Properties of RD-projective and RD-injective modules

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Abstract

In this paper, we first study RD-projective and RD-injective modules using, among other things, covers and envelopes. Some new characterizations for them are obtained. Then we introduce the RD-projective and RD-injective dimensions for modules and rings. The relations between the RD-homological dimensions and other homological dimensions are also investigated.

Key word and phrases: RD-projective module, RD-injective module, RD-flat module, RD-projective dimension, RD-injective dimension, (pre)envelope, (pre)cover.

1. Introduction

Following [20], an exact sequence $0 \to A \to B \to C \to 0$ of left $R$-modules is called RD-exact if for every $a \in R$, the sequence $\text{Hom}(R/Ra, B) \to \text{Hom}(R/Ra, C) \to 0$ is exact, or equivalently, the sequence $0 \to (R/aR) \otimes A \to (R/aR) \otimes B$ is exact. A left $R$-module $M$ is said to be RD-projective if for every RD-exact sequence $0 \to A \to B \to C \to 0$ of left $R$-modules, the sequence $0 \to \text{Hom}(M, A) \to \text{Hom}(M, B) \to \text{Hom}(M, C) \to 0$ is exact. A left $R$-module $N$ is called RD-injective if for every RD-exact sequence $0 \to A \to B \to C \to 0$ of left $R$-modules, the sequence $0 \to \text{Hom}(C, N) \to \text{Hom}(B, N) \to \text{Hom}(A, N) \to 0$ is exact. According to [3], a right $R$-module $F$ is called RD-flat if for every RD-exact sequence $0 \to A \to B \to C \to 0$ of left $R$-modules, the sequence $0 \to F \otimes A \to F \otimes B \to F \otimes C \to 0$ is exact. For more details about RD-projective, RD-injective and RD-flat modules, we refer the reader to [2, 3, 6, 15, 16, 19, 20].

Though the RD-property is most important and well known in the commutative case, so far not much is known about the RD-property in the theory of modules over non-commutative rings. In this paper, we will establish several basic results for RD-projective, RD-injective and RD-flat modules over a general ring.

In Section 2 of this paper, we obtain some properties of RD-projective and RD-injective modules in terms of, among other things, covers and envelopes. New characterizations for them are presented. For example, we prove that, if $M$ is a submodule of an RD-injective left $R$-module $E$, then $E$ is an RD-injective hull $M$ in the sense of Warfield if and only if the inclusion $M \to E$ is an RD-injective envelope in the sense of Enochs. Also, we show that $M$ is an RD-projective left $R$-module if and only if $M$ is projective relative to every RD-exact sequence $0 \to K \to E \to F \to 0$ of left $R$-modules with $E$ RD-injective. Dually, $M$ is an RD-injective

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left $R$-module if and only if $M$ is injective relative to every $RD$-exact sequence $0 \to K \to P \to L \to 0$ of left $R$-modules with $P$ $RD$-projective. In addition, we get that the class of $RD$-injective left $R$-modules is closed under extensions if and only if every Warfield cotorsion left $R$-module is $RD$-injective. Finally, we prove that the following are equivalent for a ring $R$ and an integer $n \geq 0$: (1) Every $RD$-flat left $R$-module has flat dimension $\leq n$. (2) Every $RD$-projective left $R$-module has flat dimension $\leq n$. (3) Every $RD$-injective right $R$-module has injective dimension $\leq n$. As a consequence, we obtain several new characterizations of left $PP$ rings and von Neumann regular rings.

In Section 3, we introduce and study the $RD$-derived functor $\text{Ext}_{RD}^n(\cdot, \cdot)$ of $\text{Hom}(\cdot, \cdot)$, and $RD$-projective and $RD$-injective dimensions of modules and rings. We first prove that $\text{Ext}_{RD}^1(M, N) \to \text{Ext}^1(M, N)$ is a monomorphism for any ring $R$; $R$ is a von Neumann regular ring if and only if $\text{Ext}_{RD}^1(M, N) \cong \text{Ext}^1(M, N)$ for all left $R$-modules $M$ and $N$. Then we get that the left global $RD$-projective dimension $lRD - PD(R)$ is equal to the left global $RD$-injective dimension $lRD - ID(R)$. For a left strongly $P$-coherent ring $R$, we prove that $\text{sup}\{id(M) : M \text{ is any divisible left } R\text{-module}\} \leq lRD - ID(R)$, and $\text{sup}\{pd(M) : M \text{ is any torsionfree left } R\text{-module}\} \leq lRD - ID(R)$. Finally, it is shown that $lID(R) \leq lRD - ID(R) + \text{sup}\{id(M) : M \text{ is any } RD\text{-injective left } R\text{-module}\} \leq lRD - ID(R) + wD(R)$.

Throughout this paper, $R$ is an associative ring with identity and all modules are unitary. We write $RM$ to indicate a left $R$-module. The character module $\text{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$ of $M$ is denoted by $M^+$. $lD(R)$ (resp. $wD(R)$) stands for the left (resp. the weak) global dimension of $R$. $pd(M)$ (resp. $id(M)$, $fd(M)$) denotes the projective (resp. injective, flat) dimension of $M$. Let $M$ and $N$ be $R$-modules. $\text{Hom}(M, N)$ (resp. $\text{Ext}_R^n(M, N)$) means $\text{Hom}_R(M, N)$ (resp. $\text{Ext}_R^n(M, N)$), and similarly $M \otimes_R N$ (resp. $\text{Tor}_n(M, N)$) denotes $M \otimes_R N$ (resp. $\text{Tor}_R^n(M, N)$) for an integer $n \geq 1$. For unexplained concepts and notations, we refer the reader to [1, 5, 6, 7, 11, 17, 21, 22].

2. $RD$-projective and $RD$-injective modules

We begin with the following lemmas.

Lemma 2.1 Let $R$ be a ring.

(1) [6, Lemma VI 12.1] For any left $R$-module $M$, there exists an $RD$-exact sequence $0 \to N \to C \to M \to 0$, where $C$ is a direct sum of cyclically presented left $R$-modules.

(2) [20, Corollary 1] and [3, Proposition 1.3] A left $R$-module $M$ is $RD$-projective if and only if $M$ is a direct summand of a direct sum of cyclically presented left $R$-modules if and only if $M$ is $RD$-flat and pure-projective.

(3) [3, Proposition 1.4] A right $R$-module $F$ is $RD$-flat if and only if $F^+$ is $RD$-injective.

Lemma 2.2 The following are equivalent:

(1) $0 \to A \to B \to C \to 0$ is an $RD$-exact sequence of left $R$-modules.
(2) The sequence $0 \to \text{Hom}(M, A) \to \text{Hom}(M, B) \to \text{Hom}(M, C) \to 0$ is exact for any RD-projective left $R$-module $M$.

(3) The sequence $0 \to \text{Hom}(C, N) \to \text{Hom}(B, N) \to \text{Hom}(A, N) \to 0$ is exact for any RD-injective left $R$-module $N$.

**Proof.** (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial.

(2) $\Rightarrow$ (1) is clear since $R/Ra$ is RD-projective for any $a \in R$.

