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Products of multiplication, composition and differentiation between weighted Bergman-Nevanlinna and Bloch-type spaces

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Abstract

Let φ and ψ be holomorphic maps on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Let C_φ, M_ψ and D be the composition, multiplication and differentiation operators, respectively. In this paper, we consider linear operators induced by products of these operators from Bergman-Nevanlinna spaces $\mathcal{A}^\beta_\mathcal{N}$ to Bloch-type spaces. In fact, we prove that these operators map $\mathcal{A}^\beta_\mathcal{N}$ compactly into Bloch-type spaces if and only if they map $\mathcal{A}^\beta_\mathcal{N}$ boundedly into these spaces.

Key word and phrases: Composition operator, Multiplication operator, Differentiation operator, Bergman space, Bloch space, Growth space.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the algebra of all functions holomorphic on $\mathbb{D}$. Let $dA(z) = \dfrac{1}{\pi}dxdy = \dfrac{1}{\pi}rdrd\theta$ be the normalized area measure on $\mathbb{D}$. For each $\beta \in (-1, \infty)$, we set $d\nu_\beta(z) = (\beta+1)(1-|z|^2)^\beta dA(z)$, $z \in \mathbb{D}$. Then $d\nu_\beta$ is a probability measure on $\mathbb{D}$. For $0 < p < \infty$ the weighted Bergman space $\mathcal{A}^p_\beta$ is defined as

$$\mathcal{A}^p_\beta = \Big\{ f \in H(\mathbb{D}) : ||f||_{\mathcal{A}^p_\beta} = \Big(\int_\mathbb{D} |f(z)|^p d\nu_\beta(z) \Big)^{1/p} < \infty \Big\}.$$

Note that $||f||_{\mathcal{A}^p_\beta}$ is a norm only if $1 \leq p < \infty$. When $0 < p < 1$, $\mathcal{A}^p_\beta$ is a an F-space with respect to the translation invariant metric defined by $d^\beta_p(f, g) = ||f - g||_{\mathcal{A}^p_\beta}$. The weighted Bergman-Nevanlinna class $\mathcal{A}^\beta_\mathcal{N}$ is defined by

$$\mathcal{A}^\beta_\mathcal{N} = \Big\{ f \in H(\mathbb{D}) : \int_\mathbb{D} \log^+ |f(z)| d\nu_\beta(z) < \infty \Big\},$$

where

$$\log^+ x = \begin{cases} \log x & \text{if} \quad x \geq 1 \\ 0 & \text{if} \quad x < 1. \end{cases}$$

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The space $\mathcal{A}_{\mathcal{N}}^{\beta}$ appears in the limit as $p \to 0$ of the weighted Bergman space $\mathcal{A}_{\beta}^{p}$ in the sense of

$$\lim_{p\to 0} \frac{t^p - 1}{p} = \log t, \qquad 0 < t < \infty.$$

The Bergman-Nevanlinna space $\mathcal{A}_{\mathcal{N}}^{\beta}$ contains all the Bergman spaces $\mathcal{A}_{\beta}^{p}$. Obviously, the inequalities

$$\log^+ x \le \log(1+x) \le 1 + \log^+ x, \quad x \ge 0 \tag{1.1}$$

imply that $f \in \mathcal{A}_{\mathcal{N}}^{\beta}$ if and only if

$$||f||_{\mathcal{A}_{\mathcal{N}}^{\beta}} = \int_{\mathbb{D}} \log(1 + |f(z)|) d\nu_\beta(z) < \infty.$$

Of course, we are abusing the term norm since $||f||_{\mathcal{A}_{\mathcal{N}}^{\beta}}$ fails to satisfy the properties of norm, but in this case $(f, g) \to ||f - g||_{\mathcal{A}_{\mathcal{N}}^{\beta}}$ defines a translation invariant metric on $\mathcal{A}_{\mathcal{N}}^{\beta}$ and this turns $\mathcal{A}_{\mathcal{N}}^{\beta}$ into a complete metric space. Also, by the subharmonicity of $\log(1 + |f(z)|)$, we have

$$\log(1 + |f(z)|) \le C_\beta \frac{||f||_{\mathcal{A}_{\mathcal{N}}^{\beta}}}{(1 - |z|^2)^{\beta+2}}, \qquad z \in \mathbb{D} \tag{1.2}$$

for all $f \in \mathcal{A}_{\mathcal{N}}^{\beta}$. In particular, (1.2) tells us that if $f_n \to f$ in $\mathcal{A}_{\mathcal{N}}^{\beta}$, then $f_n \to f$ locally uniformly. Here, locally uniform convergence refers to the uniform convergence on every compact subset of $\mathbb{D}$. For general background on weighted Bergman spaces $\mathcal{A}_{\beta}^{p}$ and weighted Bergman-Nevanlinna spaces $\mathcal{A}_{\mathcal{N}}^{\beta}$ one may consult [**3**], [**9**] and references therein.

Let $\alpha > 0$. A function f holomorphic in $\mathbb{D}$ is in α-Bloch space $\mathcal{B}^\alpha$ if

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty$$

and in the little α-Bloch Space $\mathcal{B}_0^\alpha$ if

$$\lim_{|z|\to 1} (1 - |z|^2)^\alpha |f'(z)| = 0.$$

For $f \in \mathcal{B}^\alpha$ define

$$||f||_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|.$$

With this norm $\mathcal{B}^\alpha$ is a Banach space and the little α-Bloch Space $\mathcal{B}_0^\alpha$ is a closed subspace of the α-Bloch Space. Note that $\mathcal{B}^1 = \mathcal{B}$, the usual Bloch space.

