}
\]

and

\[
V X^{(i)} = V(dx^i) = V(\delta_h^i dx^h) = \begin{pmatrix} 0 \\ \delta_h^i \end{pmatrix}
\]

\( i = 1, \ldots, n; \bar{i} = n+1, \ldots, 2n \) span the module of vector fields in \( \pi^{-1}(U) \). Hence, any tensor fields is determined in \( \pi^{-1}(U) \) by their actions on \( C X \) and \( V \theta \) for any \( X \in \mathcal{S}_0(M_n) \) and \( \theta \in \mathcal{S}_1(M_n) \). The complete lift \( C\varphi \) has the properties

\[
\begin{align*}
C\varphi(C X) &= C(\varphi(X)) + \gamma(L_X \varphi), \\
C\varphi(V \theta) &= V(\varphi(\theta)),
\end{align*}
\]

which characterize \( C\varphi \), where \( \varphi(\theta) \in \mathcal{S}_1(M_n) \).

**Theorem 3**  Let \( M_n \) be a Riemannian manifold with an almost complex structure \( \varphi \). Then the complete lift \( C\varphi \in \mathcal{S}_1(T^*(M_n)) \) of \( \varphi \), when restricted to the cross-section determined by an almost analytic 1-form \( \omega \) on \( M_n \), is an almost complex structure.

**Proof.** Let \( \varphi, \psi \in \mathcal{S}_1(M_n) \) and \( N \in \mathcal{S}_2(M_n) \), using (1.7), (1.8) and (2.6), we have

\[
\gamma(\varphi \mp \psi) = \gamma(\varphi) \mp \gamma(\psi), \quad C\varphi(\gamma(\psi)) = \gamma(\varphi \circ \varphi), \quad (\gamma N)(C X) = \gamma N X
\]

(2.7)

where \( N_X \) is the tensor field of type \((1,1)\) on \( M_n \) defined by \( N_X(Y) = N(X,Y) \) for any \( Y \in \mathcal{S}_0(M_n) \).

If \( X \in \mathcal{S}_0(M_n) \), then from (2.6) and (2.7), we have

\[
\begin{align*}
(C\varphi)^2(C X) &= (C\varphi \circ C\varphi)(C X) = C\varphi(C\varphi(C X)) = C\varphi(C(\varphi(X))) \\
+ \gamma(L_X \varphi) &= C\varphi(C(\varphi(X))) + C\varphi(\gamma(L_X \varphi)) = (C\varphi(\varphi(X))) \\
+ \gamma(L_X \varphi) + \gamma((L_X \varphi) \circ \varphi) &= C((\varphi \circ \varphi)(X)) + \gamma(L_X \varphi) + (L_X \varphi) \circ \varphi
\end{align*}
\]

\[
= C(\varphi^2)(C X) - \gamma(L_X (\varphi \circ \varphi)) + \gamma(L_X \varphi + (L_X \varphi) \circ \varphi)
\]

\[
= C(\varphi^2)(C X) + \gamma(L_X \varphi - \varphi(L_X \varphi)) = C(\varphi^2)(C X) + \gamma(N_{\varphi,X})
\]
\[ (C\varphi^2)(C^X) = (C\varphi)(C^X), \quad (2.8) \]

where \( N_{\varphi,X}(Y) = (L_{\varphi,X}\varphi - \varphi(L_{\varphi,X}\varphi))(Y) = [\varphi X, \varphi Y] - \varphi [X, \varphi Y] - \varphi [\varphi X, Y] + \varphi^2 [X, Y] = N_\varphi(X, Y) \) is nothing but the Nijenhuis tensor constructed by \( \varphi \) and \( \gamma N_\varphi \) has local coordinates of the form \( \gamma N_\varphi = \begin{pmatrix} 0 & 0 \\ N^h_{ij} \omega_h & 0 \end{pmatrix} \) (see (1.8)).

Similarly, if \( \theta \in \mathcal{I}_1^0(M_n) \), then by (2.6), we have

\[
(C\varphi)^2(V\theta) = (C\varphi \circ C\varphi)(V\theta) = C\varphi(C\varphi(V\theta)) = C\varphi(V(\varphi(\theta)) = V(\varphi(\varphi(\theta))) = C(\varphi^2)(V\theta) \quad (2.9)
\]

By virtue of Theorem 2, we can easily say that \( \gamma N_\varphi = 0 \). From (2.8), (2.9) and linearity of the complete lift, we have

\[
(C\varphi)^2 = C(\varphi^2) = C(-I_{M_n}) = -I_{T^*(M_n)}.
\]

This completes the proof. \( \square \)

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**References**