(3) $\Rightarrow$ (1) Let $a \in R$. By Lemma 2.1 (3), $(R/aR)^+$ is RD-injective. So by (3), we get the exact sequence

$$\text{Hom}(B, (R/aR)^+) \to \text{Hom}(A, (R/aR)^+) \to 0,$$

which gives the exactness of the sequence

$$((R/aR) \otimes B)^+ \to ((R/aR) \otimes A)^+ \to 0.$$

Therefore we obtain the exact sequence

$$0 \to (R/aR) \otimes A \to (R/aR) \otimes B.$$

So the sequence $0 \to A \to B \to C \to 0$ is RD-exact.

According to [8, 11], a left $R$-module $M$ is said to be **divisible** if $\text{Ext}^1(R/Ra, M) = 0$ for all $a \in R$. A right $R$-module $N$ is called **torsionfree** if $\text{Tor}_1(N, R/Ra) = 0$ for all $a \in R$. It is clear that a right $R$-module $N$ is torsionfree if and only if $N^+$ is divisible by the standard isomorphism $\text{Ext}^1(R/Ra, N^+) \cong \text{Tor}_1(N, R/Ra)^+$ for all $a \in R$.

Next we characterize divisible and torsion-free modules in terms of RD-projective and RD-injective modules.

**Proposition 2.3** The following are equivalent for a left $R$-module $M$:

(1) $M$ is divisible.

(2) Every left $R$-module exact sequence $0 \to M \to E \to F \to 0$ is RD-exact.

(3) There exists an RD-exact sequence $0 \to M \to B \to C \to 0$ with $B$ divisible.

(4) $\text{Ext}^1(N, M) = 0$ for any RD-projective left $R$-module $N$.

(5) For every RD-injective left $R$-module $G$, any homomorphism $M \to G$ factors through an injective left $R$-module.

**Proof.** (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are routine.

(1) $\Rightarrow$ (4) follows from Lemma 2.1 (2). (4) $\Rightarrow$ (1) is clear.

(2) $\Rightarrow$ (5) is easy since $M$ embeds in an injective $R$-module.

(5) $\Rightarrow$ (3) There exists an exact sequence $0 \to M \xrightarrow{f} E \to L \to 0$ with $E$ injective. Let $a \in R$. Then $(R/aR)^+$ is RD-injective. For any $f : M \to (R/aR)^+$, there exist an injective left $R$-module $Q$ and
\[ g : M \to Q \text{ and } h : Q \to (R/aR)^+ \text{ such that } f = hg \text{ by (5). Thus there exists } \alpha : E \to Q \text{ such that } g = \alpha i, \]
and so \( f = (h\alpha)i \). Therefore we get the exact sequence
\[ \text{Hom}(E, (R/aR)^+) \to \text{Hom}(M, (R/aR)^+) \to 0, \]
which leads to the exactness of the sequence
\[ ((R/aR) \otimes E)^+ \to ((R/aR) \otimes M)^+ \to 0. \]
It follows that \( 0 \to (R/aR) \otimes M \to (R/aR) \otimes E \) is exact, as required.

\begin{proposition}
The following are equivalent for a right \( R \)-module \( N \):
\begin{enumerate}
\item \( N \) is torsionfree.
\item Every right \( R \)-module exact sequence \( 0 \to K \to P \to N \to 0 \) is \( RD \)-exact.
\item There exists a right \( R \)-module \( RD \)-exact sequence \( 0 \to K \to T \to N \to 0 \) with \( T \) torsionfree.
\item \( \text{Ext}^1(N, M) = 0 \) for any \( RD \)-injective right \( R \)-module \( M \).
\item For every \( RD \)-projective right \( R \)-module \( F \), every homomorphism \( f : F \to N \) factors through a projective right \( R \)-module.
\item \( \text{Tor}_1(N, M) = 0 \) for any \( RD \)-flat left \( R \)-module \( M \).
\end{enumerate}
\end{proposition}

\begin{proof}
(1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3) \( \Leftrightarrow \) (4) are straightforward.
(2) \( \Rightarrow \) (5) is clear since there is an exact sequence \( P \to N \to 0 \) with \( P \) projective.
(5) \( \Rightarrow \) (1) follows from [13, Lemma 3.9].
(1) \( \Leftrightarrow \) (6) holds by the fact that every \( RD \)-flat module is a direct limit of finite direct sums of cyclically presented modules (see [3, Proposition I.1]).
\end{proof}

\begin{corollary}
The following are true for any ring \( R \):
\begin{enumerate}
\item A divisible \( RD \)-injective left \( R \)-module is injective.
\item A torsionfree \( RD \)-projective right \( R \)-module is projective.
\item A torsionfree \( RD \)-flat right \( R \)-module is flat.
\end{enumerate}
\end{corollary}

\begin{proof}
(1) follows from Proposition 2.3. (2) holds by Proposition 2.4.
(3) Let \( N \) be a torsionfree \( RD \)-flat right \( R \)-module. Then \( N^+ \) is divisible \( RD \)-injective by Lemma 2.1 (3), and so is injective by (1). Thus \( N \) is flat.
\end{proof}

Following [6], an \( RD \)-injective hull of an \( R \)-module \( M \) is defined as an \( RD \)-injective \( R \)-module \( E \) such that \( M \) is an \( RD \)-essential submodule of \( E \), where \( M \) is called an \( RD \)-essential submodule of \( E \) if \( M \) is
an \(RD\)-submodule of \(E\), and there is no nonzero submodule \(K\) of \(E\) with \(K \cap M = 0\) and \((K + M)/K\) an \(RD\)-submodule of \(E/K\).

By [6, Theorem 1.6], any \(R\)-module admits an \(RD\)-injective hull.

Let \(C\) be a class of \(R\)-modules and \(M\) an \(R\)-module. According to Enochs [4], a homomorphism \(\phi : C \to M\) is a \(C\)-precover of \(M\) if \(C \in C\) and the abelian group homomorphism \(\text{Hom}(C', \phi) : \text{Hom}(C', C) \to \text{Hom}(C', M)\) is surjective for every \(C' \in C\). A \(C\)-precover \(\phi : C \to M\) is said to be a \(C\)-cover of \(M\) if every endomorphism \(g : C \to C\) such that \(\phi g = \phi\) is an isomorphism. Dually we have the definitions of a \(C\)-preenvelope and a \(C\)-envelope. \(C\)-covers (\(C\)-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism.

**Theorem 2.6** Let \(R\) be a ring.

(1) Every \(R\)-module has an \(RD\)-projective precover.

(2) Every \(R\)-module has an \(RD\)-flat cover.

(3) Every \(R\)-module has an \(RD\)-injective envelope.