For any $\alpha > 0$, the space $\mathcal{A}^{-\alpha}$ consists of analytic functions f in $\mathbb{D}$ such that

$$||f||_{\mathcal{A}^{-\alpha}} = \sup\{(1 - |z|^2)^\alpha |f(z)| : z \in \mathbb{D}\} < \infty.$$

Each $\mathcal{A}^{-\alpha}$ is a non-separable Banach space with the norm defined above and contains all bounded analytic functions on $\mathbb{D}$. The closure in $\mathcal{A}^{-\alpha}$ of the set of polynomials will be denoted by $\mathcal{A}_0^{-\alpha}$, which is a separable Banach space and consists of exactly those functions f in $\mathcal{A}^{-\alpha}$ with

$$\lim_{|z|\to 1^-} (1 - |z|^2)^\alpha |f(z)| = 0.$$

If $\alpha > 1$, it is known that $f \in \mathcal{B}^\alpha$ if and only if $f \in \mathcal{A}^{-(\alpha-1)}$ or the antiderivative of f is in $\mathcal{B}^{\alpha-1}$. As $0 < \alpha < 1$, the space $\mathcal{B}^\alpha = Lip^{1-\alpha}$, the analytic Lipschitz space which contains analytic functions f on $\mathbb{D}$ satisfying

$$|f(z) - f(w)| \leq C|z - w|^{1-\alpha}$$

for any $z, w \in \mathbb{D}$. A good source for such spaces and their connection to Lipschitz spaces is [**10**]. See also [**2**] and [**5**].

Let φ be a holomorphic self-map on $\mathbb{D}$. The composition operator C_φ induced by φ is defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. The study of composition operators lie at the interface of analytic functions and operator theory. By the Littlewood subordination theorem every holomorphic self-map φ of $\mathbb{D}$ induces a bounded composition operator on Hardy and Bergman spaces while the boundedness on Bloch space follows by Schwarz-Pick Lemma.

For a fixed $\psi \in H(\mathbb{D})$, define the linear operator

$$\psi C_\varphi f = \psi(f \circ \varphi), \qquad f \in H(\mathbb{D}).$$

The operator ψC_φ is known as the weighted composition operator. The weighted composition operator is a generalization of the composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ and the multiplication operator M_ψ defined by $M_\psi f = \psi f$. Composition and weighted composition operators have gained increasing recognition during the last three decades, mainly due to the fact that they provide–just as, for example, Hankel and Toeplitz operators–ways and means to link classical function theory to functional analysis and operator theory. For general background on composition operators, we refer to [**1**], [**7**] and references therein.

Let D be the differentiation operator defined by

$$Df = f', \qquad f \in H(\mathbb{D}).$$

Hibschweiler and Portnoy [**4**] defined the linear operators DC_φ and $C_\varphi D$ and investigated the boundedness and compactness of these operators between Bergman spaces using Carleson-type measures. S. Ohno [**6**] discussed boundedness and compactness of $C_\varphi D$ between Hardy spaces. We can define products of these operators in the following six ways:

$$\begin{aligned}
(M_\psi C_\varphi D f)(z) &= \psi(z) f'(\varphi(z)), \\
(M_\psi D C_\varphi f)(z) &= \psi(z)\varphi'(z) f'(\varphi(z)), \\
(C_\varphi M_\psi D f)(z) &= \psi(\varphi(z)) f'(\varphi(z)), \\
(D M_\psi C_\varphi f)(z) &= \psi'(z) f(\varphi(z)) + \psi(z)\varphi'(z) f'(\varphi(z)), \\
(C_\varphi D M_\psi f)(z) &= \psi'(\varphi(z)) f(\varphi(z)) + \psi(\varphi(z)) f'(\varphi(z))
\end{aligned}$$

and

$$(D C_\varphi M_\psi f)(z) = \psi'(\varphi(z)) f(\varphi(z))\varphi'(z) + \psi(\varphi(z)) f'(\varphi(z))\varphi'(z)$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. X. Zhu in [**11**] have considered triple operators defined above from weighted Bergman spaces to Bers spaces.

In this paper, we study the boundedness and compactness of these operators from weighted Bergman-Nevanlinna to Bloch-type spaces. Note that the operator $M_\psi C_\varphi D$ induces many known operators. If $\psi(z)=1$, then $M_\psi C_\varphi D = C_\varphi D$ which was studied in [**4**] and [**6**]. When $\psi(z)=\varphi'(z)$, then we get the operator DC_φ, which was also studied in [**4**]. If we put $\varphi(z)=z$, then $M_\psi C_\varphi D = M_\psi D$, that is product of differentiation operator and multiplication operator. Also note that $M_\psi DC_\varphi = M_{\psi\varphi'} C_\varphi D$ and $C_\varphi M_\psi D = M_{\psi\circ\varphi} C_\varphi D$. Thus the corresponding characterizations of boundedness and compactness of $M_\psi DC_\varphi$ and $C_\varphi M_\psi D$ can be obtained by replacing ψ respectively by $\psi\varphi'$ and $\psi\circ\varphi$ in the results stated for $M_\psi C_\varphi D$. Also, the operator $DM_\psi C_\varphi$ induces many known operators. If $\psi(z)=1$, then $DM_\psi C_\varphi = DC_\varphi$. When $\varphi(z)=z$, then $DM_\psi C_\varphi = DM_\psi$. Throughout this paper, constants are denoted by C, they are positive and not necessary the same at each occurrence.

2. Boundedness and compactness

In this section, we characterize the boundedness and compactness of operators induced by products of composition, multiplication and differentiation from weighted Bergman-Nevanlinna to Bloch-type spaces. A subset E of $\mathcal{A}^\beta_\mathcal{N}$ is bounded if it is bounded for the defining F-norm $\|.\|_{\mathcal{A}^\beta_\mathcal{N}}$. Given a Banach space $\mathcal{Y}$, we say that a linear map $T:\mathcal{A}^\beta_\mathcal{N}\to\mathcal{Y}$ is bounded if $T(E)\subset\mathcal{Y}$ is bounded for every bounded subset E of $\mathcal{A}^\beta_\mathcal{N}$. In addition, we say that T is compact if $T(E)\subset\mathcal{Y}$ is relatively compact for every bounded set $E\subset\mathcal{A}^\beta_\mathcal{N}$.

The following criterion for compactness is a useful tool to us and it follows from standard arguments, for example, to those outlined in Proposition 3.11 of [**1**].