**Proof.** (1) follows from Lemma 2.1 (1).

(2) We first prove that the class of \(RD\)-flat \(R\)-modules is closed under pure quotient modules. Let \(0 \to A \to B \to C \to 0\) be a pure exact sequence with \(B \ RD\)-flat. Then we get the split exact sequence \(0 \to C^+ \to B^+ \to A^+ \to 0\). Since \(B^+\) is \(RD\)-injective by Lemma 2.1 (3), \(C^+\) is \(RD\)-injective. So \(C\) is \(RD\)-flat. In addition, the class of \(RD\)-flat \(R\)-modules is clearly closed under direct limits. Thus every \(R\)-module has an \(RD\)-flat cover by [9, Theorem 2.5].

(3) Since every \(R\)-module admits an \(RD\)-injective hull, every \(R\)-module admits an \(RD\)-injective preenvelope. On the other hand, any direct limit of \(RD\)-exact sequences is \(RD\)-exact (see [6, Exercise I 7.15]). By a proof similar to that of [22, Theorem 2.3.8 or 2.2.6], every \(R\)-module has an \(RD\)-injective envelope. \(\square\)

**Theorem 2.7** Suppose that \(M\) is a submodule of an \(RD\)-injective left \(R\)-module \(E\). Then the following are equivalent:

(1) \(i : M \to E\) is an \(RD\)-injective envelope (here \(i\) is the inclusion).

(2) \(E\) is an \(RD\)-injective hull of \(M\).

**Proof.** (1) \(\Rightarrow\) (2) Suppose that there is a nonzero submodule \(K\) of \(E\) such that \(K \cap M = 0\) and \((K + M)/K\) is an \(RD\)-submodule of \(E/K\). Since \((K + M)/K \cong M\) and \(E\) is \(RD\)-injective, there is \(\beta : E/K \to E\) such that the following diagram is commutative, where \(\pi : E \to E/K\) is the natural map:

\[
\begin{array}{cccccc}
E & \to & E/K & \to & E/(K \oplus M) & \to & 0 \\
\downarrow \pi \downarrow & & \downarrow \beta & & \downarrow & & \\
0 & \to & M & \to & E/K & \to & 0.
\end{array}
\]
Hence $\beta \pi i = i$. Since $i$ is an envelope, $\beta \pi$ is an isomorphism, whence $\pi$ is an isomorphism. But this is impossible because $\pi(K) = 0$. So $E$ is an $RD$-injective hull of $M$.

(2) $\Rightarrow$ (1) Let $E$ be an $RD$-injective hull of $M$. Clearly the inclusion $i : M \to E$ is an $RD$-injective preenvelope. By Theorem 2.6 (3), $M$ has an $RD$-injective envelope $\sigma : M \to N$. Thus there exist $f : N \to E$ and $g : E \to N$ such that the following diagram is commutative.

$$
\begin{array}{ccc}
0 & \to & M & \xrightarrow{\sigma} & N \\
& & i \downarrow & & \downarrow f \\
& & M & \xrightarrow{g} & E.
\end{array}
$$

So $gf\sigma = gi = \sigma$. Hence $gf$ is an isomorphism. Without loss of generality, we may assume $gf = 1$. Thus $E = \text{im}(f) \oplus \ker(g)$. Note that $M \cap \ker(g) = 0$ and $M$ is an $RD$-submodule of $\text{im}(f)$. So $(M \oplus \ker(g))/\ker(g)$ is an $RD$-submodule of $E/\ker(g)$ by [6, p.39]. Hence $\ker(g) = 0$ by (2). Thus $g$ is an isomorphism. Therefore $i : M \to E$ is an $RD$-injective envelope.

Now we give new characterizations of $RD$-projective and $RD$-injective modules.

**Theorem 2.8** The following are equivalent for a left $R$-module $M$:

1. $M$ is $RD$-projective.
2. Every $RD$-exact sequence $0 \to K \to N \to M \to 0$ of left $R$-modules is split.
3. $M$ is projective relative to every $RD$-exact sequence $0 \to K \to E \to F \to 0$ of left $R$-modules with $E$ $RD$-injective.

**Proof.** (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear.

(2) $\Rightarrow$ (1) By Lemma 2.1 (1), there exists an $RD$-exact sequence $0 \to N \to C \to M \to 0$ with $C$ $RD$-projective. So $M$ is $RD$-projective by (2).

(3) $\Rightarrow$ (1) Let $0 \to A \to B \to C \to 0$ be an $RD$-exact sequence of left $R$-modules. By Theorem 2.6 (3), $B$ has an $RD$-injective envelope $\lambda : B \to H$. Then we have the following pushout diagram:
Thus $\alpha = \lambda \iota$, and so $0 \to A \to H \to D \to 0$ is an RD-exact sequence. Let $\psi : M \to C$ be any homomorphism. By (3), there exists $\gamma : M \to H$ such that $\beta \gamma = \varphi \psi$. Since $\rho \gamma = \delta \beta \gamma = \delta \varphi \psi = 0$, we have $\text{im}(\gamma) \subseteq \ker(\rho) = \text{im}(\lambda)$. So we can define $\theta : M \to B$ by

$$\theta(x) = \lambda^{-1} \gamma(x) \text{ for any } x \in M.$$ 

Thus $\varphi \psi = \beta \gamma = \beta \lambda \theta = \varphi \pi \theta$.

So $\psi = \pi \theta$ since $\varphi$ is monic. Hence $M$ is RD-projective. \hfill \Box

**Theorem 2.9** The following are equivalent for a left $R$-module $M$:

1. $M$ is RD-injective.
2. Every RD-exact sequence $0 \to M \to E \to F \to 0$ of left $R$-modules is split.
3. $M$ is injective relative to every RD-exact sequence $0 \to K \to P \to L \to 0$ of left $R$-modules with $P$ RD-projective.

**Proof.**  (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear.

(2) $\Rightarrow$ (1) By [6, Theorem 1.6], there exists an RD-exact sequence $0 \to M \to B \to N \to 0$ with $B$ RD-injective. So $M$ is RD-injective by (2).

(3) $\Rightarrow$ (1) Let $0 \to A \to B \to C \to 0$ be an RD-exact sequence of left $R$-modules. By Lemma 2.1 (1), there is an RD-exact sequence $0 \to D \to P \to B \to 0$ with $P$ RD-projective. Then we have the following pullback diagram:

Thus $\pi = \beta \rho$, and so $0 \to Q \to P \to C \to 0$ is an RD-exact sequence. Let $\psi : A \to M$ be any homomorphism. By (3), there exists $\gamma : P \to M$ such that $\psi \phi = \gamma \iota$. Since $\gamma \iota \delta = \psi \phi \delta = 0$, we have

$$\ker(\rho) = \text{im}(\lambda) = \text{im}(\iota \delta) \subseteq \ker(\gamma).$$

So there exists $\theta : B \to M$ such that $\theta \rho = \gamma$. Thus

$$\psi \phi = \theta \rho \mu = \theta \alpha \varphi.$$ 

Therefore $\psi = \theta \alpha$ since $\phi$ is epic. Hence $M$ is RD-injective. \hfill \Box

RD-injective and RD-flat modules over a commutative ring can be characterized as follows.
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Proposition 2.10 Let $R$ be a commutative ring. The following are equivalent for an $R$-module $M$:

(3) $M$ is an RD-injective $R$-module.

(4) $	ext{Hom}(F, M)$ is an RD-injective $R$-module for any flat $R$-module $F$.

Proof. (1) $\Rightarrow$ (2) Let $0 \to A \to B \to C \to 0$ be an RD-exact sequence of $R$-modules. For any flat $R$-module $F$, we get the exact sequence

$$0 \to F \otimes A \to F \otimes B \to F \otimes C \to 0.$$ 

It is easy to verify that the sequence is RD-exact. Since $M$ is RD-injective, we obtain the exact sequence

$$\text{Hom}(F \otimes B, M) \to \text{Hom}(F \otimes A, M) \to 0,$$

which yields the exact sequence

$$\text{Hom}(B, \text{Hom}(F, M)) \to \text{Hom}(A, \text{Hom}(F, M)) \to 0.$$

Thus $\text{Hom}(F, M)$ is an RD-injective $R$-module.

(2) $\Rightarrow$ (1) is clear by letting $F = R$. $\square$

Proposition 2.11 Let $R$ be a commutative ring. The following are equivalent for an $R$-module $N$:

(1) $N$ is an RD-flat $R$-module.

(2) $\text{Hom}(N, E)$ is an RD-injective $R$-module for any injective $R$-module $E$.

Proof. (1) $\Rightarrow$ (2) Let $E$ be any injective $R$-module. Then there is a split exact sequence

$$0 \to E \to \Pi R^+.$$ 

So we get the split exact sequence

$$0 \to \text{Hom}(N, E) \to \text{Hom}(N, \Pi R^+) \cong \Pi \text{Hom}(N, R^+) \cong \Pi N^+.$$ 

By (1), $N^+$ is RD-injective, and so $\Pi N^+$ is RD-injective. Thus $\text{Hom}(N, E)$ is RD-injective.

(2) $\Rightarrow$ (1) is obvious by letting $E = R^+$. $\square$

Recall that a right $R$-module $M$ is Warfield cotorsion [6, 7] if $\text{Ext}^1(F, M) = 0$ for every torsionfree right $R$-module $F$. Clearly, any RD-injective module is Warfield cotorsion by Proposition 2.4.

The following theorem exhibits the homological property of RD-projective, RD-injective and RD-flat modules.

Theorem 2.12 The following are equivalent for a ring $R$ and an integer $n \geq 0$:

(1) Every RD-flat left $R$-module has flat dimension $\leq n$.

(2) Every RD-projective left $R$-module has flat dimension $\leq n$. 

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(3) Every Warfield cotorsion right $R$-module has injective dimension $\leq n$.

(4) Every $RD$-injective right $R$-module has injective dimension $\leq n$.

**Proof.** (1) $\Rightarrow$ (2) is clear by Lemma 2.1 (2).

(2) $\Rightarrow$ (3) Let $M$ be a Warfield cotorsion right $R$-module and $N$ any right $R$-module. Then there is an exact sequence

$$0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to N \to 0$$

with each $P_i$ projective. By (2), for any $a \in R$, we have

$$\text{Tor}_1(K_n, R/Ra) \cong \text{Tor}_{n+1}(N, R/Ra) = 0.$$ 

Thus $K_n$ is torsionfree, and so

$$\text{Ext}^{n+1}(N, M) \cong \text{Ext}^1(K_n, M) = 0.$$ 

It follows that $M$ has injective dimension $\leq n$.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1) For every $RD$-flat left $R$-module $A$, $A^+$ is $RD$-injective. By (4), for every right $R$-module $B$, we have

$$\text{Tor}_{n+1}(B, A)^+ \cong \text{Ext}^{n+1}(B, A^+) = 0.$$ 

So $\text{Tor}_{n+1}(B, A) = 0$, and hence $A$ has flat dimension $\leq n$. \qed

Recall that a ring $R$ is left $PP$ if every principal left ideal of $R$ is projective. $R$ is called left $P$-coherent [15] in case each principal left ideal of $R$ is finitely presented.

**Corollary 2.13** The following are equivalent for a ring $R$:

(1) $R$ is a left $PP$ ring.

(2) $R$ is a left $P$-coherent ring and every submodule of a torsionfree right $R$-module is torsionfree.

(3) Every quotient module of a divisible left $R$-module is divisible.

(4) Every $RD$-projective left $R$-module has projective dimension $\leq 1$.

(5) $R$ is a left $P$-coherent ring and every $RD$-injective right $R$-module has injective dimension $\leq 1$.

(6) $R$ is a left $P$-coherent ring and every $RD$-flat left $R$-module has flat dimension $\leq 1$.

**Proof.** (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) hold by [14, Theorem 5.1].

(3) $\Rightarrow$ (4) Let $M$ be an $RD$-projective left $R$-module and $N$ any left $R$-module. Then there is an exact sequence $0 \to N \to E \to L \to 0$ with $E$ injective. By (3), $L$ is divisible, and so $\text{Ext}^2(M, N) \cong \text{Ext}^1(M, L) = 0$ by Proposition 2.3. It follows that $M$ has projective dimension $\leq 1$.

(4) $\Rightarrow$ (1) Let $a \in R$. Since $R/Ra$ has projective dimension $\leq 1$, $Ra$ is projective.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) follow from Theorem 2.12 and the equivalence of (4) and (1).
(6) ⇒ (1) Let \( a \in R \). Since \( R/Ra \) has flat dimension \( \leq 1 \), \( Ra \) is flat. So \( Ra \) is projective since \( Ra \) is finitely presented.

In general, \( RD \)-projective (\( RD \)-injective) modules need not be projective (injective). For example, \( \mathbb{Z}_2 \) is an \( RD \)-projective (\( RD \)-injective) \( \mathbb{Z} \)-module, but it is not a projective (injective) \( \mathbb{Z} \)-module. In fact, we have the following result.

**Corollary 2.14** The following are equivalent for a ring \( R \):

1. \( R \) is a von Neumann regular ring.
2. Every \( RD \)-projective left \( R \)-module is projective.
3. Every \( RD \)-flat left \( R \)-module is flat.
4. Every \( RD \)-injective right \( R \)-module is injective.
5. Every left \( R \)-module exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) is \( RD \)-exact.

**Proof.** (1) ⇒ (2) By Lemma 2.1 (2), an \( RD \)-projective left \( R \)-module is a direct summand of a direct sum of cyclically presented left \( R \)-modules. Since every cyclically presented left \( R \)-module is projective by (1), every \( RD \)-projective left \( R \)-module is projective.

(2) ⇒ (3) ⇒ (4) follow from Theorem 2.12 by letting \( n = 0 \).

(4) ⇒ (5) holds by Lemma 2.2.

(5) ⇒ (1) By (5) and Proposition 2.3, every left \( R \)-module is divisible. So \( R \) is a von Neumann regular ring.

Recall that a left \( R \)-module \( M \) is absolutely pure [12] if \( M \) is a pure submodule of every module which contains \( M \) as a submodule.