Lemma 2.1 *Let $\beta\in(-1,\infty)$ and $\mathcal{Y}=\mathcal{B}^\alpha$, $\mathcal{B}^\alpha_0$, $\mathcal{A}^{-\alpha}$ or $\mathcal{A}^{-\alpha}_0$. Let $T:\mathcal{A}^\beta_\mathcal{N}\to\mathcal{Y}$ be any one of the operators defined in the introduction. Then $T:\mathcal{A}^\beta_\mathcal{N}\to\mathcal{Y}$ is compact if and only if for any sequence $\{f_n\}$ in $\mathcal{A}^\beta_\mathcal{N}$ with $\sup\limits_n\|f_n\|_{\mathcal{A}^\beta_\mathcal{N}}=M<\infty$ and which converges to zero locally uniformly on $\mathbb{D}$, we have $\lim_{n\to\infty}\|Tf_n\|_\mathcal{Y}\to 0$ in $\mathcal{Y}$.*

Theorem 2.2 *Let $\alpha>0$, $\beta>-1, \psi\in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) $M_\psi C_\varphi D$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *boundedly into* $\mathcal{A}^{-\alpha}$.

(ii) $M_\psi C_\varphi D$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *compactly into* $\mathcal{A}^{-\alpha}$.

(iii) *For all* $c>0$,

$$\lim_{|\varphi(z)|\to1}\frac{(1-|z|^2)^\alpha|\psi(z)|}{1-|\varphi(z)|^2}\exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0$$

and $\psi\in\mathcal{A}^{-\alpha}$.

Proof. It suffices to check two implications: *(i)* $\Rightarrow$ *(iii)* and *(iii)* $\Rightarrow$ *(ii)*.

(i) $\Rightarrow$ *(iii)*. Suppose *(i)* holds. By taking $f(z)=z$ in $\mathcal{A}^\beta_\mathcal{N}$, we get $\psi\in\mathcal{A}^{-\alpha}$. Fix $z_0\in\mathbb{D}$. For $c>0$ and $w=\varphi(z_0)$, consider the function

$$f_w(z)=\exp\left\{\frac{c(1-|w|^2)^{\beta+2}}{(1-\overline{w}z)^{2(\beta+2)}}\right\}.$$

Using (1.1), we have

$$\|f_w\|_{\mathcal{A}^\beta_\mathcal{N}} \le 1 + c\int_{\mathbb{D}} \frac{(1-|w|^2)^{\beta+2}}{|1-\overline{w}z|^{2(\beta+2)}} d\nu_\beta(z) = 1 + c.$$

Since $M_\psi C_\varphi D$ maps $\mathcal{A}^\beta_\mathcal{N}$ boundedly into $\mathcal{A}^{-\alpha}$ and

$$f_w'(z) = \frac{2(\beta+2)c\overline{w}(1-|w|^2)^{\beta+2}}{(1-\overline{w}z)^{2(\beta+2)+1}} \exp\Big[\frac{c(1-|w|^2)^{\beta+2}}{(1-\overline{w}z)^{2(\beta+2)}}\Big],$$

there is a constant $N > 0$ depending only on c and β such that

$$\begin{aligned} N &\ge (1-|z_0|^2)^\alpha |\psi(z_0)||f_w'(\varphi(z_0))| \\ &= (1-|z_0|^2)^\alpha \frac{2(\beta+2)c|\varphi(z_0)||\psi(z_0)|}{(1-|\varphi(z_0)|^2)^{\beta+3}} \exp\Big[\frac{c(1-|w|^2)^{\beta+2}}{(1-|\varphi(z_0)|^2)^{2(\beta+2)}}\Big]. \end{aligned}$$

That is,

$$\frac{(1-|z_0|^2)^\alpha|\psi(z_0)|}{1-|\varphi(z_0)|^2} \exp\Big[\frac{c}{(1-|\varphi(z_0)|^2)^{\beta+2}}\Big] \le \frac{N(1-|\varphi(z_0)|^2)^{\beta+2}}{2(\beta+2)c|\varphi(z_0)|}.$$

Taking $\lim_{|\varphi(z_0)|\to 1}$ on both sides of above inequality, we get (iii).

$(iii) \Rightarrow (ii)$. Assume that (iii) is valid for all $c > 0$. Note that if $f \in \mathcal{A}^\beta_\mathcal{N}$, then by (1.1) and Cauchy integral formula for derivatives, we have

$$\begin{aligned} (1-|z|^2)|f'(z)| &\le \frac{2}{\pi}\int_{\partial\mathbb{D}} |f(z + \frac{1}{2}(1-|z|)\zeta)||d\zeta| \\ &\le \exp\Big[\frac{C_\beta\|f\|_{\mathcal{A}^\beta_\mathcal{N}}}{(1-|z|^2)^{\beta+2}}\Big]. \end{aligned}$$

Choose any sequence $\{f_n\}$ in $\mathcal{A}^\beta_\mathcal{N}$ such that $\|f_n\|_{\mathcal{A}^\beta_\mathcal{N}} \le M$ and $f_n \to 0$ locally uniformly on $\mathbb{D}$. By Lemma 2.1, it is sufficient to show that $||M_\psi C_\varphi D f_n||_{\mathcal{A}^{-\alpha}} \to 0$ as $n \to \infty$. For $r \in (0,1)$, we have

$$\begin{aligned} \sup_{|\varphi(z)|\le r}(1-|z|^2)^\alpha|M_\psi C_\varphi D f_n(z)| &= \sup_{|\varphi(z)|\le r}(1-|z|^2)^\alpha|\psi(z)||f_n'(\varphi(z))| \\ &\le A\sup_{|\varphi(z)|\le r}|f_n'(\varphi(z))| \quad \to 0 \end{aligned}$$

as $n \to \infty$, where $A = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\psi(z)| < \infty$. On the other hand, whenever $r \to 1$, we have

$$\sup_{|\varphi(z)|>r}(1-|z|^2)^\alpha|M_\psi C_\varphi D f_n(z)|$$

$$\le \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^\alpha|\psi(z)|}{(1-|\varphi(z)|^2)} \exp\Big[\frac{C_\beta}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] \quad \to 0.$$

Combining the above estimates, we see that $\|M_\psi C_\varphi D f_n\|_{\mathcal{A}^{-\alpha}} \to 0$ as $n \to \infty$. This completes the proof. □

Corollary 2.3 *Let $\alpha > 0, \beta > -1$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) $C_\varphi D$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *boundedly into* $\mathcal{A}^{-\alpha}$.

(ii) $C_\varphi D$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *compactly into* $\mathcal{A}^{-\alpha}$.

(iii) For all $c > 0$,
$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha}{1-|\varphi(z)|^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0.$$

Corollary 2.4. *Let $\alpha > 0, \beta > -1$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) DC_φ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *boundedly into* $\mathcal{A}^{-\alpha}$.

(ii) DC_φ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *compactly into* $\mathcal{A}^{-\alpha}$.

(iii) For all $c > 0$
$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0$$
and $\varphi' \in \mathcal{A}^{-\alpha}$.

Proof. Since $DC_\varphi = M_{\varphi'} C_\varphi D$, the result follows by replacing ψ by φ' in Theorem 2.2. □

Corollary 2.5 *Let $\alpha > 0, \beta > -1$ and $\psi \in H(\mathbb{D})$. Then the following are equivalent:*

(i) $M_\psi D$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *boundedly into* $\mathcal{A}^{-\alpha}$.

(ii) $M_\psi D$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *compactly into* $\mathcal{A}^{-\alpha}$.

(iii) $\psi \equiv 0$.

Proof. We only need to prove that $(i) \Rightarrow (iii)$. Since $M_\psi D = M_\varphi C_\varphi D$, where $\varphi(z) = z$, so by Theorem 2.2 $M_\psi D$ maps $\mathcal{A}^\beta_\mathcal{N}$ boundedly into $\mathcal{A}^{-\alpha}$ if and only if for every $c > 0$
$$\lim_{|z|\to 1} (1-|z|^2)^{\alpha-1}|\psi(z)| \exp\left[\frac{c}{(1-|z|^2)^{\beta+2}}\right] = 0$$
and $\psi \in \mathcal{A}^{-\alpha}$, which is possible only if $\psi = 0$. □

Theorem 2.6 *Let $\alpha > 0, \beta > -1, \psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) ψC_φ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *boundedly into* $\mathcal{A}^{-\alpha}$.

(ii) ψC_φ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *compactly into* $\mathcal{A}^{-\alpha}$.

(iii) *For all* $c>0$,

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|\psi(z)|\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big]=0$$

and $\psi\in\mathcal{A}^{-\alpha}$.

Proof. It suffices to check only two implications: $(i)\Rightarrow(iii)$ and $(iii)\Rightarrow(ii)$.

$(i)\Rightarrow(iii)$. Suppose (i) holds. By taking $f(z)=1$, the constant function 1 in $\mathcal{A}^\beta_\mathcal{N}$, we get $\psi\in\mathcal{A}^{-\alpha}$. Fix $z_0\in\mathbb{D}$ and $c>0$ and let $w=\varphi(z_0)$. Consider the function

$$f_w(z)=\Big(\frac{1-|w|^2}{(1-\overline{w}z)^2}\Big)^{\beta+2}\exp\Big\{\frac{c(1-|w|^2)^{\beta+2}}{(1-\overline{w}z)^{2(\beta+2)}}\Big\}.$$

Using (1.1) and the inequalities, $\log(1+x)\le x$ and

$$\log(1+xy)\le\log(1+x)+\log(1+y)\quad\text{for}\quad x,y\ge 0,$$

we have

$$\begin{aligned}\log(1+|f_w(z)|) &\le \log\Big[1+\Big(\frac{1-|w|^2}{|1-\overline{w}z|^2}\Big)^{\beta+2}\Big]+1+\Big\{\frac{c(1-|w|^2)^{\beta+2}}{|1-\overline{w}z|^{2(\beta+2)}}\Big\}\\ &\le 1+(1+c)\Big(\frac{1-|w|^2}{|1-\overline{w}z|^2}\Big)^{\beta+2},\end{aligned}$$

so

$$\|f_w\|_{\mathcal{A}^\beta_\mathcal{N}}\le 1+(1+c)\int_\mathbb{D}\Big(\frac{1-|w|^2}{|1-\overline{w}z|^2}\Big)^{\beta+2}d\nu_\beta(z)=2+c.$$

Since ψC_φ maps $\mathcal{A}^\beta_\mathcal{N}$ boundedly into $\mathcal{A}^{-\alpha}$, there is a constant $M>0$ such that

$$\begin{aligned}M &\ge (1-|z_0|^2)^\alpha|\psi(z_0)||f_w(\varphi(z_0))|\\ &= \frac{(1-|z_0|^2)^\alpha|\psi(z_0)|}{(1-|\varphi(z_0)|^2)^{\beta+2}}\exp\Big[\frac{c}{(1-|\varphi(z_0)|^2)^{\beta+2}}\Big].\end{aligned}$$

That is,

$$(1-|z_0|^2)^\alpha|\psi(z_0)|\exp\Big\{\frac{c}{(1-|\varphi(z_0)|^2)^{\beta+2}}\Big\}\le M(1-|\varphi(z_0)|^2)^{\beta+2}.$$

Taking the limit $\lim\limits_{|\varphi(z_0)|\to 1}$ on both sides of the above inequality, we get (iii).

$(iii)\Rightarrow(ii)$. Assume that (iii) is valid for all $c>0$. Using the estimate

$$|f(z)|\le\exp\Big\{\frac{C_\beta\|f\|_{\mathcal{A}^\beta_\mathcal{N}}}{(1-|z|^2)^{\beta+2}}\Big\}$$

and proceeding as in Theorem 2.2, we get (ii). □

Corollary 2.7 *Let* $\alpha>0,\beta>-1$ *and* φ *be a holomorphic self-map of* $\mathbb{D}$. *Then the following are equivalent:*

(i) C_φ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *boundedly into* $\mathcal{A}^{-\alpha}$.