**Proposition 2.15** Consider the following conditions for a ring \( R \):

1. Every \( RD \)-exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) of left \( R \)-modules is pure.
2. Every pure injective left \( R \)-module is \( RD \)-injective.
3. Every pure projective left \( R \)-module is \( RD \)-projective.
4. Every finitely presented left \( R \)-module is a summand of a direct sum of cyclically presented left \( R \)-modules.
5. Every divisible left \( R \)-module is absolutely pure.

Then (1) ⇔ (2) ⇔ (3) ⇔ (4) ⇒ (5).

**Proof.** The equivalence of (1) through (4) follow from [3, Theorem I.4].

(1) ⇒ (5) holds by Proposition 2.3.

In [2], some examples of pure-injective modules that fail to be \( RD \)-injective were given for commutative rings. The following example gives an \( RD \)-exact sequence which is not pure over a non-commutative ring, and so there exists a pure-injective left module, which is not \( RD \)-injective.
Example 2.16 Let $K$ be a field and $\rho$ an isomorphism of $K$ onto a subfield $L$ such that $K \neq L$ and $K$ has finite vector space dimension over $L$. $K[X; \rho]$ will denote the ring of twisted right polynomials over $K$, i.e., $K[X; \rho]$ is the set of all formal polynomials in commuting indeterminate $X$ with coefficients from $K$ write on the right. Equality and addition are defined in the usual fashion and multiplication by assuming the associate and distributive laws and the rule
\[ aX = X\rho(a) \]
for all $a \in K$.

Let $R = K[X; \rho]/(X^2)$. Then by [18, Example 1], $rR$ is divisible, and $R$ is a two-sided Artinian ring, but is not a quasi-Frobenius ring. Thus $rR$ is not absolutely pure (and so is not $RD$-injective by Corollary 2.5 (1)). Let $E(rR)$ denote the injective envelope of $rR$. Then by Proposition 2.3, the left $R$-module exact sequence
\[ 0 \to rR \to E(rR) \to E(rR)/rR \to 0 \]
is an $RD$-exact sequence, but it is not pure. Thus by Proposition 2.15, there exists a pure injective left $R$-module which is not $RD$-injective, and there exists a pure projective left $R$-module which is not $RD$-projective.

By the way, the class of $RD$-flat left $R$-modules coincides with the class of $RD$-projective left $R$-modules by [3, Theorem III.1] since $R$ is left Artinian.

Remark 2.17 We note that some properties of $RD$-projective and $RD$-injective modules over commutative rings can be generalized to non-commutative cases. For example, by [6, Theorem XIII 1.1 and Example VI 12.5], for a commutative domain $R$, every $RD$-injective $R$-module has injective dimension $\leq 1$, and every $RD$-projective $R$-module has projective dimension $\leq 1$. By replacing “commutative domain” with “left $PP$ ring”, Corollary 2.13 extends the above result to a more general setting.

However, there seems to be some difference between the commutative and the non-commutative cases when we consider the projectivity and injectivity for $RD$. For instance, if $R$ is a commutative domain, then by [6, Proposition IX 3.4 and Theorem XIII 2.8], all conditions in Proposition 2.15 are equivalent (which exactly characterizes Prüfer domain). But for a non-commutative ring, we do not know whether the conditions (4) and (5) in Proposition 2.15 are equivalent. However, by [7, Corollary 3.2.4], the condition (5) in Proposition 2.15 is equivalent to the condition that every finitely presented left $R$-module is a direct summand in a left $R$-module $N$ such that $N$ is a union of a continuous chain, $(N_\alpha : \alpha < \lambda)$, for a cardinal $\lambda$, $N_0 = 0$ and $N_{\alpha+1}/N_\alpha$ is cyclically presented for all $\alpha < \lambda$.

Although the class of $RD$-injective left $R$-modules is closed under direct products and direct summands, the class of $RD$-injective left $R$-modules is not closed under direct sums in general. In fact, if $R$ is not a left Artinian ring, then the class of $RD$-injective left $R$-modules is not closed under direct sums by [3, Theorem II. 1].

Next we will consider when the class of $RD$-injective left $R$-modules is closed under extensions.

Theorem 2.18 The following are equivalent for a ring $R$:

(1) The class of $RD$-injective left $R$-modules is closed under extensions.

(2) Every Warfield cotorsion left $R$-module is $RD$-injective.
Proof. (1) ⇒ (2) Let $M$ be a Warfield cotorsion left $R$-module. Then by Theorem 2.6 (3), we have an exact sequence $0 \to M \to N \to L \to 0$, where $M \to N$ is an $RD$-injective envelope of $M$. By (1) and Wakamatsu’s Lemma (see [22, Lemma 2.1.2]), $\text{Ext}^1(L, C) = 0$ for every $RD$-injective left $R$-module $C$, and so $L$ is torsionfree by Proposition 2.4. Therefore $\text{Ext}^1(L, M) = 0$, and hence the exact sequence $0 \to M \to N \to L \to 0$ is split. Thus $M$ is $RD$-injective.

(2) ⇒ (1) is obvious because the class of Warfield cotorsion left $R$-modules is closed under extensions. □

Remark 2.19 (1) In general, the class of $RD$-injective $R$-modules is not closed under extensions. For example, [22, p. 75, Example] constructs a cotorsion $Z$-module which is not pure injective. Since torsionfree $Z$-modules coincide with flat $Z$-modules, Warfield cotorsion $Z$-modules need not be $RD$-injective. So the class of $RD$-injective $Z$-modules is not closed under extensions by Theorem 2.18.

(2) If $R$ is a left pure-semisimple ring, then the equivalent conditions of Theorem 2.18 are clearly satisfied.

(3) If $R$ is a von Neumann regular ring, then every $RD$-injective left $R$-module is injective by Corollary 2.14. So the equivalent conditions of Theorem 2.18 are also satisfied.

(4) If $R$ is a Prüfer domain, then the equivalent conditions of Theorem 2.18 hold if and only if the class of $RD$-injective $R$-modules is closed under cokernels of monomorphisms by [16, Proposition 4.5] and [22, Theorem 3.5.1].

3. $RD$-derived functors of $\text{Hom}(-, -)$ and $RD$-homological dimensions

By Theorem 2.6 (1), every left $R$-module has an $RD$-projective precover. So every left $R$-module $M$ has a left $RD$-projective resolution, that is, there is an exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ with each $P_i$ $RD$-projective and such that $\text{Hom}(N, -)$ leaves the sequence exact whenever $N$ is an $RD$-projective left $R$-module, equivalently, there exists an $RD$-exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ with each $P_i$ $RD$-projective by Lemma 2.2. Write $K_0 = M$, $K_1 = \ker(P_0 \to M)$, $K_i = \ker(P_{i-1} \to P_{i-2})$ for $i \geq 2$. The $n$th kernel $K_n$ ($n \geq 0$) is called the $n$th $RD$-projective syzygy of $M$.