(ii) C_φ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *compactly into* $\mathcal{A}^{-\alpha}$.

(iii) $\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0.$

Corollary 2.8 *Let* $\alpha>0, \beta>-1$ *and* $\psi\in H(\mathbb{D})$. *Then the following are equivalent:*

(i) M_ψ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *boundedly into* $\mathcal{A}^{-\alpha}$.

(ii) M_ψ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *compactly into* $\mathcal{A}^{-\alpha}$.

(iii) $\psi\equiv 0$.

Proof. The proof follows on same lines as the proof of Corollary 2.5. We omit the details. □

Lemma 2.9 *Let* $\alpha>0, \beta>-1, \psi\in H(\mathbb{D})$ *and* φ *be a holomorphic self-map of* $\mathbb{D}$. *Then the following are equivalent:*

(i) For all $c>0$,

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha|\psi(z)|}{1-|\varphi(z)|^2}\exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0. \tag{2.1}$$

(ii) For all $c>0$,

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|\psi(z)|}{1-|\varphi(z)|^2}\exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0 \tag{2.2}$$

and $\psi\in\mathcal{A}_0^{-\alpha}$.

Proof. $(i)\Rightarrow(ii)$ Suppose that (i) holds. Then

$$(1-|z|^2)^\alpha|\psi(z)|\le c\,\frac{(1-|z|^2)^\alpha|\psi(z)|}{1-|\varphi(z)|^2}\exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]\to 0$$

as $|z|\to 1$. Hence $\psi\in\mathcal{A}_0^{-\alpha}$. If $|\varphi(z)|\to 1$, then $|z|\to 1$, from which it follows that

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|\psi(z)|}{1-|\varphi(z)|^2}\exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right]=0.$$

$(ii)\Rightarrow(i)$ Suppose that (ii) holds, but (2.1) is not true for some $c>0$, then there are c_0 and ϵ_0 and a sequence $\{z_n\}$ tending to $\partial\mathbb{D}$ such that

$$\frac{(1-|z_n|^2)^\alpha|\psi(z_n)|}{1-|\varphi(z_n)|^2}\exp\left[\frac{c_0}{(1-|\varphi(z_n)|^2)^{\beta+2}}\right]\ge\epsilon_0. \tag{2.3}$$

Since $\psi\in\mathcal{A}_0^{-\alpha}$, (2.3) indicates that $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ with $|\varphi(z_{n_k})|\to 1$. Thus (2.2) produces the following limit:

$$\frac{(1-|z_{n_k}|^2)^\alpha|\psi(z_{n_k})|}{1-|\varphi(z_{n_k})|^2}\exp\left[\frac{c}{(1-|\varphi(z_{n_k})|^2)^{\beta+2}}\right]\to 0,$$

which contradicts (2.3). Hence we are done. □

Theorem 2.10 *Let $\alpha > 0, \beta > -1, \psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) $M_\psi C_\varphi D$ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *boundedly into* $\mathcal{A}_0^{-\alpha}$.

(ii) $M_\psi C_\varphi D$ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *compactly into* $\mathcal{A}_0^{-\alpha}$.

(iii) For all $c > 0$,

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |\psi(z)|}{1-|\varphi(z)|^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0.$$

Proof. It suffices to check two implications: $(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$. Using the same test functions as in Theorem 2.2 we see that

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\psi(z)|}{1-|\varphi(z)|^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0.$$

Again, since $M_\psi C_\varphi D$ maps $\mathcal{A}_\beta^0$ boundedly into $\mathcal{A}_0^{-\alpha}$, so by taking $f(z) = z$ in $\mathcal{A}_\beta^0$, we have $\psi \in \mathcal{A}_0^{-\alpha}$ and hence by Lemma 2.9, we have (iii).

$(iii) \Rightarrow (ii)$. The proof follows on same lines as the proof of the corresponding case of Theorem 2.2. So we omit the details. □

Corollary 2.11 *Let $\alpha > 0, \beta > -1$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) $C_\varphi D$ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *boundedly into* $\mathcal{A}_0^{-\alpha}$.

(ii) $C_\varphi D$ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *compactly into* $\mathcal{A}_0^{-\alpha}$.

(iii) For all $c > 0$,

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha}{1-|\varphi(z)|^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0.$$

Corollary 2.12 *Let $\alpha > 0, \beta > -1$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) DC_φ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *boundedly into* $\mathcal{A}_0^{-\alpha}$.

(ii) DC_φ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *compactly into* $\mathcal{A}_0^{-\alpha}$.

(iii) For all $c > 0$,

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0.$$

Corollary 2.13 *Let $\alpha > 0, \beta > -1$ and $\psi \in H(\mathbb{D})$. Then the following are equivalent:*

(i) $M_\psi D$ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *boundedly into* $\mathcal{A}_0^{-\alpha}$.

(ii) $M_\psi D$ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *compactly into* $\mathcal{A}_0^{-\alpha}$.

(iii) $\psi \equiv 0$.

Theorem 2.14 *Let* $\alpha > 0, \beta > -1, \psi \in H(\mathbb{D})$ *and* φ *be a holomorphic self-map of* $\mathbb{D}$. *Then the following are equivalent:*

(i) ψC_φ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *boundedly into* $\mathcal{A}_0^{-\alpha}$.

(ii) ψC_φ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *compactly into* $\mathcal{A}_0^{-\alpha}$.

(iii) *For all* $c > 0$,

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha |\psi(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0$$

and $\psi \in \mathcal{A}_0^{-\alpha}$.

(iv) *For all* $c > 0$,

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |\psi(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0.$$

Proof. That (iii) and (iv) are equivalent follows on the same lines as the proof of Lemma 2.9, whereas the proof of the implications $(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$ follows on the same lines as the proof of Theorem 2.10. We omit the details. □

Corollary 2.15 *Let* $\alpha > 0, \beta > -1$ *and* φ *be a holomorphic self-map of* $\mathbb{D}$. *Then the following are equivalent:*

(i) C_φ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *boundedly into* $\mathcal{A}_0^{-\alpha}$.