Dually, by Theorem 2.6 (3), every left $R$-module $N$ has an $RD$-injective envelope. So $N$ has a right $RD$-injective resolution, that is, there is an exact sequence $0 \to N \to E^0 \to E^1 \to \cdots$ with each $E^i$ $RD$-injective and such that $\text{Hom}(-, M)$ leaves the sequence exact whenever $M$ is an $RD$-injective left $R$-module, equivalently, there is an $RD$-exact sequence $0 \to N \to E^0 \to E^1 \to \cdots$ with each $E^i$ $RD$-injective by Lemma 2.2. Write $L^0 = N$, $L^1 = \text{coker}(N \to E^0)$, $L^i = \text{coker}(E^{i-2} \to E^{i-1})$ for $i \geq 2$. The $n$th cokernel $L^n$ ($n \geq 0$) is called the $n$th $RD$-injective cosyzygy of $N$.

Note that $\text{Hom}(-, -)$ is right balanced by $\{\text{the class of all } RD\text{-projective left } R\text{-modules}\} \times \{\text{the class of all } RD\text{-injective left } R\text{-modules}\}$ (see [5, Definition 8.2.13]). Let $\text{Ext}_{RD}^n(-, -)$ denote the $n$th right derived functor of $\text{Hom}(-, -)$ with respect to $\{\text{the class of all } RD\text{-projective left } R\text{-modules}\} \times \{\text{the class of all } RD\text{-injective left } R\text{-modules}\}$. Then, for two left $R$-modules $M$ and $N$, $\text{Ext}_{RD}^n(M, N)$ can be computed using a left $RD$-projective resolution of $M$ or a right $RD$-injective resolution of $N$.

For any family $\{M_i\}$ of left $R$-modules, it is easy to check that the natural map $\text{Ext}_{RD}^n(\oplus M_i, N) \to$
Theorem 3.1 Let $R$ be a ring such that the class of $RD$-injective left $R$-modules is closed under direct sums. If $N$ is a finitely generated left $R$-module, $\{M_i\}$ is a family of left $R$-modules, then $\Ext^n_{RD}(N, \oplus M_i) \cong \oplus \Ext^n_{RD}(N, M_i)$ for any $n \geq 0$.

**Proof.** Every $M_i$ has a right $RD$-injective resolution

$$0 \to M_i \to E^0_i \to E^1_i \to E^2_i \to \cdots.$$  
Then by hypothesis and [22, Proposition 1.2.4],

$$0 \to \oplus M_i \to \oplus E^0_i \to \oplus E^1_i \to \oplus E^2_i \to \cdots$$

is a right $RD$-injective resolution of $\oplus M_i$. Applying $\Hom(N, -)$, we have the following commutative diagram of complexes:

$$0 \to \oplus \Hom(N, E^0_i) \to \oplus \Hom(N, E^1_i) \to \oplus \Hom(N, E^2_i) \to \cdots$$

Since $N$ is finitely generated, every $\theta_i$ is an isomorphism by [1, Exercise 16.3]. So $\Ext^n_{RD}(N, \oplus M_i) \cong \oplus \Ext^n_{RD}(N, M_i)$ for any $n \geq 0$ by [17, Exercise 6.7].

We now compare the $RD$-derived functor $\Ext^n_{RD}(-, -)$ with the usual derived functor $\Ext^n(-, -)$. There is a natural transformation $\Ext^n_{RD}(-, -) \to \Ext^n(-, -)$.

**Theorem 3.2** The following are true for any ring $R$.

1. $\Ext^0_{RD}(M, N) \cong \Hom(M, N) \cong \Ext^0(M, N)$ for all left $R$-modules $M$ and $N$.

2. $\Ext^1_{RD}(M, N) \to \Ext^1(M, N)$ is a monomorphism for all left $R$-modules $M$ and $N$.

**Proof.** Let

$$0 \to N \xrightarrow{\epsilon} D^0 \xrightarrow{d^0} D^1 \xrightarrow{d^1} D^2 \xrightarrow{d^2} \cdots$$

be a right $RD$-injective resolution of $N$. Since $D^0$ can be embedded in an injective left $R$-module $E^0$, $N$ admits a right injective resolution

$$0 \to N \xrightarrow{\lambda} E^0 \xrightarrow{e^0} E^1 \xrightarrow{e^1} E^2 \xrightarrow{e^2} \cdots.$$  

So we can complete the following commutative diagram uniquely up to homotopy, where $\tau_0$ is a monomorphism:

$$0 \quad 0 \quad 0 \quad 0 \quad 0$$

$$\xrightarrow{\epsilon} \quad \xrightarrow{\lambda} \quad \xrightarrow{\epsilon} \quad \xrightarrow{\lambda} \quad \xrightarrow{\epsilon} \quad \cdots$$

$$\xrightarrow{d^0} \quad \xrightarrow{d^0} \quad \xrightarrow{d^0} \quad \xrightarrow{d^0} \quad \cdots$$

$$\xrightarrow{d^1} \quad \xrightarrow{d^1} \quad \xrightarrow{d^1} \quad \xrightarrow{d^1} \quad \cdots$$

$$\xrightarrow{d^2} \quad \xrightarrow{d^2} \quad \xrightarrow{d^2} \quad \xrightarrow{d^2} \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
Applying $\text{Hom}(M,-)$ for any left $R$-module $M$, we have the following commutative diagram of complexes:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Hom}(M,D^0) & \overset{d^0}{\longrightarrow} & \text{Hom}(M,D^1) & \overset{d^1}{\longrightarrow} & \text{Hom}(M,D^2) & \overset{d^2}{\longrightarrow} & \cdot \cdot \cdot \\
& & \downarrow{\tau_{0*}} & & \downarrow{\tau_{1*}} & & \downarrow{\tau_{2*}} & & \\
0 & \longrightarrow & \text{Hom}(M,E^0) & \overset{e^0}{\longrightarrow} & \text{Hom}(M,E^1) & \overset{e^1}{\longrightarrow} & \text{Hom}(M,E^2) & \overset{e^2}{\longrightarrow} & \cdot \cdot \cdot 
\end{array}
$$

(1) It is clear that $\text{Ext}_{RD}^0(M,N) \cong \text{Hom}(M,N) \cong \text{Ext}^0(M,N)$.

(2) Note that $\text{Ext}_{RD}^1(M,N) = \ker(d^1)/\text{im}(d^0)$ and $\text{Ext}^n(M,N) = \ker(e^1)/\text{im}(e^0)$.

Define $\theta : \text{Ext}_{RD}^1(M,N) \rightarrow \text{Ext}^n(M,N)$ via $\theta(\tau(\alpha)) = \tau_1(\alpha)$ for any $\alpha \in \ker(d^1)$.