(ii) C_φ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *compactly into* $\mathcal{A}_0^{-\alpha}$.

(iii) *For all* $c > 0$,

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\right] = 0.$$

The above result was recently proved by Xiao in [**8**].

Corollary 2.16 *Let* $\alpha > 0, \beta > -1$ *and* $\psi \in H(\mathbb{D})$. *Then the following are equivalent:*

(i) M_ψ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *boundedly into* $\mathcal{A}_0^{-\alpha}$.

(ii) M_ψ *maps* $\mathcal{A}_{\mathcal{N}}^{\beta}$ *compactly into* $\mathcal{A}_0^{-\alpha}$.

(iii) $\psi \equiv 0$.

Theorem 2.17 *Let* $\alpha > 0, \beta > -1, \psi \in H(\mathbb{D})$ *and* φ *be a holomorphic self-map of* $\mathbb{D}$. *Then the following are equivalent:*

(i) $DM_\psi C_\varphi$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *boundedly into* $\mathcal{A}^{-\alpha}$.

(ii) $DM_\psi C_\varphi$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *compactly into* $\mathcal{A}^{-\alpha}$.

(iii) For all $c>0$,

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|\psi'(z)|\exp[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}]=0,$$

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)}\exp[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}]=0,$$

$\psi'\in\mathcal{A}^{-\alpha}$ *and* $\psi\varphi'\in\mathcal{A}^{-\alpha}$.

Proof. Suppose (i) holds. By taking $f(z)=c$, a constant function in $\mathcal{A}^\beta_\mathcal{N}$, we get $\psi'\in\mathcal{A}^{-\alpha}$. Again by taking $f(z)=z$ in $\mathcal{A}^\beta_\mathcal{N}$, we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\psi'(z)\varphi(z)+\psi(z)\varphi'(z)|<\infty.$$

Since $\psi'\in\mathcal{A}^{-\alpha}$ and $|\varphi(z)|<1$, we get

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\psi(z)\varphi'(z)|<\infty.$$

For $c>0$ and $\lambda\in\mathbb{D}$, consider the function

$$f_\lambda(z)=\Big\{\frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{3(1-\overline{\varphi(\lambda)}z)^{3(\beta+2)}}-\frac{1}{2}(\frac{1-|\varphi(\lambda)|^2}{(1-\overline{\varphi(\lambda)}z)^2})^{\beta+2}\Big\}$$

$$\exp\Big[6c\Big\{\frac{1}{2}(\frac{1-|\varphi(\lambda)|^2}{(1-\overline{\varphi(\lambda)}z)^2})^{\beta+2}-\frac{1}{3}\frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{(1-\overline{\varphi(\lambda)}z)^{3(\beta+2)}}\Big\}\Big].$$

It can be shown that every $f_\lambda\in\mathcal{A}^\beta_\mathcal{N}$ and $\|f_\lambda\|_{\mathcal{A}^\beta_\mathcal{N}}\le M$ for some $M>0$.
Moreover,

$$f_\lambda(\varphi(\lambda))=\frac{-1}{6}\frac{1}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\exp\Big[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\Big].$$

Also

$$f'_\lambda(z)=\Big[\Big\{\frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{(1-\overline{\varphi(\lambda)}z)^{3(\beta+2)+1}}-\frac{(1-|\varphi(\lambda)|^2)^{\beta+2}}{(1-\overline{\varphi(\lambda)}z)^{2(\beta+2)+1}}\Big\}(\beta+2)(\overline{\varphi(\lambda)}).$$

$$+6c(\beta+2)(\overline{\varphi(\lambda)})\Big\{\frac{(1-|\varphi(\lambda)|^2)^{\beta+2}}{(1-\overline{\varphi(\lambda)}z)^{2(\beta+2)+1}}-\frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{(1-\overline{\varphi(\lambda)}z)^{3(\beta+2)+1}}\Big\}$$

$$\Big\{\frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{3(1-\overline{\varphi(\lambda)}z)^{3(\beta+2)}}-\frac{1}{2}\Big(\frac{1-|\varphi(\lambda)|^2}{(1-\overline{\varphi(\lambda)}z)^2}\Big)^{\beta+2}\Big\}\Big]$$

$$\exp\Big[6c\Big\{\frac{1}{2}(\frac{1-|\varphi(\lambda)|^2}{(1-\overline{\varphi(\lambda)}z)^2})^{\beta+2}-\frac{1}{3}\frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{(1-\overline{\varphi(\lambda)}z)^{3(\beta+2)}}\Big\}\Big].$$

Moreover,

$$f'_\lambda(\varphi(\lambda)) = 0.$$

Since $DM_\psi C_\varphi$ maps $\mathcal{A}^0_\beta$ boundedly into $\mathcal{A}^{-\alpha}$, we can find some $M_0 > 0$ such that

$$\begin{aligned} M_0 &\geq (1-|\lambda|^2)^\alpha |\psi'(\lambda) f_\lambda(\varphi(\lambda)) + \psi(\lambda)\varphi'(\lambda) f'_\lambda(\varphi(\lambda))| \\ &= \frac{(1-|\lambda|^2)^\alpha |\psi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\beta+2}} \exp[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}] \end{aligned}$$

and so

$$(1-|\lambda|^2)^\alpha |\psi'(\lambda)| \ \exp\Big[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\Big] \leq M_0 (1-|\varphi(\lambda)|^2)^{\beta+2}.$$

Taking $\lim\limits_{|\varphi(\lambda)|\to 1}$ on both sides of the above inequality we get

$$\lim_{|\varphi(\lambda)|\to 1} (1-|\lambda|^2)^\alpha |\psi'(\lambda)| \ \exp\Big[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\Big] = 0.$$

Now consider the function

$$g_\lambda(z) = (z-\varphi(\lambda))\Big(\frac{1-|\varphi(\lambda)|^2}{(1-\overline{\varphi(\lambda)}z)^2}\Big)^{\beta+2} \exp\Big[6c\Big\{\frac{1}{2}\Big(\frac{1-|\varphi(\lambda)|^2}{(1-\overline{\varphi(\lambda)}z)^2}\Big)^{\beta+2} - \frac{1}{3}\frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{(1-\overline{\varphi(\lambda)}z)^{3(\beta+2)}}\Big\}\Big].$$

It can be shown that every $g_\lambda \in \mathcal{A}^\beta_\mathcal{N}$ and $\|g_\lambda\|_{\mathcal{A}^\beta_\mathcal{N}} \leq M$ for some $M > 0$.