Let $\theta(\tau(\alpha)) = \tau_1(\alpha) = 0$ for some $\alpha \in \ker(d^1)$. Then

$$
\tau_1(\alpha) = \tau_1 \alpha \in \text{im}(e^0).
$$

So there exists $\beta \in \text{Hom}(M,E^0)$ such that

$$
\tau_1 \alpha = e^0(\beta) = e^0 \beta.
$$

Since $d^1 \alpha = d^1_\ast(\alpha) = 0$, we have $\alpha(x) \in \ker(d^1) = \text{im}(d^0)$ for any $x \in M$. Thus there exists $y \in D^0$ such that $\alpha(x) = d^0(y)$. Hence

$$
e^0 \beta(x) = \tau_1 \alpha(x) = \tau_1 d^0(y) = e^0 \tau_0(y),$$

and so

$$
\beta(x) - \tau_0(y) \in \text{ker}(e^0) = \text{im}(\lambda) = \text{im}(\tau_0 e).
$$

Therefore there exists $t \in N$ such that

$$
\beta(x) - \tau_0(y) = \tau_0 e(t).
$$

Thus $\beta(x) = \tau_0(y + e(t))$. Define $\gamma : M \rightarrow D^0$ via

$$
\gamma(x) = y + e(t).
$$

Then $\gamma$ is well defined since $\tau_0$ is a monomorphism. Note that $\alpha = d^0_\ast(\gamma)$, and so $\overline{\tau} = 0$. It follows that $\theta : \text{Ext}_{RD}^1(M,N) \rightarrow \text{Ext}^1(M,N)$ is a monomorphism.

In general, $\text{Ext}_{RD}^1(M,N) \rightarrow \text{Ext}^1(M,N)$ need not be an epimorphism. In fact, $\text{Ext}_{RD}^1(M,N) \rightarrow \text{Ext}^1(M,N)$ is an epimorphism if and only if $R$ is a von Neumann regular ring as shown by the following proposition.

**Proposition 3.3** The following are equivalent for a ring $R$:

(1) $R$ is a von Neumann regular ring.

(2) $\text{Ext}_{RD}^n(M,N) \rightarrow \text{Ext}^n(M,N)$ is an isomorphism for all left $R$-modules $M$ and $N$ and $n \geq 1$.

(3) $\text{Ext}_{RD}^1(M,N) \rightarrow \text{Ext}^1(M,N)$ is an isomorphism for all left $R$-modules $M$ and $N$. 

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Proof. (1) ⇒ (2) By (1) and Corollary 2.14, the class of \( RD \)-injective left \( R \)-modules coincides with the class of injective left \( R \)-modules. So \( \text{Ext}^n_{RD}(M, N) \cong \text{Ext}^n(M, N) \) for all left \( R \)-modules \( M \) and \( N \) and \( n \geq 1 \).

(2) ⇒ (3) is trivial.

(3) ⇒ (1) Let \( N \) be any \( RD \)-injective left \( R \)-module. Then \( \text{Ext}^1_{RD}(M, N) = 0 \) for any left \( R \)-module \( M \) since there exists a right \( RD \)-injective resolution \( 0 \to N \to N \to 0 \to 0 \to \cdots \). So \( \text{Ext}^1(M, N) = 0 \) by (3). Thus \( N \) is injective. Hence \( R \) is a von Neumann regular ring by Corollary 2.14. \( \square \)

Next we introduce the \( RD \)-projective and \( RD \)-injective dimensions for modules and rings.

Definition 3.4 Let \( R \) be a ring. For a left \( R \)-module \( M \), let \( RD - pd(M) = \inf \{ n : \text{there exists a left} \ RD \)-projective resolution \( 0 \to P_n \to \cdots \to P_0 \to M \to 0 \} \) and call \( RD - pd(M) \) the \( RD \)-projective dimension of \( M \). If no such sequence exists for any \( n \), set \( RD - pd(M) = \infty \).

Put \( lRD - PD(R) = \sup \{ RD - pd(M) : M \text{ ranges over all left } R \text{-modules} \} \) and call \( lRD - PD(R) \) the left global \( RD \)-projective dimension of the ring \( R \).

Dually, we can define the \( RD \)-injective dimension \( RD - id(M) \) of a left \( R \)-module \( M \), and the left global \( RD \)-injective dimension \( lRD - ID(R) \) of the ring \( R \).

Proposition 3.5 The following are equivalent for a left \( R \)-module \( M \) and an integer \( n \geq 0 \):

1. \( RD - pd(M) \leq n \).
2. \( \text{Ext}^{n+j}_{RD}(M, N) = 0 \) for all left \( R \)-modules \( N \) and \( j \geq 1 \).
3. \( \text{Ext}^{n+1}_{RD}(M, N) = 0 \) for all left \( R \)-modules \( N \).
4. Every \( n \)th \( RD \)-projective syzygy of \( M \) is \( RD \)-projective.

Proof. (1) ⇒ (2) By (1), \( M \) admits a left \( RD \)-projective resolution

\[
0 \to P_n \to \cdots \to P_0 \to M \to 0.
\]

Then \( \text{Hom}(P_{n+j}, N) = 0 \) for all left \( R \)-modules \( N \) and \( j \geq 1 \). So \( \text{Ext}^{n+j}_{RD}(M, N) = 0 \).

(2) ⇒ (3) is trivial.

(3) ⇒ (4) Let

\[
\cdots \to P_{n+2} \to P_{n+1} \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0
\]

be a left \( RD \)-projective resolution of \( M \) with \( K_n = \ker(P_{n-1} \to P_{n-2}) \) and \( K_{n+1} = \ker(P_n \to P_{n-1}) \). Then we have the following exact commutative diagram:

\[
\begin{array}{ccccccccc}
\cdots & P_{n+2} & g & P_{n+1} & f & P_n & \cdots & P_0 & M & 0 \\
& \pi & \downarrow & \lambda & & & & & & \\
& 0 & \downarrow & K_{n+1} & 0 & .
\end{array}
\]

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By (3), $\Ext^{n+1}_{RD}(M, K_{n+1}) = 0$. Thus the sequence

$$\Hom(P_n, K_{n+1}) \xrightarrow{f^*} \Hom(P_{n+1}, K_{n+1}) \xrightarrow{g^*} \Hom(P_{n+2}, K_{n+1})$$

is exact. Since $g^*(\pi) = \pi g = 0$, $\pi \in \ker(g^*) = \im(f^*)$. Thus there exists $h \in \Hom(P_n, K_{n+1})$ such that $\pi = f^*(h) = hf = h\lambda \pi$, and hence $h\lambda = 1$ since $\pi$ is epic. So the exact sequence $0 \to K_{n+1} \xrightarrow{\lambda} P_n \to K_n \to 0$ is split. Therefore $K_n$ is $RD$-projective.

$(4) \Rightarrow (1)$ is obvious. 

Dually, we have the following proposition.

**Proposition 3.6** The following are equivalent for a left $R$-module $N$ and an integer $n \geq 0$:

1. $RD - \text{id}(N) \leq n$.
2. $\Ext^{n+1}_{RD}(M, N) = 0$ for all left $R$-modules $M$.
3. Every $n$th $RD$-injective cosyzygy of $N$ is $RD$-injective.

Combining Propositions 3.5 with 3.6, we have

**Theorem 3.7** The following are equivalent for a ring $R$ and an integer $n \geq 0$:

1. $lRD - PD(R) \leq n$.
2. $lRD - ID(R) \leq n$.
3. $\Ext^{n+1}_{RD}(M, N) = 0$ for all left $R$-modules $M, N$.
4. $\Ext_{RD}^{n+1}(M, N) = 0$ for all left $R$-modules $M$ and $n \geq 1$.