Moreover,

$$g_\lambda(\varphi(\lambda)) = 0.$$

Also,

$$\begin{aligned} g'_\lambda(z) = \Big[&\Big(\frac{1-|\varphi(\lambda)|^2}{(1-\overline{\varphi(\lambda)}z)^2}\Big)^{\beta+2} + (z-\varphi(\lambda))(2\beta+4)\frac{(1-|\varphi(\lambda)|^2)^{\beta+2}}{(1-\overline{\varphi(\lambda)}z)^{2(\beta+2)+1}}(\overline{\varphi(\lambda)}) \\ &+\Big\{6c\,(z-\varphi(\lambda))(\beta+2)\frac{(1-|\varphi(\lambda)|^2)^{\beta+2}}{(1-\overline{\varphi(\lambda)}z)^{2(\beta+2)}}(\overline{\varphi(\lambda)})\Big\} \\ &\Big\{\frac{(1-|\varphi(\lambda)|^2)^{\beta+2}}{(1-\overline{\varphi(\lambda)}z)^{2(\beta+2)+1}} - \frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{(1-\overline{\varphi(\lambda)}z)^{3(\beta+2)}}\Big\}\Big] \\ &\exp\Big[6c\Big\{\frac{1}{2}\Big(\frac{1-|\varphi(\lambda)|^2}{(1-\overline{\varphi(\lambda)}z)^2}\Big)^{\beta+2} - \frac{1}{3}\frac{(1-|\varphi(\lambda)|^2)^{2(\beta+2)}}{(1-\overline{\varphi(\lambda)}z)^{3(\beta+2)}}\Big\}\Big]. \end{aligned}$$

Moreover,

$$g'_\lambda(\varphi(\lambda)) = \frac{1}{(1-|\varphi(\lambda)|^2)^{\beta+2}} \exp\Big[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\Big].$$

Since $DM_\psi C_\varphi$ maps $\mathcal{A}^0_\beta$ boundedly into $\mathcal{A}^{-\alpha}$, we can find some constant $M_1 > 0$ such that

$$\begin{aligned} M_1 &\geq (1-|\lambda|^2)^\alpha |\psi'(\lambda) g_\lambda(\varphi(\lambda)) + \psi(\lambda)\varphi'(\lambda) g'_\lambda(\varphi(\lambda))| \\ &= \frac{(1-|\lambda|^2)^\alpha |\psi(\lambda)\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\beta+2}} \exp\Big[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\Big] \end{aligned}$$

and so

$$\frac{(1-|\lambda|^2)^\alpha |\psi(\lambda)\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)} \exp\Big[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\Big] \leq M_1 (1-|\varphi(\lambda)|^2)^{\beta+1}.$$

Taking $\lim\limits_{|\varphi(\lambda)|\to 1}$ on both sides of the above inequality we get

$$\lim_{|\varphi(\lambda)|\to 1} \frac{(1-|\lambda|^2)^\alpha |\psi(\lambda)\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)} \exp\Big[\frac{c}{(1-|\varphi(\lambda)|^2)^{\beta+2}}\Big] = 0.$$

Assume that the conditions in (iii) are valid for all $c > 0$. Note that if $f \in \mathcal{A}^0_\beta$, then by (1.2) and Cauchy integral formula for derivatives

$$\begin{aligned} (1-|z|^2)|f'(z)| &\leq \frac{2}{\pi} \int_{\partial \mathbb{D}} |f(z + \frac{1}{2}(1-|z|)\zeta)||d\zeta| \\ &\leq \exp\Big[\frac{C_\beta \|f_w\|_{\mathcal{A}^\beta_\mathcal{N}}}{(1-|z|^2)^{\beta+2}}\Big]. \end{aligned}$$

Choose a sequence $\{f_n\}$ in $\mathcal{A}^\beta_\mathcal{N}$ such that $\|f_n\|_{\mathcal{A}^\beta_\mathcal{N}} \leq M'$ and $f_n \to 0$ locally uniformly on $\mathbb{D}$. Then for each $r \in (0,1)$

$$\sup_{|\varphi(z)|\leq r} (1-|z|^2)^\alpha |\psi'(z) f_n(\varphi(z)) + \psi(z)\varphi'(z) f'_n(\varphi(z))|$$

$$\leq A \sup_{|\varphi(z)|\leq r} |f_n(\varphi(z))| + B \sup_{|\varphi(z)|\leq r} |f'_n(\varphi(z))| \quad \to 0 \text{ as } n \to \infty,$$

where $A = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |\psi'(z)| < \infty$ and $B = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |\psi(z)\varphi'(z)| < \infty$. On the other hand, whenever $r \to 1$, we have

$$\sup_{|\varphi(z)|>r} (1-|z|^2)^\alpha |DM_\psi C_\varphi f_n(z)|$$

$$\leq \sup_{|\varphi(z)|>r} (1-|z|^2)^\alpha |\psi'(z)| \ \exp[\frac{C'_\beta}{(1-|\varphi(z)|^2)^{\beta+2}}]$$

$$+ \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^\alpha |\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)} \ \exp[\frac{C_\beta}{(1-|\varphi(z)|^2)^{\beta+2}}] \to 0 \text{ as } r \to 1.$$

Combining the above estimates, we see that

$$\|DM_\psi C_\varphi f_n\|_{\mathcal{A}^{-\alpha}} \to 0 \text{ as } n \to \infty.$$

Thus by Lemma 2.1, *(ii)* follows. □

Corollary 2.18 *Let $\alpha > 0, \beta > -1$ and $\psi \in H(\mathbb{D})$. Then the following are equivalent:*

(i) DM_ψ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *compactly into* $\mathcal{A}^{-\alpha}$.

(ii) DM_ψ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *boundedly into* $\mathcal{A}^{-\alpha}$.

(iii) $\psi \equiv 0$.

Corollary 2.19 *Let $\alpha > 0, \beta > -1, \psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) ψC_φ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *boundedly into* $\mathcal{B}^\alpha$.

(ii) ψC_φ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *compactly into* $\mathcal{B}^\alpha$.

(iii) *For all* $c > 0$,

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|\psi'(z)|\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

$\psi \in \mathcal{B}^\alpha$ *and* $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\psi(z)\varphi'(z)| < \infty$.

Corollary 2.20 *Let $\alpha > 0, \beta > -1$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) C_φ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *boundedly into* $\mathcal{B}^\alpha$.

(ii) C_φ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *compactly into* $\mathcal{B}^\alpha$.

(iii) *For all* $c > 0$,

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^\alpha|\varphi'(z)|}{1-|\varphi(z)|^2}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0.$$

Corollary 2.21 *Let $\alpha > 0, \beta > -1, \psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) $DM_\psi C_\varphi$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *boundedly into* $\mathcal{A}^{-\alpha}_0$.

(ii) $DM_\psi C_\varphi$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *compactly into* $\mathcal{A}^{-\alpha}_0$.

(iii) *For all* $c > 0$,

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha|\psi'(z)|\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

$\psi' \in \mathcal{A}_0^{-\alpha}$ *and* $\lim_{|z|\to 1}(1-|z|^2)^\alpha |\psi(z)\varphi'(z)| = 0.$

(iv) For all $c > 0$,

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |\psi'(z)| \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0$$

and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |\psi(z)\varphi'(z)|}{(1-|\varphi(z)|^2)} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0.$$

The equivalence of the above implications can be proved on the same lines as the proof of Theorem 2.10 and Lemma 2.9. We omit the details.

Routine calculations yield the following theorems.

Theorem 2.22 *Let* $\alpha > 0, \beta > -1, \psi \in H(\mathbb{D})$ *and* φ *be a holomorphic self-map of* $\mathbb{D}$. *Then the following are equivalent:*

(i) $C_\varphi DM_\psi$ *maps* $\mathcal{A}_\mathcal{N}^\beta$ *boundedly into* $\mathcal{A}^{-\alpha}$.

(ii) $C_\varphi DM_\psi$ *maps* $\mathcal{A}_\mathcal{N}^\beta$ *compactly into* $\mathcal{A}^{-\alpha}$.

(iii) For all $c > 0$,

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha |\psi'(\varphi(z))| \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\psi(\varphi(z))|}{1-|\varphi(z)|^2} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

$\psi' \circ \varphi \in \mathcal{A}^{-\alpha}$ *and* $\psi \circ \varphi \in \mathcal{A}^{-\alpha}$.

Theorem 2.23 *Let* $\alpha > 0, \beta > -1, \psi \in H(\mathbb{D})$ *and* φ *be a holomorphic self-map of* $\mathbb{D}$. *Then the following are equivalent:*

(i) $C_\varphi DM_\psi$ *maps* $\mathcal{A}_\mathcal{N}^\beta$ *boundedly into* $\mathcal{A}_0^{-\alpha}$.

(ii) $C_\varphi DM_\psi$ *maps* $\mathcal{A}_\mathcal{N}^\beta$ *compactly into* $\mathcal{A}_0^{-\alpha}$.

(iii) For all $c > 0$,

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)^\alpha |\psi'(\varphi(z))| \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\psi(\varphi(z))|}{1-|\varphi(z)|^2} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

$\psi' \circ \varphi \in \mathcal{A}_0^{-\alpha}$ *and* $\psi \circ \varphi \in \mathcal{A}_0^{-\alpha}$.

(iv) For all $c > 0$,

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |\psi'(\varphi(z))| \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |\psi(\varphi(z))|}{1-|\varphi(z)|^2} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0.$$

Theorem 2.24 *Let $\alpha > 0, \beta > -1, \psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) $DC_\varphi M_\psi$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *boundedly into* $\mathcal{A}^{-\alpha}$.

(ii) $DC_\varphi M_\psi$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *compactly into* $\mathcal{A}^{-\alpha}$.

(iii) *For all* $c > 0$,

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)^\alpha |\psi'(\varphi(z))\varphi'(z)| \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\psi(\varphi(z))\varphi'(z)|}{1-|\varphi(z)|^2} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

$\varphi'(\psi'\circ\varphi) \in \mathcal{A}^{-\alpha}$ *and* $\varphi'(\psi\circ\varphi) \in \mathcal{A}^{-\alpha}$.

Theorem 2.25 *Let $\alpha > 0, \beta > -1, \psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following are equivalent:*

(i) $DC_\varphi M_\psi$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *boundedly into* $\mathcal{A}^{-\alpha}_0$.

(ii) $DC_\varphi M_\psi$ *maps* $\mathcal{A}^\beta_\mathcal{N}$ *compactly into* $\mathcal{A}^{-\alpha}_0$.

(iii) *For all* $c > 0$,

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)^\alpha |\psi'(\varphi(z))\varphi'(z)| \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\alpha |\psi(\varphi(z))\varphi'(z)|}{1-|\varphi(z)|^2} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

$\varphi'(\psi'\circ\varphi) \in \mathcal{A}^{-\alpha}_0$ *and* $\varphi'(\psi\circ\varphi) \in \mathcal{A}^{-\alpha}_0$.

(iv) *For all* $c > 0$,

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |\psi'(\varphi(z))\varphi'(z)| \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0,$$

and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\alpha |\psi(\varphi(z))\varphi'(z)|}{1-|\varphi(z)|^2} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\beta+2}}\Big] = 0.$$

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