We list some corollaries of Theorem 3.7 as follows.

**Corollary 3.8** For any ring $R$, $lRD - PD(R) = lRD - ID(R)$.

**Corollary 3.9** The following are equivalent for a ring $R$:

1. $lRD - PD(R) = lRD - ID(R) = 0$.
2. Every left $R$-module is $RD$-projective.
3. Every left $R$-module is $RD$-injective.
4. $\Ext^n_{RD}(M, N) = 0$ for all left $R$-modules $M, N$.
5. $\Ext^1_{RD}(M, N) = 0$ for all left $R$-modules $M$ and $n \geq 1$. 

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(6) Every left \( R \)-module \( RD \)-exact sequence is split.

**Corollary 3.10** The following are equivalent for a ring \( R \):

1. \( lRD - PD(R) = lRD - ID(R) \leq 1 \).
2. Every \( RD \)-submodule of an \( RD \)-projective left \( R \)-module is \( RD \)-projective.
3. For any \( RD \)-submodule of an \( RD \)-injective left \( R \)-module \( M \), \( M/N \) is \( RD \)-injective.
4. \( \text{Ext}^n_{RD}(M,N) = 0 \) for all left \( R \)-modules \( M \) and \( N \) and \( n \geq 2 \).
5. \( \text{Ext}^2_{RD}(M,N) = 0 \) for all left \( R \)-modules \( M \) and \( N \).

Finally, we discuss the relations between the \( RD \)-homological dimensions and other homological dimensions.

Recall that \( R \) is left strongly \( P \)-coherent [15] if every principal left ideal of \( R \) is cyclically presented.

**Theorem 3.11** Let \( R \) be a left strongly \( P \)-coherent ring. Then

1. \( RD - \text{id}(M) = \text{id}(M) \) for a divisible left \( R \)-module \( M \).
2. \( RD - \text{pd}(M) = \text{pd}(M) \) for a torsionfree left \( R \)-module \( M \).
3. \( \sup\{\text{id}(M)\}: \text{M is any divisible left R-module}\} \leq lRD - ID(R) \).
4. \( \sup\{\text{pd}(M)\}: \text{M is any torsionfree left R-module}\} \leq lRD - PD(R) \).

**Proof.** (1) Let \( M \) be a divisible left \( R \)-module. By [15, Lemma 4.10] and Proposition 2.3, a right injective resolution of \( M \) must be its right \( RD \)-injective resolution. So \( RD - \text{id}(M) \leq \text{id}(M) \). Conversely, we may assume \( RD - \text{id}(M) = m < \infty \). There is an exact sequence

\[
0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots \rightarrow E^{m-1} \rightarrow L^m \rightarrow 0
\]

with each \( E^i \) injective. By [15, Lemma 4.10] and Proposition 2.3, the above sequence is an \( RD \)-exact sequence. Thus \( L^m \) is divisible and \( RD \)-injective by Proposition 3.6, and hence is injective by Corollary 2.5 (1). So \( \text{id}(M) \leq m \). Thus \( RD - \text{id}(M) = \text{id}(M) \).

(2) Let \( M \) be a torsionfree left \( R \)-module. By [15, Lemma 4.10] and Proposition 2.4, a left projection resolution of \( M \) must be its left \( RD \)-projective resolution. So \( RD - \text{pd}(M) \leq \text{pd}(M) \).

Conversely, we may assume \( RD - \text{pd}(M) = n < \infty \). There exists an exact sequence

\[
0 \rightarrow K_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,
\]

where each \( P_i \) is projective. By [15, Lemma 4.10] and Proposition 2.4, the above sequence is an \( RD \)-exact sequence. So \( K_n \) is torsionfree and \( RD \)-projective by Proposition 3.5, and so is projective by Corollary 2.5 (2). Thus \( \text{pd}(M) \leq n \). Hence \( RD - \text{pd}(M) = \text{pd}(M) \).

(3) follows from (1), (4) holds by (2).

Observing the following facts:
(1) If \( R \) is a von Neumann regular ring, then \( lD(R) = lRD - ID(R) \) by Corollary 2.14.

(2) If \( lRD - ID(R) = 0 \), then \( ID(R) = wD(R) \).

In general, we have the following inequalities.

**Theorem 3.12** Let \( R \) be a ring. Then
\[
lD(R) \leq lRD - ID(R) + \sup \{ \text{id}(M) : M \text{ is any RD-injective left } R\text{-module} \}
\]
\[
\leq lRD - ID(R) + wD(R).
\]

**Proof.** By Theorem 2.12, \( \sup \{ \text{id}(M) : M \text{ is any RD-injective left } R\text{-module} \} \leq \sup \{ \text{fd}(M) : M \text{ is any RD-flat right } R\text{-module} \} \leq wD(R) \). So the second inequality in the theorem holds.

Next we show that \( lD(R) \leq lRD - ID(R) + \sup \{ \text{id}(M) : M \text{ is any RD-injective left } R\text{-module} \} \). We may assume that both \( lRD - ID(R) \) and \( \sup \{ \text{id}(M) : M \text{ is any RD-injective left } R\text{-module} \} \) are finite. Let \( lRD - ID(R) = m < \infty \) and \( \sup \{ \text{id}(M) : M \text{ is any RD-injective left } R\text{-module} \} = n < \infty \). Suppose \( M \) is a left \( R\)-module, then \( M \) admits a right RD-injective resolution
\[
0 \to M \to E^0 \to E^1 \to \cdots \to E^{m-1} \to E^m \to 0.
\]

Note that \( \text{id}(E^i) \leq n \). For every left \( R\)-module \( N \), we have
\[
\text{Ext}^{n+m+1}(N, M) \cong \text{Ext}^{n+1}(N, E^m) = 0.
\]

So \( \text{id}(M) \leq n + m \). Thus \( lD(R) \leq n + m \).

We conclude this paper with the following

**Remark 3.13** (1) Let \( R = \mathbb{Z} \). Then \( D(R) = RD - ID(R) = wD(R) = 1 \).

By [21, 40.5], \( \sup \{ \text{id}(M) : M \text{ is any divisible left } R\text{-module} \} = 0 \). So the inequality \( \sup \{ \text{id}(M) : M \text{ is any divisible left } R\text{-module} \} \leq lRD - ID(R) \) in Theorem 3.11 may be strict.

On the other hand, by Corollaries 2.13 and 2.14, \( \sup \{ \text{id}(M) : M \text{ is any RD-injective left } R\text{-module} \} = 1 \). Thus the inequality \( lD(R) \leq lRD - ID(R) + \sup \{ \text{id}(M) : M \text{ is any RD-injective left } R\text{-module} \} \) in Theorem 3.12 may be strict.

(2) The second inequality in Theorem 3.12 may be also strict. For example, by [10, Corollary, p.439], there exists a left Noetherian domain \( R \) with \( lD(R) = wD(R) = 2 \). Then \( \sup \{ \text{id}(M) : M \text{ is any RD-injective left } R\text{-module} \} = 1 \) by Corollary 2.13.

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